

Examples of ergodic infinite measure-preserving bijections

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Abstract

This very informal note aims to present different ∞ -mp ergodic bijections on standard σ -finite spaces. Constructing examples of such bijections is ‘a non-trivial task’ according to Halmos, but since Kakutani’s skyscraper construction in 1943, which to our knowledge is the first significant and non-trivial one, many other examples have been studied. We go into a bit more details and provide some proofs in the first sections about Kakutani skyscrapers, as they represent the most relevant examples for the author’s work, and remain more concise for the other sections, with some exceptions.

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Malgré moi l’infini me tourmente.
Alfred de Musset

1 Kakutani Skyscrapers

The Kakutani skyscraper construction is one of the first (if not the first) meaningful example of ∞ -mp ergodic bijection (see [Kak43]), and it is in a way the reverse of the induced bijection construction, which we recall.

Definition 1.1. Consider (X, λ) a standard σ -finite space. Let $T \in \text{Aut}(X, \lambda)$, and $A \subseteq \mathfrak{C}(T)$ be a Borel subset with $\lambda(A) > 0$. Halmos' Recurrence Theorem ensures us that for λ -almost every $x \in A$ there exists a (finite) **smallest return time** in A denoted by $n_A(x)$, *i.e.*

$$n_A(x) := \min\{n \in \mathbb{N}^* \mid T^n(x) \in A\}.$$

We can then define the **induced bijection** T_A by $T_A(x) = T^{n_A(x)}(x)$ for all $x \in A$ and $T_A(x) = x$ for all $x \notin A$. Partitioning A in sets of the form $\{x \in A \mid n_A(x) = n\}$ easily yields that $T_A \in [T]$.

We start by describing a very concrete example before giving the general construction.

1.1 Basic skyscraper example

We use probability measure-preserving ergodic bijections to construct a Kakutani skyscraper ∞ -mp ergodic bijection. Let $X_1 = [0, 1[$ be equipped with the Lebesgue measure μ , and $T = r_\theta$, the irrational rotation of angle θ in $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1[$. Define $X_n = [0, \frac{1}{n}[$ for any $n > 1$, and

$$X := \bigsqcup_{n \in \mathbb{N}^*} X_n \times \{n\}.$$

We have $X \subseteq X_1 \times \mathbb{N}$, and it is equipped with the product measure of μ with the counting measure. For (x, n) in $X_n \times \{n\}$ we then define S by

$$S(x, n) = \begin{cases} (x, n+1) & \text{if } x \in X_{n+1} \\ (T(x), 1) & \text{else.} \end{cases}$$

Since $\mu(X_n) = \frac{1}{n}$, X is an infinite σ -finite standard space, and for any x in X_1 we have $\text{orb}_T(x) \times \{1\} = (X \times \{1\}) \cap \text{orb}_S(x, 1)$, therefore S is ergodic because T is, hence S is totally conservative and aperiodic. Figure 1 shows part of the positive S -orbit of an element $(x, 1)$ of the base.

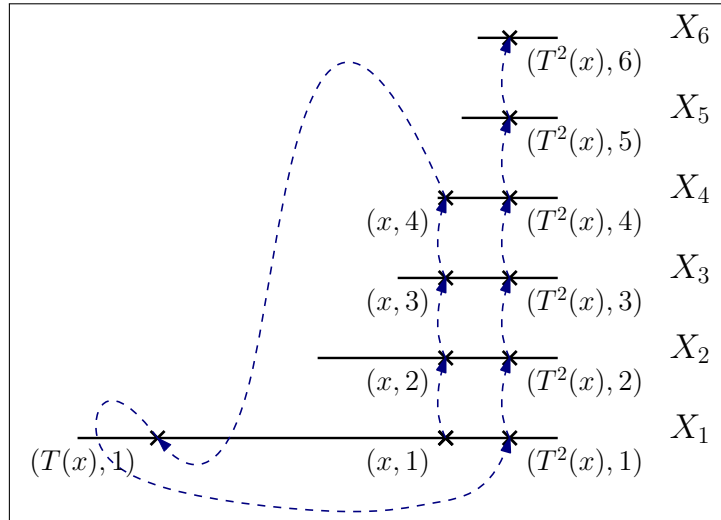


Figure 1: Positive S -orbit on the first floors of X .

Remark 1.2. Here we chose the irrational rotation r_θ as the ergodic bijection acting on the base X_1 of the skyscraper, but of course any other choice would yield a satisfying ∞ -mp ergodic bijection. For instance in [EHIP14, p. 4.1.2] the authors detail the case where the adding machine was chosen instead. We give more details about this example later.

1.2 General skyscraper construction

In this section we give the general construction, which has varied applications. As stated earlier, the Kakutani skyscraper is in a way a reverse construction to the induced bijection construction defined in Definition 1.1. We give two equivalent definitions commonly found in the literature, as we believe they provide a different intuition and nicely complement each other.

Our first definition explicitly describes the building blocks of the skyscraper through the height function. This definition builds the skyscraper vertically while the second definition does it horizontally.

Definition 1.3 (First skyscraper definition, [EHIP14, p. 4.1.1]). The base of the skyscraper is defined in the following way: we let $(A, \mathcal{B}_A, \lambda_A)$ be a standard σ -finite measure space of finite or infinite total mass, and we let T_A be a λ_A -preserving conservative self-bijection of A .

We let $(B_k)_{k \in \mathbb{N}}$ be a sequence of mutually disjoint Borel subsets of A satisfying

$$\begin{cases} A = \bigsqcup_{k \in \mathbb{N}} A_k \\ \lambda(A_k) \rightarrow 0. \end{cases}$$

We define this time the height function simply as an increasing sequence of nonnegative integers $(\varphi_k)_{k \in \mathbb{N}} = \{0 \leq \varphi_0 \leq \varphi_1 \leq \varphi_2 \leq \dots\}$, and for each $k \in \mathbb{N}$ we consider $\varphi_k + 1$ isomorphic (as measured spaces) copies of A_k (the first copy being A_k itself), that we denote by $A_k = A_k^0, A_k^1, A_k^2, \dots, A_k^{\varphi_k}$. We denote by R_k all the isomorphisms of the form

$$R_k : A_k^i \rightarrow A_k^{i+1}, \text{ for } i \text{ in } \{0, \dots, \varphi_k\} \text{ and } k \geq 0.$$

The measure space we consider is the following:

$$X := \bigsqcup_{k \in \mathbb{N}} \bigsqcup_{i=0}^{\varphi_k} A_k^i$$

and we endow it with the natural σ -algebra of all measurable subsets of X (as induced by $\mathcal{B}_A$), and the measure λ naturally defined on $(X, \mathcal{B})$ by summing the measure of the components. We can finally define the bijection T by setting

$$T(x) := \begin{cases} R(x) & \text{if } x \in A_k^i \text{ for } i \text{ in } \{0, \dots, \varphi_k\} \text{ and } k \geq 0 \\ T_A(R^{-\varphi_k}(x)) & \text{if } x \in A_k^{\varphi_k} \text{ for } k \geq 0. \end{cases}$$

The $\bigsqcup_{i=0}^{\varphi_k} A_k^i$ correspond to ‘vertical slices’, and for any $n \in \mathbb{N}^*$ the ‘horizontal’ n -th floor is

$$X_n := \bigsqcup_{k \in D_n} A_k^n = \bigsqcup_{k=l_n}^{+\infty} A_k^n$$

where $D_n := \{k \in \mathbb{N} \mid \varphi_k + 1 \geq n\}$ and $l_n := \min D_n$. We can visualise the construction on Figure 2.

This first definition is ‘external’ in a sense, we define a base and construct the skyscraper above it, with no ambient space to speak of. The point of view chosen for the second definition is ‘internal’, in that we build the skyscraper from Borel subsets of a σ -finite space (X, λ) . They are indeed equivalent.

On Figure 3 we can see the Borel subsets (A_n) coming from $\mathbb{R}_+$, which provides a nice way to visualise this construction. The maps π_n then just mean ‘going down one floor’, and in purple we represented the image by T of a point $x \in A_3 \setminus \pi_3(A_4)$.

The skyscraper construction satisfies some nice properties, some of which are given in the following proposition. In particular the last one guarantees that we can construct many ∞ -mp ergodic bijection this way. We provide the proofs of the next two results, as those properties are quite indispensable in our work. We prove it using the notations of Definition 1.4.

Proposition 1.5 ([CK79, §3]). *Let T be a skyscraper of base (A, T_A) . The following hold:*

1. $T \in \text{Aut}(X, \lambda)$;
2. T is conservative;
3. T_A is the induced bijection of T on A in the sense of Definition 1.1;
4. T is ergodic if and only if T_A is ergodic on A .

Proof. 1. It is clear by construction.

For 2. and 3. the only thing that we have to prove is that for λ -a.e. $x \in X$, there exists a positive power $n \in \mathbb{N}^*$ of T sending x to the base A . For 2. the conservativity of T_A then concludes. This holds by convergence of $(\lambda(A_n))$ to 0: since the π_n are measure preserving, for any x , $x \in A_k$ for some $k \in \mathbb{N}$ and there exists a large enough integer n such that $\pi_{n-1}^{-1} \dots \pi_{k+1}^{-1} \pi_k^{-1}(x) \in A_n \setminus \pi_n(A_{n+1})$. By definition of T , the integer $n - k + 1$ is suitable. For 3., we notice that if $k = 0$, then $n - k + 1$ is exactly the smallest return time in A of x .

4. We now move on to ergodicity. We first notice that for any T -invariant set B in X , the set $B \cap A$ is T_A -invariant. Indeed, using the fact that the family (A_n) is a Kakutani-Rokhlin partition we write

$$B \cap A = \bigsqcup_{n \in \mathbb{N}} B \cap \{x \in A \mid n_A(x) = n\} = \bigsqcup_{n \in \mathbb{N}} T^n(B) \cap \{x \in A \mid n_A(x) = n\} = T_A(B \cap A).$$

Now assume that T_A is ergodic. By the previous argument, $B \cap A$ is either null or conull. We have $\lambda(B \cap T^n(A)) = \lambda(T^n(B \cap A)) = \lambda(B \cap A)$, so it is either null or conull in A . As taking the towers above B yields a partition of the T -saturation of B , this means that B is either null or conull in X .

The converse implication can be proved by very similar arguments: if B is a T_A -invariant subset of A , then the towers above B form a T -invariant subset of X , and ergodicity of T concludes. $\square$

Those skyscraper constructions are our first examples, but they are in a way ‘universal’, as by Rokhlin’s lemma any aperiodic conservative element of $\text{Aut}(X, \lambda)$ can be represented by a skyscraper. We have the following.

Proposition 1.6 ([CK79, Prop. 2(i)]). *Let $T \in \text{Aut}(X, \lambda)$ be conservative aperiodic. For any $\varepsilon > 0$ there exists a Borel subset $C \subseteq X$ such that $\lambda(C) < \varepsilon$ and such that T is a skyscraper over (C, T_C) .*

Proof. The case of a finitely supported T is easier and taken care of by a straightforward application of Rokhlin’s lemma, so let’s assume that $\text{supp } T$ has infinite measure. Fix $\varepsilon > 0$ and let $\text{supp } T = \bigsqcup_{n \in \mathbb{N}} X_n$, with $\lambda(X_n) = \varepsilon$ for all n in $\mathbb{N}$. Do note at this point that for λ -almost all $x \in X_n$ we have $\text{orb}_{T_{X_n}}(x) = \text{orb}_T(x) \cap X_n$, therefore we can work with the T_{X_n} orbits, which are infinite by Halmos’ Recurrence Theorem. In each X_n , we apply

Rokhlin's (aperiodic) lemma (see *e.g.* [KM10, Thm. 172]) to find a subset $A_n \subseteq X_n$ such that $A_n, T_{X_n}(A_n), \dots, T_{X_n}^{2^{n+2}}(A_n)$ are pairwise disjoint, and such that B_n the complement of the tower in X_n satisfies

$$\lambda(B_n) = \lambda \left(X_n \setminus \bigsqcup_{k=0}^{2^{n+2}} T_{X_n}^k(A_n) \right) < 2^{-(n+2)}\varepsilon.$$

Define now $C_n = A_n \sqcup B_n$ and notice that C_n intersects all the T_{X_n} -orbits on X_n , so $C = \bigsqcup_n C_n$ intersects all the T -orbits. By construction $\lambda(A_n) < 2^{-(n+2)}\varepsilon$ and we then have

$$0 < \lambda(C) = \sum_{n \in \mathbb{N}} \lambda(C_n) = \sum_{n \in \mathbb{N}} (\lambda(A_n) + \lambda(B_n)) < \varepsilon,$$

which concludes the proof, as T is clearly a skyscraper over C . $\square$

Remark 1.7. The skyscraper construction is ‘universal’ in the sense of Proposition 1.6 for ∞ -mp ergodic bijections, but finer dynamical properties are encoded by the parameters of the floors. For example [EHIP14, p. 4.1.2] and [EHIP14, p. 4.3.1] exhibit very different behaviours.

Example 1.8. Using the notations of Definition 1.3, we quickly present the historical example of Hajian and Kakutani from [HK70], which is also the one exposed in [EHIP14, p. 4.1.2]. The base space X_0 is just the unit interval endowed with the usual Lebesgue measure, and T_A is the dyadic odometer. We then set $A_k = [\frac{1}{2^k}, \frac{1}{2^{k+1}}[$ and define the height function φ by $\varphi_0 = 0$ and

$$\varphi_k = 2 + 2^3 + \dots + 2^{2k-1}$$

for any $k > 0$. This was the first example of an ∞ -mp ergodic bijection for which an exhaustive weakly wandering sequence was exhibited (we recall that $(n_i)_{i \in \mathbb{N}}$ is an exhaustive weakly wandering sequence for T if there exists W of positive measure such that $X = \bigsqcup_{i \in \mathbb{N}} T^{n_i}(W)$).

2 Maharam extensions

Our second general class of examples of ∞ -mp ergodic bijections is the class of Maharam extensions of non-singular ergodic bijections. The main idea behind the Maharam extension is to compensate the ‘loss of measure-preservation’ by considering infinitely many copies of the original space (equivalently, adding a new coordinate), and by moving ‘vertically’ along those copies in a proportional manner to the ‘horizontal’ movement in the base space. The main point is that the Radon-Nikodym derivative quantifies exactly ‘how much of the measure-preservation is lost’.

We present two versions of this construction, a *continuous* Maharam extension with uncountably many copies (*i.e.* $X = Y \times \mathbb{R}$), and a *discrete* Maharam extension with countably many copies (*i.e.* $X = Y \times \mathbb{Z}$), which can only be defined in some cases.

2.1 Maharam $\mathbb{R}$ -extension

This section presents a classical variation of the historical construction from [Mah64] (see [Aar97, p. 3.4] for a more modern exposition). We start with the general definition then give a specific example also from [Aar97] (the first example of an ergodic Maharam $\mathbb{R}$ -extension is from [Kri76]).

We let $(Y, \mathcal{B}(Y), \mu)$ be a standard probability space, and $T \in \text{Aut}(Y, [\mu])$ be a non-singular bijection.

The construction. The Radon-Nikodym cocycle exactly encodes the measure-preservation perturbation that we want to compensate. We then define the space, the σ -algebra, and the extension $\tilde{T}$ of T :

$$\begin{cases} X = Y \times \mathbb{R} \\ \mathcal{B}(X) = \mathcal{B}(Y) \otimes \mathcal{B}(\mathbb{R}) \\ \tilde{T}(y, t) := (T(y), t - \ln c(T, y)). \end{cases}$$

It is easy to check that the measure λ defined by $d\lambda = d\mu \times e^t dt$ is infinite, σ -finite, and $\tilde{T}$ -invariant. The first result is the following.

Proposition 2.1 ([Mah64, Thm. 2]). *T is conservative on (X, μ) if and only if $\tilde{T}$ is conservative on (X, λ) .*

Remark 2.2. We chose here the additive notation by using the logarithm (which is more common in the literature), but it is possible to define a *multiplicative Maharam extension*, which is isomorphic to the one we just defined. For this we choose $0 < a < 2$ and consider the space

$$(X = Y \times \mathbb{R}_+^*, \mathcal{B}(X) = \mathcal{B}(Y) \otimes \mathcal{B}(\mathbb{R}_+^*), \lambda = \mu \otimes \frac{1}{t^{1+a}} dt)$$

along with the bijection $\tilde{T}_a$ defined by

$$\tilde{T}_a(y, t) = \left(T(y), t \times \left(c(T, y)^{\frac{1}{a}} \right) \right).$$

See *e.g.* [Roy12] for more details about this multiplicative version.

Remark 2.3. The technique used for the Maharam extension of one bijection can easily adapt to non-singular group actions by generalizing the definition of the Radon-Nikodym cocycle:

$$c: \begin{cases} G \times Y & \longrightarrow \mathbb{R} \\ (g, y) & \longmapsto c(g, y) = \frac{dg_*\mu}{d\mu}(y) \end{cases}$$

and in this case we talk about the Maharam $\mathbb{R}$ -extension of $G \curvearrowright Y$. We refer to [BKV21] for more details about this construction.

An example. This example is detailed at the end of § 8.4 in [Aar97]. The space is the countable product

$$Y = \prod_{n \in \mathbb{N}^*} \{0, \dots, a_n - 1\}$$

Where $a_n = 2^n$ for any $n \in \mathbb{N}$. We let $(\gamma_1, \gamma_2, \dots) = (1, \sqrt{2}, 1, \sqrt{2}, \dots)$ and (β_n) be such that

$$\beta_n(k) := \begin{cases} k\gamma_n & \text{for } 0 \leq k \leq a_n - 2 \\ 0 & \text{for } k = a_n - 1. \end{cases}$$

Instead of the usual Haar measure (*i.e.* the uniform measure $\prod_{n \in \mathbb{N}^*} (\frac{1}{a_n}, \dots, \frac{1}{a_n})$) on Y , we define the measure μ by

$$\mu := \prod_{n \in \mathbb{N}^*} p_n, \text{ where } p_n(k) := \frac{e^{\beta_n(k)}}{\sum_{j=1}^{a_n-1} e^{\beta_n(j)}}.$$

We finally define T as the odometer on (Y, μ) , and $\tilde{T}$ as the $\mathbb{R}$ -Maharam extension of T (which is well defined since $T_*\mu \in [\mu]$). It can be shown that $\tilde{T}$ is ergodic on $X = Y \times \mathbb{R}$.

We have the following, allowing us to know whether or not the Maharam extension is ergodic.

Proposition 2.4 ([DS22, Prop. 5.2]). *The Maharam extension $\tilde{T}$ of T is ergodic if and only if T is of type III₁.*

2.2 Maharam $\mathbb{Z}$ -extension

In this section we explain the example from [EHIP14, p. 4.2.3], itself based on [HIK72, Sec. 3], after briefly explaining how the general $\mathbb{R}$ -construction adapts.

The construction. In cases where the Radon-Nikodym cocycle can be written as $c(T, \cdot) = a^\phi$ for some $a \in]1, +\infty[$ and some measurable function $\phi : Y \rightarrow \mathbb{Z}$, one can define the $\mathbb{Z}$ -Maharam extension of T by setting

$$\begin{cases} X = Y \times \mathbb{Z} \\ \mathcal{B}(X) = \mathcal{B}(Y) \otimes \mathcal{B}(\mathbb{Z}) \\ \tilde{T}(y, n) := (T(y), n - \phi(y)). \end{cases}$$

One can check that the measure λ defined by $\lambda(A \times \{n\}) := a^n \mu(A)$ is $\tilde{T}$ -invariant.

An example. We closely follow [EHIP14, p. 4.2.3]. The base space is $Y = \{0, 1\}^{\mathbb{N}^*}$ with its natural σ -algebra, but we do not consider the uniform measure. Instead, we choose $\alpha \in \mathbb{R}_+^*$, we set $\beta = \frac{\alpha}{1 + \alpha}$, and we define

$$\mu_n := \beta \delta_0 + (1 - \beta) \delta_1$$

where δ_i is the Dirac measure on $\{i\}$. In other words for any $n \in \mathbb{N}^*$ we have $\mu_n(0) = \beta$ and $\mu_n(1) = 1 - \beta$. We then endow Y with μ , which is the product measure of the μ_n , and remove from Y the μ -null set of sequences with finitely many 0 or finitely many 1. We will denote by $Y[j]$ the following cylinder set

$$Y[j] := \{(y_i)_{i \in \mathbb{N}^*} \in Y \mid y_0 = 1, y_1 = 1, \dots, y_{j-1} = 1, y_j = 0\},$$

which contains all the sequences whose first 0 is at position j . It is clear that $Y = \bigsqcup_{j \in \mathbb{N}^*} Y[j]$. We then define T_Y as the (non-singular here!) odometer on Y :

$$T_Y(Y[j]) = \{(y_i)_{i \in \mathbb{N}^*} \in Y \mid y_0 = 0, y_1 = 0, \dots, y_{j-1} = 0, y_j = 1\}$$

We have

$$\begin{cases} \mu(Y[j]) = (1 - \beta)^{j-1} \beta \\ \mu(T_Y(Y[j])) = \beta^{j-1} (1 - \beta) = \alpha^{j-2} \mu(Y[j]). \end{cases}$$

We have that T_Y is ergodic and that $c(T_Y, \cdot) = \alpha^\phi$ for $\phi : Y \rightarrow \mathbb{Z}$ defined by $\phi(y) = j_y - 2$ where j_y is the unique $j \in \mathbb{N}^*$ such that $y \in \mu(Y[j])$ (in other words $j_y = \min\{n \in \mathbb{N}^* \mid y_n = 0\}$). Therefore, we can define the ‘extended’-space $X = Y \times \mathbb{Z}$, with its natural σ -algebra, and endow it with the measure λ defined by

$$\lambda(A, n) = \alpha^n \mu(A)$$

for any $(A, n) \in \mathcal{B}(Y) \times \mathbb{Z}$.

The last step is to define $\tilde{T}$, and for any $(y, n) \in Y[j] \times \mathbb{Z}$ we set

$$\tilde{T}(y, n) = (T_Y(y), n + j - 2).$$

A straightforward computation yields that

$$\lambda(\tilde{T}(A, n)) = \lambda(T_Y(A), n + j - 2) = \alpha^{n+j-2} \mu(T_Y(A)) = \alpha^n \mu(A) = \lambda(A, n)$$

for any $(A, n) \subseteq (Y[j], n)$, with j and n in $\mathbb{N}^*$ and $\mathbb{Z}$ respectively, so $\tilde{T}$ is λ -preserving.

We will not prove the following proposition, which uses the Hewitt-Savage Zero-One Law. We refer to [EHIP14, pp. 4.2.2, 4.2.3] for the proofs.

Proposition 2.5. *The $\mathbb{Z}$ -Maharam extension $\tilde{T}$ is ergodic on (X, λ) .*

Remark 2.6. The odometer T_Y on Y that we presented is non-singular because the measure μ is not the uniform one, but one may wonder if there exists an equivalent (or just μ -absolutely continuous) measure that is T_Y -invariant. Arnold's theorem states that this cannot happen (see *e.g.* [Aar97, Thm. 1.2.9]). If we look at measures that are supported on different sets, this is another matter entirely. We present in Section 4.2 an example of an infinite measure-preserving ergodic odometer, whose invariant measure is supported on a set which is null for the usual uniform measure.

3 Infinite measure rank-one bijections

Rank-one transformations, or rank-one bijections are a powerful generalization of both the odometers and the irrational rotations. Historically, the *Chacon* map, defined in 1966, opened the way to the study of these bijections, and they since then have proved to be a rich source of examples and counter-examples in ergodic theory. Sadly, they are also notorious for having many equivalent definitions, some being very technical. One can consult the impressive survey [Fer97] for more details.

A lot of authors define them in finite measure spaces only, but the definition is flexible enough to encompass the σ -finite spaces as well; the first ones to do so were most likely [AFS97] and by now even the infinite measure context has seen some substantial development (see *e.g.* [Dan04] or [RT15]). Rank-one bijections are a major source of counter-examples in the ∞ -mp context as well; for instance let us quickly consider the following as a motivating example. It is known that in the pmp context, being weakly mixing, doubly ergodic, and Cartesian square ergodic are equivalent (we refer to [AS18] for a comprehensive survey on weak mixing, with a special emphasis on infinite measure bijections). In the ∞ -mp context however, the authors of [BFMS01] proved the following:

Theorem 3.1 ([BFMS01]). *There exists rank-one bijections $S, T \in \text{Aut}(X, \lambda)$ such that S is doubly ergodic, but has non-ergodic Cartesian square and T is weak mixing, but is not doubly ergodic.*

3.1 Infinite Chacon map

Before giving the general construction we present the infinite measure version of the Chacon map, which has the advantage of being less heavy technically, while still providing some insight. It is of note that the *probability measure-preserving* version of Chacon's map was historically the first example of a transformation that is weakly mixing, but not (strongly) mixing (see [Cha69]). Such transformations were known to exist, since weakly mixing transformations are generic, while strong mixing ones form a set of the first category (see [Hal56]).

This section follows the exposition of [JRR18, § 1.2], and the construction is from [AFS97, § 2]. We let $X = \mathbb{R}_+$. The construction follows the iterative method of *cutting and stacking*.

First step: At first we consider the interval $[0, 1]$, cut into three subintervals of same length. We call this the *Tower 0*, and declare that it has height $h_0 = 1$. We take an extra interval of length $\frac{1}{3}$ (for instance $[1, \frac{4}{3}]$) and stack it on top of the middle part of Tower 0. We then take four (this four is $(3 \times h_0) + 1$) more intervals of length $\frac{1}{3}$ and stack them all on top of the right part of Tower 0. We call those additional parts *spacers*. At step 1, spacers have length $\frac{1}{3} = \frac{1}{3^1}$. We finally construct Tower 1 by stacking the middle part of the *modified* Tower 0 on top of the left part, and by stacking the right part on top of the obtained stack. The obtained Tower 1 has height $h_1 = 8$. Figure 4 shows the cutting and

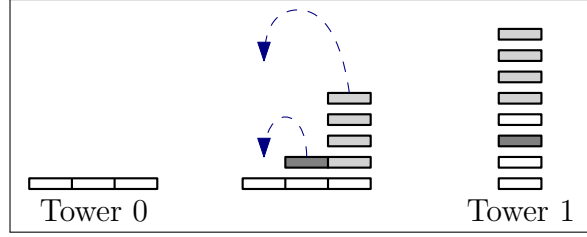


Figure 4: The first step of the construction.

stacking process of the first step. The transformation T maps each point of Tower 1 to the point exactly above it. Thus T is not yet defined on the top floor (floor 8).

($n+1$)-th step: After n steps we have a Tower n of height h_n , made of floors which are semi-open intervals of length $\frac{1}{3^n}$. We then subdivide Tower n in three subcolumns of equal length $\text{len}(\text{Tower } n)/3 = \frac{1}{3^{n+1}}$. We add one spacer of length $\frac{1}{3^{n+1}}$ on top of the middle subcolumn, and $3h_n + 1$ spacers of length $\frac{1}{3^{n+1}}$ on top of the right subcolumn. We can pick the extra intervals successively by taking for instance the leftmost interval of desired length in the unused part of X . We can then, like in the first step, stack the second subcolumn on top of the first one, and the third one on top of it all. Figure 5 shows this step.

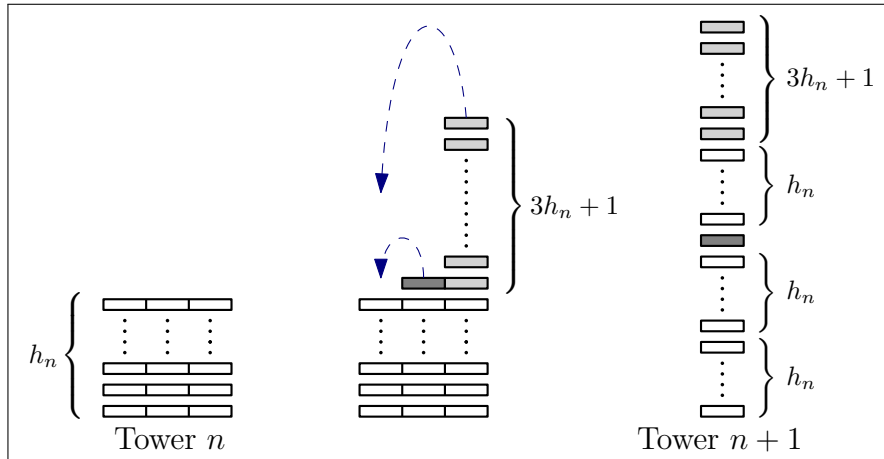


Figure 5: The $(n+1)$ -th step of the construction.

Similarly, T maps each point of Tower $n+1$ to the point exactly above it. Again, it is not defined on the top floor, which has length $\frac{1}{3^{n+1}}$.

At each step, we add two more spacers than three times the number of floors of the previous steps, in particular, the measure of the added spacers is more that the total measure of the previous floor. Clearly, the added measure of the spacers goes to infinity, so at the limit we get a $\text{Leb}_{\mathbb{R}_+}$ -measure preserving bijection. We have the following.

Theorem 3.2 ([AFS97, Thm. 2.2]). *The previously defined bijection T is ergodic, and in fact all of its Cartesian powers are ergodic.*

3.2 General geometric rank-one bijections

Here we understand *geometric* in the sense of [Fer97], who opposes this definition to the *symbolic* one. The definition we chose is the one from [Cor25]. Another notable reference

is [DV25], which presents the more general (C, F) -constructions. Their impressive work completely solves the classification of rank-one systems, which is a point we will not touch upon.

We define a rank-one transformations through its parameters.

Definition 3.3. We define a **cutting and spacing parameter** as a tuple $(q, (\sigma_0, \dots, \sigma_q))$ such that $q \in \mathbb{N}^*$ and $\sigma_k \in \mathbb{N}$ for any $k \leq q$. The parameter q will represent the ‘number of parts in which we will cut each floor of the tower’ and the numbers σ_n will represent the ‘number of spacers added in-between each floor of the next tower’. We then define

- (1) $\mathcal{P}$ the space of all possible cutting and spacing parameters;
- (2) $\mathcal{P}^{\mathbb{N}}$ the space of sequences of cutting and spacing parameters;
- (3) $\mathcal{P}^{<\mathbb{N}} = \bigcup_n \mathcal{P}^n$ the space of finite sequences of cutting and spacing parameters.

From a sequence $p = (q_n, (\sigma_0^n, \dots, \sigma_{q_n}^n))_{n \in \mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ of cutting and spacing parameters, we say that the k -th coordinate $p_k = (q_k, (\sigma_0^k, \dots, \sigma_{q_k}^k))$ is the **k -th cutting and stacking parameter** of p . We also define

1. The **number of spacers** added at step n : $(\sigma_n)_{n \in \mathbb{N}}$, with $\sigma_n = \sum_{k=0}^{q_n} \sigma_k^n$;
2. The **height of the tower** at step n : $(h_n)_{n \in \mathbb{N}}$, with h_n defined inductively by

$$\begin{cases} h_0 := 0 \\ h_{n+1} = q_n h_n + \sigma_n. \end{cases}$$

Figure 6 shows the role of the different parameters for the construction of a tower $n+1$ from a tower n .

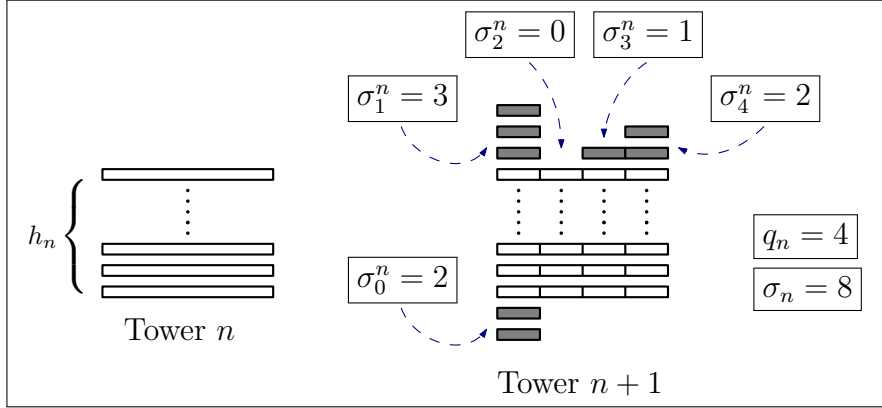


Figure 6: Visualisation of the parameters on a example.

Remark 3.4. In the example of Section 3.1 we had $p = (3, (0, 0, 1, 3h_n + 1))_{n \in \mathbb{N}}$, where the height h_n of the tower at step n was defined iteratively as explained.

Lemma 3.5 ([Cor25, Lem. 3.3]). *For any $p \in \mathcal{P}^{\mathbb{N}}$, the following are equivalent:*

- (i) $\sum_{n \in \mathbb{N}} \frac{\sigma_n}{h_{n+1}}$ converges;
- (ii) $\sum_{n \in \mathbb{N}} \frac{\sigma_n}{q_0 \dots q_n}$ converges.

The convergence series in the statement of the previous lemma correspond to the finiteness of the measure that will be preserved by the rank-one bijection. Hence we are of course specifically interested in sequences yielding divergent series.

Definition 3.6. A bijection $T \in \text{Aut}(X, \lambda)$ is a **rank-one bijection** if there exists $p = (q_n, (\sigma_0^n, \dots, \sigma_{q_n}^n))_{n \in \mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ such that there exists sequences

$$\begin{cases} (B_n)_{n \in \mathbb{N}} & \text{(the bases of the towers),} \\ \{B_{n,i} \in \mathcal{B}(X) \mid n \in \mathbb{N}, 0 \leq i \leq q_n - 1\} & \text{(the bases of the subcolumns),} \\ \{\Sigma_{n,i,j} \in \mathcal{B}(X) \mid n \in \mathbb{N}, 0 \leq i \leq q_n, 0 \leq j \leq \sigma_i^n\} & \text{(the spacers),} \end{cases}$$

of Borel subsets of X such that for any $n \in \mathbb{N}$ we have

1. $B_n, T(B_n), \dots, T^{h_n-1}(B_n)$ are pairwise disjoint;
2. $B_{n,0}, \dots, B_{n,q_n-1}$ form a partition of B_n ;
3. $T^{h_n}(B_{n,i}) = \begin{cases} \Sigma_{n,i+1,1} & \text{if } \sigma_i^n > 0 \\ B_{n,i+1} & \text{if } \sigma_i^n = 0 \text{ and } i < q_n - 1 \end{cases}$
4. if $\sigma_i^n > 0$ then $T(\Sigma_{n,i,j}) = \begin{cases} \Sigma_{n,i,j+1} & \text{if } j < \sigma_i^n \\ B_{n,i} & \text{if } j = \sigma_i^n, \end{cases}$
5. $B_{n+1} = \begin{cases} \Sigma_{n,0,1} & \text{if } \sigma_0^n > 0 \\ B_{n,0} & \text{if } \sigma_0^n = 0. \end{cases}$

Finally, we require that the σ -algebra generated by the $\mathcal{R}_n := (T^k(B_n))_{0 \leq k \leq h_n-1}$ grows to the restriction of the σ -algebra $\mathcal{B}$ of the ambient space to the support of T . In particular, it separates the points.

In more casual terms, the bijection T associated to the sequence of parameters p maps each point of any ‘non-top’ floor of the tower to the floor above (as in the last drawing of Figure 5), each point of the top floor of each ‘non-last’ subcolumn to the bottom floor of the next subcolumn (as in Figure 7), and the measure of the last floor of the last subcolumn, where T is not defined, goes to 0.

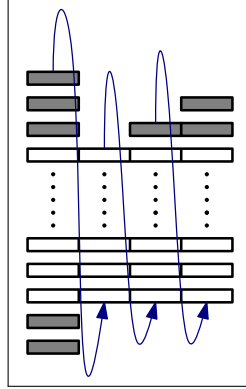


Figure 7: Behaviour of T on the top floor of each subcolumn.

Theorem 3.7 ([DS22, § 3.4]). *Every rank-one bijection is ergodic.*

Another remarkable result is the following. It has been known for a long time that rank-one bijections are generic in the group of measure-preserving bijections of a standard probability space endowed with the weak topology, and the result still holds in standard spaces of infinite measure.

Theorem 3.8 ([BSSSW15, Thm. 3.1]). *Rank-one bijections are generic in $\text{Aut}(X, \lambda)$ endowed with the weak topology.*

4 The three basic examples in the type II_∞ context

In this section we go back to three basic examples that are well-known in the probability measure-preserving and measure class-preserving contexts, but this time in the infinite measure-preserving context, with of course a particular interest towards ergodicity. We remain very brief on irrational rotations and Bernoulli shifts, but provide a lot of details for a construction of a ∞ -mp ergodic odometer, as we have not seen it fully detailed with explicit computations. The example does appear in the literature (see *e.g.* [DS22, Ex. 5.1.(iii)], [EHH98] or [AN87]), but abstract arguments are used to establish that it is of type II_∞ .

4.1 Irrational rotations and Bernoulli shifts

For the irrational rotations, we refer to [DEJLMS23]. It has been shown by Del Junco in [Del76] that pmp irrational rotations are rank-one. Although it has been known since Schmidt's work in 1977 (see [Sch77]) that there exists infinite invariant measures for rotations on the circle, we want to spotlight [DEJLMS23] (see also [DV23]), where the authors construct *explicit* ∞ -mp rank-one bijections that are isomorphic to irrational rotations. They moreover show that they are totally ergodic (T is totally ergodic if T^n is ergodic for any $n \neq 0$). We thus have the following.

Theorem 4.1 ([DEJLMS23, Theorems 3.1, 5.1 and 6.4], see also [DV23, Thm. B]). *There exists infinitely many irrational angles θ for which the rotation R_θ admits an invariant atomless σ -finite measure λ , for which it is totally ergodic, and such that (R_θ, λ) is isomorphic to an explicit ∞ -mp rank-one bijection (T_θ, λ) .*

Let us now shift our focus towards Bernoulli shifts. The first thing to keep in mind is that the situation is quite different depending on the base space. Recall that a non-singular Bernoulli shift is an action of a countable group $G \curvearrowright (X^G, \bigotimes_{g \in G} \mu_g)$ defined by $(g \cdot x)_h = x_{g^{-1}h}$, and in the case where $G = \mathbb{Z}$ the bijection we are interested in is just the action of the generator 1. When all the μ_g are equal, the action is measure-preserving, but when it is not the case, studying the properties of the action (such as establishing ergodicity and determining the Krieger type) is a complicated problem. We will not go into details, but will state the following corollary of [BKV21, Thm. B], specifically concerning type II_∞ actions. Recall that a locally finite group is a group for which every finitely generated subgroup is finite.

Theorem 4.2 ([BKV21, Thm. B]). *Let G be a countable infinite Abelian group, which is not locally finite. Any non-singular Bernoulli shift of G on a space of base $X = \{0, 1\}$ is never of type II_∞ . In particular this is the case for $\mathbb{Z}$.*

It is of note also that a locally finite G admits non-singular Bernoulli actions of type II_∞ , and in fact of every possible type (see [BKV21, Prop. 3.6]).

Now if the base space is not necessarily $\{0, 1\}$ but any standard Borel space, we have the following.

Theorem 4.3 ([BV22, Thm. A], see also [DK22, Cor. 0.2]). *Every countable infinite amenable group G admits non-singular Bernoulli actions of any Krieger type: II_1 , II_∞ and III_λ for all $\lambda \in [0, 1]$.*

In particular for $G = \mathbb{Z}$ the action of the generator provides a ∞ -mp ergodic bijection.

4.2 An infinite measure-preserving ergodic odometer construction

We once again stress the fact that no originality is claimed for this section. The odometer that we consider is the adding machine $T : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ defined as usual by the addition with $(1, 0, 0, \dots)$, with carry over on the right, and sending $(1, 1, 1, \dots)$ to $(0, 0, 0, \dots)$. Our goal is to describe an infinite measure λ on $\{0, 1\}^{\mathbb{N}}$ and explain why T is λ -measure-preserving and why it is ergodic. Let us rigorously define the space and the corresponding measure.

We describe the sequences in $\{0, 1\}^{\mathbb{N}}$ that we will be interested in by describing their behaviour in even and odd coordinates. For the even coordinates, all sequences will be allowed. We denote by X^e the following space:

$$X^e := \{0, 1\}^{\mathbb{N}}.$$

For the odd coordinates, we will only be interested in the countable set of sequences with a finite number of non-zero coordinates. We denote by X^o the following countable space:

$$X^o := \left\{ y = (y_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}} \mid \exists \text{ finitely many } n \text{ such that } y_n = 1 \right\}$$

The space X will be a countable product of copies of $X^e = \{0, 1\}^{\mathbb{N}} \simeq \{0, 1\}^{2\mathbb{N}}$, indexed by X^o . Any T -orbit will move through all these copies, which will provide ergodicity. We define X as follows:

$$X = \bigsqcup_{y \in X^o} X_y := \bigsqcup_{y \in X^o} \left\{ (x_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}} \mid (x_{2n+1})_{n \in \mathbb{N}} = y \right\} \simeq \mathbb{N} \times \{0, 1\}^{2\mathbb{N}}.$$

Each space X_y contains all the sequences with all possible even coordinates, and odd coordinates fixed to be equal to y , in particular they are all zero after a certain index.

Let us now describe the measure. We endow X^o with the counting measure c , and we let μ_e be the usual product measure on X^e , *i.e.* $\mu_e = \frac{1}{2}(\delta_0 + \delta_1)^{\otimes 2\mathbb{N}}$. We finally endow X with the product measure $\lambda := c \otimes \mu_e$. In other words $\lambda = \sum_{y \in X^o} \mu_e|_{X_y}$ where each $\mu_e|_{X_y}$ is a probability measure. In particular, (X, λ) is a standard σ -finite space. The first important remark is that this measure is not absolutely continuous with regard to the uniform product measure μ on $\{0, 1\}^{\mathbb{N}}$, nor does the converse hold: indeed μ gives full weight to the sequences whose odd coordinates take infinitely many times both values by the strong law of large numbers. Note also that the sequences that are given weight by λ take infinitely many times both values in their even coordinates, and only finitely many times the value 1 in the odd coordinates.

The difficulty in describing the dynamics of T precisely is that T weaves in and out of different X_y . We give some more precision by describing this weaving phenomenon in more details.

Notation 4.4. We define even cylinders of the form $\mathcal{C}_{2l}^e = [x_0, \dots, x_{2l}]^e$. We will write those cylinder sets more precisely when necessary, giving the information of the indices on the top line and of the coordinates on the bottom line:

$$\mathcal{C}_{2l}^e = \left[\begin{array}{cccc} 0 & 2 & \dots & 2l \\ x_0 & x_2 & \dots & x_{2l} \end{array} \right].$$

It clearly has measure $\mu_e(\mathcal{C}_{2l}^e) = \frac{1}{2^{\text{len}(\mathcal{C}_{2l}^e)}} = \frac{1}{2^{l+1}}$. We will use similar notations for odd sequences when necessary, using the exponent o to indicate that we look at odd coordinates. When a cylinder has no exponent, it means that we look at an even-odd cylinder, considering both even and odd coordinates.

The measure μ_e is determined by the even cylinders $\mathcal{C}_{2l}^e$. We have

$$\mathcal{C}_{2l}^e = \bigsqcup_{y \in X^o} \mathcal{C}_{2l}^e \cap X_y,$$

so $T(\mathcal{C}_{2l}^e) = \bigsqcup_{y \in X^o} T(\mathcal{C}_{2l}^e \cap X_y)$. Therefore our main goal is to understand the dynamics of T on sets of the form $\mathcal{C}_{2l}^e \cap X_y$, and to this end we introduce another notation.

Notation 4.5. We denote by S_{2k}^e the even cylinder giving the information of the index of the first 0 appearing in even coordinates. We have

$$S_{2k}^e = \begin{bmatrix} 0 & 2 & \dots & 2k-2 & 2k \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}.$$

Similarly, S_{2k+1}^o and S_k are naturally defined as the analogous odd cylinders and even-odd cylinders.

We fix an even cylinder $\mathcal{C}_{2l}^e$ and an odd sequence $y \in X^o$. There are two cases to consider, but one is very easy to describe.

Case 1. The first zero that appears in either $\mathcal{C}_{2l}^e$ or y appears at index at most $2l+1$, *i.e.* $\mathcal{C}_{2l}^e \cap X_y \subseteq S_k$ with $k \leq 2l+1$.

The odometer action is very well understood on an even-odd cylinder with only ones in the first $k-1$ coordinates, and a zero at index k :

$$T\left(\begin{bmatrix} 0 & 1 & \dots & k-1 & k \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 & 1 & \dots & k-1 & k \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}\right).$$

Therefore, the T -image of $\mathcal{C}_{2l}^e \cap X_y$ is another set of the same form, in which only the coordinates of index less than $2l+1$ have changed: the even cylinder has different coordinates, but the same length, and the sequence y is changed but uniquely determined.

Case 2. The even cylinder only has coordinates equal to one, and the first zero that appears in y appears at index larger than $2l+3$, *i.e.* $\mathcal{C}_{2l}^e \cap X_y \subseteq S_{2k+1}^o$ with $2k+1 \geq 2l+3$.

In this case we have to look at the even coordinates of index $2l < i < 2k+1$. We provide an example to show why this case is more troublesome.

Example 4.6. We consider $\mathcal{C}_2^e = [1, 1]^e$ and $y = (1, 1, 1, 1, 0, *, \dots)$, where y is any sequence in X^o with these first five coordinates fixed. In other words we have $l = 1$ and $k = 4$. Applying T to $\mathcal{C}_2^e \cap X_y$ changes the coordinates of indices 0 to 3 from ones into zeroes, but we then have to look at the even coordinates of indices 4 to 8. We have

$$\mathcal{C}_2^e = [1, 1, 0]^e \sqcup [1, 1, 1, 0]^e \sqcup [1, 1, 1, 1, 0]^e \sqcup [1, 1, 1, 1, 1]^e.$$

It is easy to manually check that

$$\begin{aligned} T(\mathcal{C}_2^e \cap X_y) &= T([1, 1, 0]^e \cap X_y) \\ &\sqcup T([1, 1, 1, 0]^e \cap X_y) \\ &\sqcup T([1, 1, 1, 1, 0]^e \cap X_y) \\ &\sqcup T([1, 1, 1, 1, 1]^e \cap X_y) \\ &= [0, 0, 1]^e \cap X_{y_1} \text{ where } y_1 = (0, 0, 1, 1, 0, *, \dots) \\ &\sqcup [0, 0, 0, 1]^e \cap X_{y_2} \text{ where } y_2 = (0, 0, 0, 1, 0, *, \dots) \\ &\sqcup [0, 0, 0, 0, 1]^e \cap X_{y_3} \text{ where } y_3 = (0, 0, 0, 0, 0, *, \dots) \\ &\sqcup [0, 0, 0, 0, 0]^e \cap X_{y_4} \text{ where } y_4 = (0, 0, 0, 0, 1, *, \dots). \end{aligned}$$

It is also easy to manually check that the measure is preserved: $\lambda(\mathcal{C}_2^e \cap X_y) = \mu_e(\mathcal{C}_2^e) = \frac{1}{4}$, while $\lambda([0, 0, 1]^e \cap X_{y_1}) = \mu_e([0, 0, 1]^e) = \frac{1}{8}$, then the second line of the T -image has measure $\frac{1}{16}$, and the last two have measure $\frac{1}{32}$ each.

We now write the general statement. We denote by d the difference $(2k + 1) - 2l$ (in the previous example we had $d = 3$). We have:

$$\mathcal{C}_{2l}^e \cap X_y = \left(\bigsqcup_{1 \leq i \leq d} \mathcal{C}_{2l}^e \cap S_{2l+2i}^e \right) \sqcup \begin{bmatrix} 0 & 2 & \dots & 2k-2 & 2k \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

This decomposition of $\mathcal{C}_{2l}^e \cap X_y$ keeps track of the position of the first zero, it is at index $2l + 2i$ in each part of the disjoint union indexed by i , and at index $2k + 1$ otherwise. Applying T is then straightforward:

$$\begin{aligned} & T(\mathcal{C}_{2l}^e \cap X_y) \\ &= T \left(\left(\bigsqcup_{1 \leq i \leq d} \mathcal{C}_{2l}^e \cap S_{2l+2i}^e \right) \sqcup \begin{bmatrix} 0 & 2 & \dots & 2k-2 & 2k \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & \dots & 2l & 2l+2 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_1} \\ &\quad \text{where } y_1 = \begin{pmatrix} 1 & 3 & \dots & 2l+1 & 2l+3 & 2l+5 & \dots & 2k-1 & 2k+1 & 2k+3 & 2k+5 & \dots \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 & * & * & \dots \end{pmatrix} \\ &\sqcup \begin{bmatrix} 0 & \dots & 2l+2 & 2l+4 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_2} \\ &\quad \text{where } y_2 = \begin{pmatrix} 1 & 3 & \dots & 2l+1 & 2l+3 & 2l+5 & \dots & 2k-1 & 2k+1 & 2k+3 & 2k+5 & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 0 & * & * & \dots \end{pmatrix} \\ &\dots \\ &\sqcup \begin{bmatrix} 0 & \dots & 2k-4 & 2k-2 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_{d-1}} \\ &\quad \text{where } y_{d-1} = \begin{pmatrix} 1 & 3 & \dots & 2k-3 & 2k-1 & 2k+1 & 2k+3 & 2k+5 & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 & * & * & \dots \end{pmatrix} \\ &\sqcup \begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_d} \\ &\quad \text{where } y_d = \begin{pmatrix} 1 & 3 & \dots & 2k-1 & 2k+1 & 2k+3 & 2k+5 & \dots \\ 0 & 0 & \dots & 0 & 0 & * & * & \dots \end{pmatrix} \\ &\sqcup \begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 0 & \dots & 0 & 0 \end{bmatrix} \cap X_{y_{d+1}} \\ &\quad \text{where } y_{d+1} = \begin{pmatrix} 1 & 3 & \dots & 2k-1 & 2k+1 & 2k+3 & 2k+5 & \dots \\ 0 & 0 & \dots & 0 & 1 & * & * & \dots \end{pmatrix}. \end{aligned}$$

In y_1 the first 1 appears at index $2l + 3$, and there are $d - 1$ ones among its $2k + 1$ first coordinates, in y_2 it appears at index $2l + 5$, and there are $d - 2$ ones among its $2k + 1$ first coordinates. In y_{d-1} it first appears at index $2k - 1$, and it is the only one in the $2k + 1$ first coordinates. In the d -th line, the one in the even cylinder appears so late that not a single one appears the $2k + 1$ first coordinates of y_d . Finally in y_{d+1} the only one has replaced the zero at index $2k + 1$.

In order to completely understand the T -action, let us also describe (for convenience) the preimage of a set of the form $\mathcal{C}_{2l}^e \cap X_y$ where $\mathcal{C}_{2l}^e = [0, 0, \dots, 0]^e$ and $y =$

$(0, 0, \dots, 0, 1, *, *, \dots) \in S_{2k+1}^o$. We still denote by d the difference $(2k+1) - 2l$. Again, we advise the reader to manually test the case where for instance $\mathcal{C}_{2l}^e = [0, 0]^e$ and y is of the form $(0, 0, 0, 0, 1, *, *, \dots)$. The exact same computations (we just switch the 0s and the 1s) yield that

$$\begin{aligned} T^{-1}(\mathcal{C}_{2l}^e \cap X_y) &= \begin{bmatrix} 0 & \dots & 2l & 2l+2 \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_1} \text{ where } y_1 = (1, 1, \dots, 1, 0, \dots, 0, 1, *, *, \dots) \\ &\sqcup \begin{bmatrix} 0 & \dots & 2l+2 & 2l+4 \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_2} \text{ where } y_2 = (1, 1, \dots, 1, 1, 0, \dots, 0, 1, *, *, \dots) \\ &\dots \\ &\sqcup \begin{bmatrix} 0 & \dots & 2k-4 & 2k-2 \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_{d-1}} \text{ where } y_{d-1} = (1, 1, \dots, 1, 0, 1, *, *, \dots) \\ &\sqcup \begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_d} \text{ where } y_d = (1, 1, \dots, 1, 1, *, *, \dots) \\ &\sqcup \begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 1 & \dots & 1 & 1 \end{bmatrix} \cap X_{y_{d+1}} \text{ where } y_{d+1} = (1, 1, \dots, 1, 0, *, *, \dots). \end{aligned}$$

The only extra case that we have to treat is the case where there is actually not a single 1 appearing in y , *i.e.* where y is the constant sequence where all coordinates are 0. This case follows from what we have already done:

$$\begin{aligned} T^{-1}(\mathcal{C}_{2l}^e \cap X_y) &= \bigsqcup_{j=1}^{+\infty} \begin{bmatrix} 0 & \dots & 2l & 2l+2j \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_j} \\ &\text{where } y_j = \begin{pmatrix} 1 & \dots & 2l+2j-1 & 2l+2j+1 & 2l+2j+3 & \dots \\ 1 & \dots & 1 & 0 & 0 & \dots \end{pmatrix} \end{aligned}$$

We can now manually check that the measure is preserved by T .

Proposition 4.7. *The odometer T preserves the measure λ on X .*

Proof. As previously stated, it suffices to show that $\lambda(T(\mathcal{C}_{2l}^e \cap X_y)) = \lambda(\mathcal{C}_{2l}^e \cap X_y)$ for any set of the form $\mathcal{C}_{2l}^e \cap X_y$. We clearly have

$$\lambda(\mathcal{C}_{2l}^e \cap X_y) = \mu_e(\mathcal{C}_{2l}^e) = \frac{1}{2^{l+1}}.$$

If $\mathcal{C}_{2l}^e \cap X_y$ is such that we are in the Case 1, there is nothing to check as it is clear that the T -image is a set of the same form, and the measure is then only defined by the length of the even cylinder. If $\mathcal{C}_{2l}^e \cap X_y$ is such that we are in the Case 2 we have

$$\begin{aligned}
\lambda(T(\mathcal{C}_{2l}^e \cap X_y)) &= \lambda\left(\left[\begin{array}{cccc} 0 & \dots & 2l & 2l+2 \\ 0 & \dots & 0 & 1 \end{array}\right] \cap X_{y_1}\right) \\
&+ \lambda\left(\left[\begin{array}{cccc} 0 & \dots & 2l+2 & 2l+4 \\ 0 & \dots & 0 & 1 \end{array}\right] \cap X_{y_2}\right) \\
&+ \dots \\
&+ \lambda\left(\left[\begin{array}{cccc} 0 & \dots & 2k-4 & 2k-2 \\ 0 & \dots & 0 & 1 \end{array}\right] \cap X_{y_{d-1}}\right) \\
&+ \lambda\left(\left[\begin{array}{cccc} 0 & \dots & 2k-2 & 2k \\ 0 & \dots & 0 & 1 \end{array}\right] \cap X_{y_d}\right) \\
&+ \lambda\left(\left[\begin{array}{cccc} 0 & \dots & 2k-2 & 2k \\ 0 & \dots & 0 & 0 \end{array}\right] \cap X_{y_{d+1}}\right) \\
&= \frac{1}{2^{l+2}} + \frac{1}{2^{l+3}} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^{l+1}}.
\end{aligned}$$

□

The next step is checking the ergodicity of our infinite dyadic odometer. As it is not a product odometer, the usual arguments do not hold, and we have to check it manually. The proof however still boils down to the fact that T acts transitively on cylinders. We provide some additional details.

Proposition 4.8. *The odometer T is ergodic on (X, λ) .*

Proof. Consider a T -invariant set A such that $\lambda(A) > 0$, and assume towards contradiction that $\lambda(X \setminus A) > 0$. By definition of λ , since there exists at least one X_{y_1} (resp. X_{y_2}) such that $\lambda(A \cap X_{y_1}) > 0$ (resp. $\lambda((X \setminus A) \cap X_{y_2}) > 0$) there exists $\mathcal{C}_{2l_1}$ and $\mathcal{C}_{2l_2}$ with $l+1 := \text{len}(\mathcal{C}_{2l_1}) = \text{len}(\mathcal{C}_{2l_2})$, as well as $p \in]0, 1[$ such that

$$\begin{cases} \lambda(A \cap (\mathcal{C}_{2l_1} \cap X_{y_1})) = \mu_{e|X_{y_1}}(A \cap \mathcal{C}_{2l_1}) > p \frac{1}{2^{l+1}} \\ \lambda((X \setminus A) \cap (\mathcal{C}_{2l_2} \cap X_{y_2})) = \mu_{e|X_{y_2}}(A \cap \mathcal{C}_{2l_2}) > (1-p) \frac{1}{2^{l+1}}. \end{cases}$$

Without loss of generality we can assume that $2l$ is larger than the index of the last 1 appearing in both y_{k_1} and y_{k_2} (indeed by definition those are finite numbers, and we can then consider an even cylinder as long as we want for the argument). We now use the transitivity of the T -action on the cylinders: denote by $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the even-odd cylinder whose fixed even coordinates are those of $\mathcal{C}_{2l_1}$ (resp. $\mathcal{C}_{2l_2}$) and whose fixed odd coordinates are the l first coordinates of y_1 (resp. y_2). In other words, for $i = 1, 2$ we have

$$\mathcal{C}_i = \left[\begin{array}{cccccc} 0 & 1 & 2 & \dots & 2l-1 & 2l \\ (\mathcal{C}_{2l_i})_0 & (y_i)_1 & (\mathcal{C}_{2l_i})_2 & \dots & (y_i)_{2l-1} & (\mathcal{C}_{2l_i})_{2l} \end{array} \right].$$

It is easy to send $\mathcal{C}_1$ to $\mathcal{C}_2$ via a power of T , but we insist on two main points. The first one is that our assumption that $2l$ is larger than the non-zero part of y_1 and y_2 ensures that we send X_{y_1} to X_{y_2} by applying our power of T , as long as we do not modify the coordinates after $2l$. The second point is that we can always do that by using either T or T^{-1} : indeed, if the coordinates of $\mathcal{C}_2$ are larger than the coordinates of $\mathcal{C}_1$ for the natural binary order, (e.g. $[0, 1, 1, 1] > [1, 1, 0, 1]$) then a positive power of T works, and a negative power of T works for the converse case. Denote by j this power. We thus have

$$T^j(\mathcal{C}_1 \cap X_{y_1}) = \mathcal{C}_2 \cap X_{y_2},$$

so by T -invariance of A we have

$$\lambda(T^j(A \cap (\mathcal{C}_{2l_1} \cap X_{y_1}))) = \lambda(A \cap (\mathcal{C}_{2l_2} \cap X_{y_2})),$$

and since T preserves λ , we have

$$\lambda(A \cap (\mathcal{C}_{2l_2} \cap X_{y_2})) = \lambda(T^j(A \cap (\mathcal{C}_{2l_1} \cap X_{y_1}))) = \lambda(A \cap (\mathcal{C}_{2l_1} \cap X_{y_1})) = \mu_{e|X_{y_1}}(A \cap \mathcal{C}_{2l_1}) > p \frac{1}{2^{l+1}}.$$

This is a contradiction since $\lambda(\mathcal{C}_{2l_2} \cap X_{y_2}) = \mu_{e|X_{y_2}}(\mathcal{C}_{2l_2}) = \frac{1}{2^{l+1}}$ but

$$\lambda((X \setminus A) \cap (\mathcal{C}_{2l_2} \cap X_{y_2})) = \mu_{e|X_{y_2}}((X \setminus A) \cap \mathcal{C}_{2l_2}) > (1-p) \frac{1}{2^{l+1}}.$$

Therefore T is ergodic. $\square$

5 A random walk on $\mathbb{Z}$

5.1 Definition via the Baker's map

This section is based on [EHIP14, p. 4.3.2] and presents a very simple example of an ∞ -mp ergodic bijection, requiring almost no preliminary knowledge, except for basic understanding of the Baker's map, of which we recall the definition.

There are several ways to define the (unfolded) Baker's map, with one of the fastest being the following:

$$B(y_1, y_2) := \left(2y_1 - \lfloor 2y_1 \rfloor, \frac{y_2 + \lfloor 2y_1 \rfloor}{2} \right) = \begin{cases} (2y_1, \frac{y_2}{2}) & \text{if } 0 \leq y_1 < \frac{1}{2}, \\ (2y_1 - 1, \frac{y_2 + 1}{2}) & \text{if } \frac{1}{2} \leq y_1 < 1, \end{cases}$$

where (y_1, y_2) is in the unit square $Y = [0, 1]^2$. It is easy to check that $B : Y \rightarrow Y$ is an ergodic measure-preserving bijection of Y endowed with the usual (on $\mathbb{R}^2$) Lebesgue measure $\text{Leb}|_Y$. We now consider the standard σ -finite measure space

$$(X, \lambda) = (\mathbb{Z} \times Y, c \otimes \text{Leb}|_Y) \cong (\mathbb{R} \times [0, 1[, \text{Leb}|_{\mathbb{R} \times [0, 1[}],$$

where c is the counting measure on $\mathbb{Z}$. Our ∞ -mp ergodic bijection will move the points of the 'strip' X horizontally while applying the Baker's map to each 'cell' of length 1, *i.e.* each copy of Y . We thus need to decide whether to go left or right, and this is the role of the function $\Phi : Y \rightarrow \{-1, 1\}$ defined as follows:

$$\Phi(y_1, y_2) = \begin{cases} -1 & \text{if } 0 \leq y_1 < \frac{1}{2}, \\ +1 & \text{if } \frac{1}{2} \leq y_1 < 1. \end{cases}$$

We can finally define the ∞ -mp bijection as the skew-product $\tilde{T} : X \rightarrow X$ defined by

$$\tilde{T}(n, (y_1, y_2)) := (n + \Phi(y_1, y_2), B(y_1, y_2))$$

for any $(n, (y_1, y_2)) \in \mathbb{Z} \times Y = X$. Figure 8 illustrates the action of $\tilde{T}$ on one cell of X .

Let us sketch the proof of the ergodicity of $\tilde{T}$, so as to establish that it is indeed an ∞ -mp ergodic bijection. We first notice that

$$\tilde{T}^k(n, (y_1, y_2)) = (n + S_k(\Phi(y_1, y_2)), B^k(y_1, y_2))$$

for any $(n, (y_1, y_2)) \in X$ and any $k \in \mathbb{N}$, where S_k is the sum of k independent Bernoulli random variables. We consider a positive measure $\tilde{T}$ -invariant subset A of X . Looking at the second coordinate, it is clear by ergodicity of the Baker's map that the Y -component of A has to be the whole set Y . Now looking at the first coordinate and from what we just established, it is clear that S_k can take any value between 0 and k , so the $\mathbb{Z}$ -component of A also has to be the whole set $\mathbb{Z}$, hence A is conull.

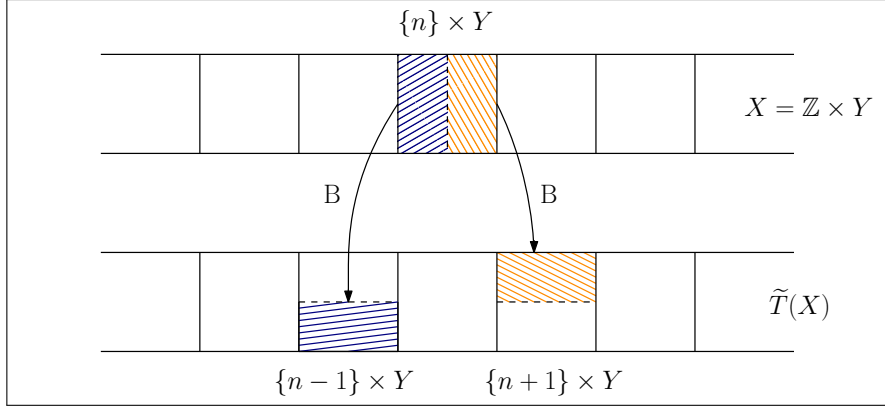


Figure 8: Visualisation of the application of $\tilde{T}$ on a cell of X .

5.2 Definition via the two-sided shift

The reader who is familiar with it would have recognized that the Baker's transformation is just another avatar of the two-sided shift operator on $\{0,1\}^{\mathbb{Z}}$. Indeed, encoding the information of two real numbers y_1 and y_2 in $[0,1[$ by their dyadic series, *i.e.* writing

$$y_1 = \sum_{n \in \mathbb{N}} b_{-n} \frac{1}{2^{n+1}} \quad \text{and} \quad y_2 = \sum_{n \in \mathbb{N}} a_{n+1} \frac{1}{2^{n+1}}$$

and realising the sequence $(b_{-n})_{n \in \mathbb{N}}$ as the negative-index coordinates of $y \in \{0,1\}^{\mathbb{Z}}$ while $(a_{n+1})_{n \in \mathbb{N}}$ corresponds to the positive-index coordinates, we bijectively identify Y with $\{0,1\}^{\mathbb{Z}}$.

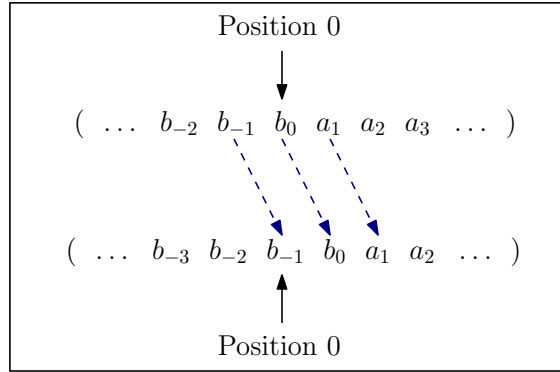


Figure 9: Application of the shift on $y \in \{0,1\}^{\mathbb{Z}}$.

We can observe in Figure 9 the effect on the shift on a sequence, and we check that

$$2y_1 - \lfloor 2y_1 \rfloor = b_0 + \sum_{n \in \mathbb{N}} b_{-n+1} \frac{1}{2^{n+1}} - b_0 = \sum_{n \in \mathbb{N}} b_{-n+1} \frac{1}{2^{n+1}}$$

while

$$\frac{y_2 + \lfloor 2y_1 \rfloor}{2} = \sum_{n \in \mathbb{N}} a_{n+1} \frac{1}{2^{n+2}} + \frac{b_0}{2} = \frac{b_0}{2} + \sum_{n \in \mathbb{N}^*} a_n \frac{1}{2^{n+1}},$$

so it is immediate to see that the shift exactly corresponds to the unfolded Baker's transformation B applied to (y_1, y_2) . With this point of view we obtain an ∞ -mp ergodic bijection on the space

$$\tilde{X} = \mathbb{Z} \times \{0,1\}^{\mathbb{Z}},$$

which is a realization of a random walk on $\mathbb{Z}$.

Seeing this ∞ -mp ergodic bijection as a shift lets us open the door to the very interesting world of *symbolic dynamics*, which we will just understand as shifts on sequences of abstract symbols. This provides many examples of ∞ -mp ergodic bijection, and we present a few, that we believe are accessible enough, at least in their basic concepts.

6 Markov shifts

Markov shifts are classical examples of infinite measure-preserving systems, that have in particular been studied by Kakutani and Parry in [KP63] in order to exhibit transformations with different *ergodic index*. We also refer to [Aar97, § 4.5]. This time we start with the general theory before giving an example.

6.1 General theory

Let P be an infinite stochastic matrix, *i.e.* a matrix $P = (P_{i,j})_{i,j \in \mathbb{Z}}$ such that all $P_{i,j} \geq 0$, and each line sums to one: $\sum_{j \in \mathbb{Z}} P_{i,j} = 1$. We denote by $(P_{i,j}^{(n)})_{i,j \in \mathbb{Z}}$ the coefficients of the n -th power P^n of P . We also assume that for any (i,j) there exists $n \geq 0$ such that $P_{i,j}^{(n)} > 0$ (it is sometimes said that P has only one **ergodic class**). The matrix P is called the **transition matrix**, giving the probabilities of going from one symbol -or state- to another.

We also assume that there exists a left eigenvector $(\lambda_n)_{n \in \mathbb{Z}}$ with eigenvalue 1 (we actually give a bit more details about the theory of infinite matrices in Section 7.2, and refer to [Kit98, Sec. 7.1] for more on this subject), and such that

$$\begin{cases} \lambda_i \geq 0 \text{ for all } i \in \mathbb{Z}, \\ \sum_{i \in \mathbb{Z}} \lambda_i = +\infty. \end{cases}$$

We now define the ambient space to be $X = \mathbb{Z}^{\mathbb{Z}}$ (for more generality, one might consider $X = S^{\mathbb{Z}}$ with S being a countable set of *symbols* -or *states*-), endowed with the σ -algebra generated by the usual x -cylinder sets of the form

$$[x]_{m,n} = \{y \in X \mid \forall m \leq k \leq n \text{ we have } y_k = x_k\},$$

where x is an element of X , and $m \leq n$ are in $\mathbb{Z}$ (in particular this makes X into a standard Borel space). We now define the measure λ on X as follows:

$$\lambda([x]_{m,n}) = \lambda_{x_m} \prod_{k=m}^{n-1} P_{x_k, x_{k+1}},$$

(where the product is empty when $m = n$). Let us explain in details how this measure works on a concrete example. Consider $\bar{x}$ to be an element of X such that

$$\begin{cases} \bar{x}_0 = 1 \\ \bar{x}_1 = -1 \\ \bar{x}_2 = 3 \\ \bar{x}_3 = 2. \end{cases}$$

We are interested in the measure of the cylinder defined by these coefficients. In other words we want to compute $\lambda([\bar{x}]_{0,3})$. We can rewrite $[\bar{x}]_{0,3}$ as $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & -1 & 3 & 2 \end{bmatrix}$ where the

first line represents the indices, and the second line the corresponding coefficients, *i.e.* the symbols. We have

$$\lambda([\bar{x}]_{0,3}) = \lambda_{\bar{x}_0} P_{\bar{x}_0, \bar{x}_1} P_{\bar{x}_1, \bar{x}_2} P_{\bar{x}_2, \bar{x}_3} = \lambda_1 P_{1,-1} P_{-1,3} P_{3,2}.$$

We interpret this as λ giving the initial weight of the ‘starting point’ of the cylinder, and then the transition probabilities are given by the matrix P , by following the dynamic on $\mathbb{Z}$ given by the word $\bar{x}$. Notice that the measure only takes into account the symbols, not their indices. We go from 1 to -1 , then to 3 and then to 2, which reads for the measure as illustrated by Figure 10.

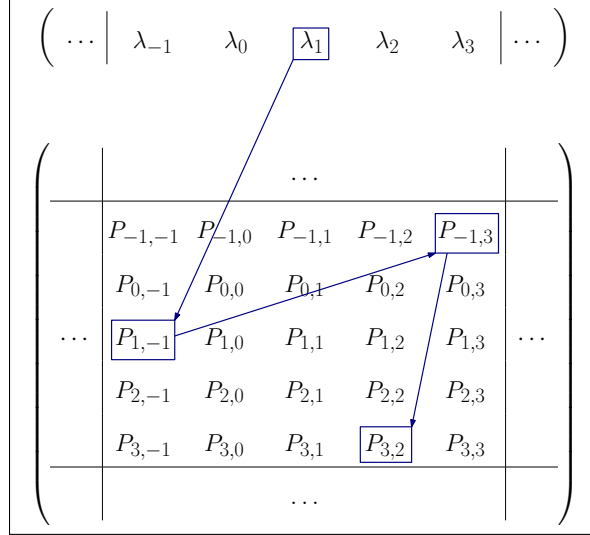


Figure 10: Computation of the measure of $[\bar{x}]_{0,3}$.

We have that $X = \bigsqcup_{n \in \mathbb{Z}} X_n$, where $X_n = \{x \in X \mid x_0 = n\}$, and $\lambda(X_n) = \lambda([n]_{0,0}) = \lambda_n$. We moreover have $\lambda(X) = \sum_{n \in \mathbb{Z}} \lambda(X_n) = \sum_{n \in \mathbb{Z}} \lambda_n = +\infty$, which in particular means that (X, λ) is a standard σ -finite space.

Our measure-preserving bijection on X is the shift T , defined by $(T(x))_k = x_{k+1}$ (we advise the reader to be careful, as this shift does not go in the same direction as the one used in Section 5). The shift T preserves λ . Indeed,

$$T([x]_{m,n}) = T \left(\begin{bmatrix} m & m+1 & \dots & n \\ x_m & x_{m+1} & \dots & x_n \end{bmatrix} \right) = \begin{bmatrix} m-1 & m & \dots & n-1 \\ x_m & x_{m+1} & \dots & x_n \end{bmatrix}$$

and so $\lambda(T([x]_{m,n})) = \lambda([x]_{m,n})$, as both the starting point and the transitions are the same, they are just not at the same index.

The two properties that will be of interest to us are the following. We give the general version, as it can be of interest, but for simplicity one might want to only consider $k = 1$ for both the conditions and the conclusion of Theorem 6.3.

Definition 6.1. We say that the transition matrix P is:

- **k -irreducible** if for every $i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}$ there exists $n \in \mathbb{N}$ such that

$$P_{i_1, j_1}^{(n)} \times \dots \times P_{i_k, j_k}^{(n)} > 0,$$

(for $k = 1$, we think of this as meaning that it is possible to go from i to j in n steps).

- **k -recurrent** if for some $i \in \mathbb{Z}$ (hence for all $i \in \mathbb{Z}$ if P is also k -irreducible) we have

$$\sum_{n=1}^{+\infty} \left(P_{i,i}^{(n)} \right)^k = +\infty,$$

(for $k = 1$, we think of this as meaning that the symbol i will be visited infinitely many times).

Remark 6.2. The ‘hence’ in the previous definition holds if P is k -irreducible. Indeed, let P be k -irreducible, and i be a symbol that is k -recurrent. We want to show that $i \neq j \in \mathbb{Z}$ is also k -recurrent. We prove it for $k = 1$. By irreducibility we let l, m be such that $P_{j,i}^{(l)} > 0$ and $P_{i,j}^{(m)} > 0$. We then have

$$\sum_{n=1}^{+\infty} P_{j,j}^{(n)} \geq \sum_{n=1}^{+\infty} P_{j,j}^{(l+n+m)} \geq \sum_{n=1}^{+\infty} P_{j,i}^{(l)} P_{i,i}^{(n)} P_{i,j}^{(m)} = +\infty.$$

The main theorem is the following.

Theorem 6.3 ([KP63],[Aar97, Thm. 4.5.3]). *If the transition matrix P is k -irreducible and k -recurrent, then $T \times \dots \times T$ (k times) is ergodic on $(X^k, \lambda^{\otimes k})$.*

6.2 The biased random walk on $\mathbb{Z}$

The example is detailed in [KP63] (but for their purposes they take the squared random walk), and originally from [Gil56]. It is a generalization of the unbiased random walk on $\mathbb{Z}$: indeed, one simply has to take $\varepsilon = 0$ in the following discussion to recover this case. We fix $-\frac{1}{2} < \varepsilon < \frac{1}{2}$ and consider the matrix P defined by

$$\begin{cases} P_{0,1} = P_{0,-1} = \frac{1}{2}; \\ P_{i,i+1} = \frac{1 - \frac{\varepsilon}{i}}{2} = \frac{i - \varepsilon}{2i}; \\ P_{i,i-1} = \frac{1 + \frac{\varepsilon}{i}}{2} = \frac{i + \varepsilon}{2i}; \\ P_{i,j} = 0 \text{ for } |i - j| \neq 1. \end{cases}$$

This transition matrix is the band matrix that appears in Figure 12 (we represented the same block $-1 \leq i, j \leq 3$). It corresponds to the random walk represented in Figure 11.

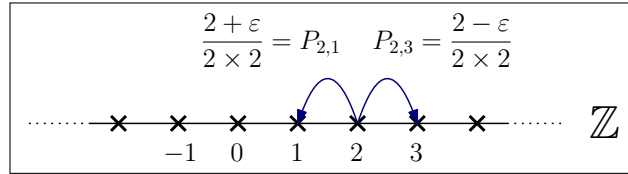


Figure 11: The biased Random Walk of parameter ε on $\mathbb{Z}$.

We now define the vector $(\lambda_n)_{n \in \mathbb{Z}}$ by

$$\begin{cases} \lambda_0 = 1 \\ \lambda_i = \lambda_{-i} = \frac{\Gamma(1 + \varepsilon) i \Gamma(i - \varepsilon)}{\Gamma(1 - \varepsilon) \Gamma(i + 1 + \varepsilon)} \text{ for } i \in \mathbb{N}^*. \end{cases}$$

By an immediate induction using the property of Euler’s Γ function, we obtain that $\Gamma(i \pm \varepsilon) = (i - 1 \pm \varepsilon)(i - 2 \pm \varepsilon) \dots (2 \pm \varepsilon)(1 \pm \varepsilon)\Gamma(1 \pm \varepsilon)$, so we have

$$\lambda_i = \frac{i(i - 1 - \varepsilon) \dots (1 - \varepsilon)}{(i + \varepsilon)(i - 1 + \varepsilon) \dots (1 + \varepsilon)}.$$

$$\left(\begin{array}{c|ccccc|c} & & & \dots & & & \\ \hline & 0 & \frac{-1-\varepsilon}{2 \times (-1)} & 0 & 0 & 0 & \\ & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \\ \dots & 0 & \frac{1+\varepsilon}{2 \times 1} & 0 & \frac{1-\varepsilon}{2 \times 1} & 0 & \dots \\ & 0 & 0 & \frac{2+\varepsilon}{2 \times 2} & 0 & \frac{2-\varepsilon}{2 \times 2} & \\ & 0 & 0 & 0 & \frac{3+\varepsilon}{2 \times 3} & 0 & \\ \hline & & & \dots & & & \end{array} \right)$$

Figure 12: The transition matrix of Kakutani and Parry's example.

We can now show that (λ_i) is a left eigenvector for P . The computation is straightforward, if somewhat tedious.

$$\begin{aligned} (\lambda P)_i &= P_{i-1,i} \lambda_{i-1} + P_{i+1,i} \lambda_{i+1} \\ &= \frac{(i-1-\varepsilon)}{2(i-1)} \frac{(i-1)(i-2-\varepsilon) \dots (1-\varepsilon)}{(i-1+\varepsilon)(i-2+\varepsilon) \dots (1+\varepsilon)} \\ &\quad + \frac{(i+1+\varepsilon)}{2(i+1)} \frac{(i-\varepsilon)(i-1-\varepsilon) \dots (1-\varepsilon)}{(i+1+\varepsilon)(i+\varepsilon) \dots (1+\varepsilon)} \\ &= \frac{1}{2} \frac{(i-1-\varepsilon) \dots (1-\varepsilon)}{(i-1+\varepsilon) \dots (1+\varepsilon)} \left(1 + \frac{i-\varepsilon}{i+\varepsilon} \right) \\ &= \frac{1}{2} \frac{(i-1-\varepsilon) \dots (1-\varepsilon)}{(i-1+\varepsilon) \dots (1+\varepsilon)} \times \frac{2i}{(i+\varepsilon)} \\ &= \lambda_i. \end{aligned}$$

It can moreover be checked that $\lambda_n C \times n^{-2\varepsilon}$ as $n \rightarrow \infty$, so the series $\sum_{n \in \mathbb{Z}} \lambda_n$ diverges whenever $\varepsilon \leq \frac{1}{2}$, thus making $(\mathbb{Z}^{\mathbb{Z}}, \lambda)$ a standard σ -finite space. It is clear that P is k -irreducible for every k , and it can be shown (see [Gil56] and the references therein) that P is 1-recurrent. From Theorem 6.3 we can deduce that our example is an ∞ -mp ergodic bijection as wanted.

7 Substitution Dynamics on infinite alphabets

We go back to presenting an example before presenting the general theory.

7.1 The Drunken man substitution

We stumble into the realm of substitution dynamics with the (*squared*) *drunken man substitution*. We will follow the exposition from [Fer06], and refer to [Que87] and [FBFMS02] for more on this vast subject, that we will barely brush. We recall the basics of a substitution dynamical system, and present Ferenczi's example. We also refer to [BJS24]. We provide quite a lot of details, as this type of construction is very different from the ones seen until here.

Definition 7.1. We start with an infinite countable set $\mathcal{A}$, that we call an **alphabet**, and elements of $\mathcal{A}$ are called **letters** (of $\mathcal{A}$). A **word** w of length $\text{len}(w) = k$ (on $\mathcal{A}$) is a finite string $w = a_1 \dots a_k$ of letters, and the **concatenation** of two words w, w' is the string

$ww' := a_1 \dots a_{\text{len}(w)} a'_1 \dots a'_{\text{len}(w')}$. We say that a word w **occurs at place** i in an infinite string u (i.e. a one-sided sequence $(u_n)_{n \in \mathbb{N}^*}$) if for any $i \leq j \leq i + \text{len}(w) - 1$ we have that $u_j = a_j$. For an infinite string u we moreover denote by

- $L_n(u)$ the set of words of length $n \geq 1$ occurring in u ;
- $L(u) = \bigcup_{n \in \mathbb{N}^*} L_n(u)$ the set of words occurring in u : the **language of** u .

A **substitution** is a map σ from $\mathcal{A}$ to the set of finite words $\mathcal{A}^* = \bigcup_{k \in \mathbb{N}^*} \mathcal{A}^k$ on $\mathcal{A}$, which we extend to a homomorphism for the concatenation by setting $\sigma(ww') = \sigma(w)\sigma(w')$. A **fixed point** for σ is an infinite string u such that $\sigma(u) = u$. We say that a word w **occurs minimally** in a word v if w does not occur in any subword v' of v (i.e. words v' such that $v = a_1 \dots a_k v' b_1 \dots b_l$, for $k, l \in \mathbb{N}$, $a_i, b_i \in \mathcal{A}$).

For any two words v and w , we say that v is an **ancestor** (under σ) of w with multiplicity $m_w(v)$ if w occurs minimally in $\sigma(v)$ at m different places. We denote by $\text{Anc}(w)$ the set of all ancestors of w .

We finally define the **(one-sided) shift** T by setting $T(x_1 x_2 x_3 \dots) = x_2 x_3 x_4 \dots$, for any $x_1 x_2 x_3 \dots \in \mathcal{A}^{\mathbb{N}}$.

The existence of fixed points is important for the rest of the study, and while it can be explicitly constructed for the example we will be interested in, there is a general criterion.

Proposition 7.2 ([Mau06]). *There exists a fixed point u for the substitution σ on the infinite alphabet $\mathcal{A}$ if and only if the three following conditions are satisfied:*

1. $\exists \iota \in \mathcal{A}, \sigma(\iota) \in \iota \mathcal{A}^*$;
2. $\text{len}(\sigma(\iota)) \geq 2$;
3. $\text{len}(\sigma(a)) \geq 1$ for every $a \in \mathcal{A}$.

We now give the construction of the drunken man substitution, and of the substitution dynamical system associated with it. For periodicity reasons, that we will not detail, the substitution we give is called the *squared* drunken man substitution. It is however the correct one to consider (see the next section), and for the sake of simplicity, we will just call it the drunken man substitution.

Definition 7.3. Let $\mathcal{A} = 2\mathbb{Z}$ be an alphabet. The **drunken man substitution** is the map σ taking defined as follows:

$$\begin{aligned} \sigma : \mathcal{A} &\longrightarrow \mathcal{A}^* \\ n &\longmapsto (n-2)nn(n+2), \end{aligned}$$

and in order for σ to satisfy the conditions of Proposition 7.2, we (artificially) adjoin to $\mathcal{A}$ the letter ι , and set $\sigma(\iota) = \iota 0$. We still call the alphabet $\mathcal{A}$ and the substitution σ , despite the changes. The fixed point for σ , which exists by Proposition 7.2, will thereafter be denoted by u .

We now define X to be the subset of $\mathcal{A}^{\mathbb{N}^*}$ comprised of all the sequences $x = x_1 x_2 x_3 \dots$ such that every word occurring in x is a word of the language $L(u)$, equivalently, such that every word occurring in x occurs in $\sigma^n(a)$ for some $a \in \mathcal{A}$ and some $n \in \mathbb{N}$. Notice now that $T(X) \subseteq X$, and as such, we say that (X, T) is the **symbolic dynamical system** associated with σ . As X is a closed subset of the (non-compact) product Polish space $\mathcal{A}^{\mathbb{N}}$ (for the discrete topology on $\mathcal{A}$), the space X is a standard Borel space. We also use the usual notion of cylinders: for a word $w = a_1 \dots a_{\text{len}(w)}$, the set $[w] \subseteq X$ is the **w -cylinder** and is defined by

$$[w] = \{x \in X \mid x_1 = a_1, \dots, x_{\text{len}(w)} = a_{\text{len}(w)}\}.$$

Those cylinders generate both the topology and the σ -algebra on X .

Remark 7.4. The fixed point u of σ can easily be constructed by induction, starting with the letter ι . Indeed we have

$$\begin{aligned}\sigma(\iota) &= \iota 0 \\ \sigma(\iota 0) &= \iota 0.(-2)002 \\ \sigma(\iota 0.(-2)002) &= \iota 0.(-2)002.(-4)(-2)(-2)0.(-2)002.(-2)002.0224 \\ &\dots\end{aligned}$$

(the dots have been added for legibility). On Figure 13 we can observe the self-similarity phenomenon occurring in u (the letter ι is not represented).

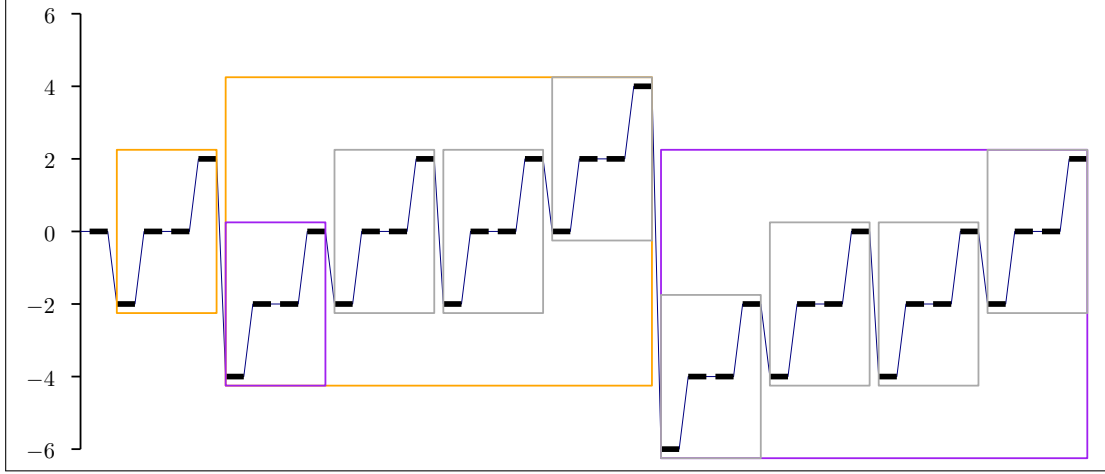


Figure 13: The first terms of the fixed point u of σ .

We explicitly describe how the infinite measure that is preserved by T is constructed.

Definition 7.5. We say that λ is a **natural measure** for (X, T) if it is T -invariant, and if for any cylinder $[w]$ we have

$$\lambda([w]) = \frac{1}{4} \sum_{v \in \text{Anc}(w)} \lambda([v]) m_w(v). \quad (*)$$

Proposition 7.6 ([Fer06, Prop. 2.11]). *There exists a unique infinite natural measure λ for the substitution dynamical system (X, T) .*

Proof. We define λ on one-letter-word cylinders, such that the measure of the cylinder $[0]$ is equal to 1: by definition of naturality we have

$$4\lambda([2n]) = \lambda([2n-2]) + 2\lambda([2n]) + \lambda([2n+2]). \quad (*)$$

Equation $(*)$ ensures that all one-letter-word cylinders have equal measure. Indeed, assume the contrary, and consider $(*)$ for $n = 0$. Without loss of generality we can assume that $\lambda([0]) > \lambda([2])$. By Equation $(*)$, the measure of any cylinder $[2n+2]$ (for $n \in \mathbb{N}^*$) has to be strictly less than that of $[2n]$. By $(*)$ again there exists $m \in \mathbb{N}^*$ such that $\lambda([2m]) < 0$, a contradiction. We then choose this constant value to be 1.

For cylinders of length 2, we define their measure recursively. Words of the form $(2n)(2n)$ or $(2n)(2n+2)$ only have ancestors of length 1, so $\lambda([(2n)(2n)]) = \frac{1}{4}$ and $\lambda([(2n)(2n+2)]) = \frac{1}{2}$. Words of the form $(2n)(2n-2p)$, for $p > 0$, only have ancestors of the form $(2n-2)(2n-2p+2)$, *i.e.* of the form $(2m)(2m-2p+4)$. Therefore, a straightforward induction shows that their measures are entirely determined by what we

have already done: $\lambda([(2n)(2n-2p)]) = 2^{-p-2}$.

For cylinders of length $n \geq 3$, we first prove the following intermediate result ([Fer06, Lem. 2.7]), which establishes that σ is **left-determined**.

Fact: any word w in the language $L(u)$ of length at least 3 has a unique decomposition $w = w_1 \dots w_k$, where each (4-letter word) w_i is a $\sigma(a_i)$ for some $a_i \in \mathcal{A}$, except that w_1 may be only a suffix of $\sigma(a_1)$ (i.e. a word v such that $\sigma(a_1) = b_1 \dots b_l v$ for $1 \leq l \leq 3$ and $b_i \in \mathcal{A}$), and w_k may be only a prefix of $\sigma(a_k)$. Note that in this case the a_i ($1 \leq i \leq k-1$) are unique.

Proof of the Fact: Any word w of length at least 3 contains (at least) one of the following:

- $nn(n+2)$;
- $n(n+2)(n+2)$;
- $n(n+2)m$ for some $m \neq (n+2)$;
- $mn(n+2)$ for some $m \neq n$.

Any of those words allow us to separate w into the w_i in a unique way, and the last letter of a word $\sigma(a)$ completely determines a , so they are also unique. $\diamond$

By the previously proved fact, all words w of length $k \geq 3$ only have ancestors of length at most $k-1$. It is then possible to define $\lambda([w])$ recursively thanks to (*).

Now we check that it is T -invariant, i.e. that for any word w we have

$$\lambda([w]) = \sum_{a \in \mathcal{A}} \lambda([aw]) = \sum_{a \in \mathcal{A}} \lambda([wa]).$$

By construction of λ and Equation (*) we only have to check this equality for 1-letter words. Since $\lambda([(2n)(2n-2p)]) = \lambda([(2n+2p)(2n)]) = 2^{-p-2}$, we have

$$\lambda([2n]) = 1 = \sum_{p=-1}^{+\infty} \lambda([(2n)(2n-2p)]) = \sum_{p=-1}^{+\infty} \lambda([(2n+2p)(2n)]),$$

and since the measure λ is clearly infinite on X (do note also that it is atomless by construction), the proof is over. $\square$

We do not provide the proofs of the following facts. In particular the proof of the ergodicity is quite involved.

Proposition 7.7 ([Fer06, Prop. 2.3 and 2.9]). *The shift T is not surjective on X , but it is bijective on a set of countable complement. In particular, T is a measure-preserving bijection on a λ -conull set.*

Proposition 7.8 ([Fer06, Prop. 2.14]). *The shift T is ergodic on (X, λ) .*

Remark 7.9. It is also possible to define the whole system as a two-sided shift, i.e. a shift on bi-infinite words (on $\mathcal{A}^{\mathbb{Z}}$ instead of $\mathcal{A}^{\mathbb{N}}$). The two systems are measure-theoretically isomorphic, see [Fer06, Sec. 2.4]. This is also the point of view adopted in [BJS24], in which the authors develop the general theory of ∞ -mp (ergodic) bijections coming from substitution dynamics.

7.2 General theory

In [Fer97], the author opened the door to the study of substitution dynamics on infinite alphabets, and constructed the base of the theory. We succinctly present the framework that generalizes the beginning of Section 7.1.

Definition 7.10. The **matrix of a substitution** σ on a countably infinite alphabet $\mathcal{A}$ is $M = (M_{i,j})_{i,j \in \mathcal{A}}$, where $M_{i,j}$ is the number of occurrences of j in the word $\sigma(i)$.

Example 7.11. The substitution matrix of the system presented in Section 7.1 is the one given in Figure 14 below. As in Section 6.2, we notice that it is a band matrix. This is in fact linked to the following property.

$$(M_{i,j})_{i,j \in 2\mathbb{Z}} = \begin{pmatrix} & & & \dots & & \\ & 2 & 1 & 0 & 0 & 0 \\ & 1 & 2 & 1 & 0 & 0 \\ \dots & 0 & 1 & 2 & 1 & 0 & \dots \\ & 0 & 0 & 1 & 2 & 1 \\ & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$(M_{i,j})_{j \in 2\mathbb{Z} \cup \{t\}} = \left(\dots \mid 0 \quad \boxed{1} \quad \boxed{1} \quad 0 \quad 0 \mid \dots \right)$$

$\uparrow$
 $M_{t,t}$

$\uparrow$
 $M_{t,0}$

Figure 14: The substitution matrix of the squared drunken man substitution.

Definition 7.12. A substitution σ on $\mathcal{A}$ is:

1. of **bounded length** if there exists $L \geq 2$ such that for any $n \in \mathcal{A}$ we have $|\sigma(n)| \leq L$ (we moreover say that σ is of **constant length** if $|\sigma(n)| = L$);
2. of **bounded size** if it is of bounded length and if there exists a positive integer $t \in \mathbb{Z}$ (which is independent of n) such that for any $n \in \mathbb{Z}$, all the letters of $\sigma(a)$ are in $\{n - t, n - t + 1, \dots, n, \dots, n + t - 1, n + t\}$, where we identify $\mathcal{A}$ and $\mathbb{Z}$. When it is the case, the smallest t verifying this is called the **size** of σ .

It is then easy to see that a substitution matrix of a σ of bounded size is a band matrix.

This notion of being of bounded size is used in [BJS24] to construct Bratteli-Vershik models of the shifts. As it is beyond the scope of this appendix, we do not expand on this subject.

One other notion that we will need however, is the notion of left-determinacy, already mentioned in Section 7.1. We recall it, in greater generality, after giving a few more definitions. Although we will not really need it, we in particular use this occasion to define *subshifts* (following [BJS24]), as this is a very active area of research.

Definition 7.13. The **language of a substitution** σ is denoted by L_σ and defined by

$$L_\sigma = \{\text{words } w \mid w \text{ occurs in } \sigma^n(a) \text{ for some } n \in \mathbb{Z}, a \in \mathcal{A}\}.$$

Notice that this time, the definition does not presuppose the existence of a fixed point for σ . We also define $X_\sigma = \{x \in \mathcal{A}^{\mathbb{Z}} \mid L(x) \subseteq L_\sigma\} \subseteq \mathcal{A}^{\mathbb{Z}}$, and then X_σ is closed in X , and T -invariant. The dynamical system (X_σ, T) is called the **subshift** on $\mathcal{A}$ associated with σ .

Definition 7.14. A substitution σ on $\mathcal{A}$ is **left-determined** if there exists a positive $N_\sigma \in \mathbb{N}$ such that any word $w \in L_\sigma$ of length at least N_σ has a unique decomposition $w = w_1 \dots w_s$ where each $w_i = \sigma(a_i)$ for a unique $a_i \in \mathcal{A}$, except that w_1 may be only a suffix of $\sigma(a_1)$ and w_s may be only a prefix of $\sigma(a_s)$.

We have the following theorem, which fixes the topological framework for the study of subshifts of infinite alphabets.

Theorem 7.15 ([BJS24, Thm. 3.3]). *If σ is left-determined, then σ is a homeomorphism from X_σ to $\sigma(X_\sigma)$.*

The next step of the theory would be to extend the Perron-Frobenius theory of matrices to countable matrices. This is done in [Kit98, Sec. 7.1], and we do not recall details. Although the terminology comes from Random Walks and Markov shifts, in the following definition, the matrix is not stochastic.

Definition 7.16. As in Definition 6.1, we say that an infinite matrix $M = (M_{i,j})_{i,j \in \mathcal{A}}$ is **irreducible** if for every couple $(i, j) \in \mathcal{A}^2$, there exists $n \geq 1$ such that $M_{i,j}^{(n)} > 0$. An irreducible matrix M has **period** d if for every $i \in \mathcal{A}$, we have that $d = \gcd \left\{ n \in \mathbb{N} \mid M_{i,i}^{(n)} > 0 \right\}$, and is **aperiodic** if $d = 1$. By the generalized Perron-Frobenius Theorem (see [Kit98, Thm. 7.1.3]), an irreducible aperiodic matrix M admits a left eigenvalue λ_∞ defined by

$$\lambda_\infty = \lim_{n \rightarrow \infty} \left(M_{i,i}^{(n)} \right)^{\frac{1}{n}},$$

i.e. the generalized spectral radius (it is independent of the choice of i). Note also that the eigenvectors are unique up to constant multiples. We will restrict ourself to the case of $\lambda_\infty < \infty$. We then have different possible behaviours:

1. M is **transient** if $\sum_{n \in \mathbb{Z}} M_{i,j}^{(n)} \lambda_\infty^{-n} < +\infty$ for some $i, j \in \mathcal{A}$ (equivalently all of them by irreducibility, see [Kit98, Lem. 7.1.8]), and **recurrent** otherwise. If M is recurrent, we then define for any $i, j \in \mathcal{A}$: $l_{i,j}(1) = M_{i,j}$ and $l_{i,j}(n+1) = \sum_{k \neq i} l_{i,k}(n) M_{k,j}$.
- 2.a) M is **null recurrent** if $\sum_{n \in \mathbb{N}} n l_{i,i}(n) \lambda_\infty^{-n} < +\infty$ for some $i \in \mathcal{A}$ (equivalently all of them, see [Kit98, Lem. 7.1.13]);
- 2.b) M is **positive recurrent** if $\sum_{n \in \mathbb{N}} n l_{i,i}(n) \lambda_\infty^{-n} = +\infty$.

Armed with all these definitions, we can state the main results. Ferenczi ([Fer06, Sec. 3]) classifies substitution systems through the behaviour of their substitution matrices, generalized by Bezuglyi-Jorgensen-Sanadhya ([BJS24, Sec. 7]) via generalized Bratteli diagrams techniques.

Theorem 7.17 ([Fer06, Sec. 3]). *Let σ be of constant length, left-determined, with an irreducible aperiodic substitution matrix M .*

1. *If M is positive recurrent, then (X, T) admits an ergodic invariant natural probability measure μ .*
2. *If M is null recurrent, then (X, T) admits an infinite σ -finite natural measure λ .*

Theorem 7.18 ([BJS24, Thm. 7.6]). *Let σ be a bounded size left-determined substitution on a countably infinite alphabet $\mathcal{A}$, and (X_σ, T) be the associated subshift. Assume that the substitution matrix M associated with σ is irreducible, aperiodic and recurrent.*

1. *There exists a T -invariant measure λ on X_σ .*

2. The measure λ is finite if and only if the Perron-Frobenius left eigenvector $(\lambda_i)_{i \in \mathcal{A}}$ satisfies $\sum_{i \in \mathcal{A}} \lambda_i < +\infty$.

Of course, while it is solved for probability measures, the question of ergodicity is not answered for invariant infinite measures. As pointed out by Ferenczi, the ergodicity for the squared drunken man substitution (*i.e.* Proposition 7.8) is obtained through precise and technical considerations and estimates, which are not generalizable. Bezuglyi-Jorgensen-Sanadhya also obtain ergodicity as a corollary of their result only for the finite measure case, and in the positive recurrent case (see [BJS24, Rem. 7.8]). To our knowledge, there does not exist any general ergodicity result in infinite measure for substitution dynamics on infinite alphabets.

Remark 7.19. We conclude this appendix by a few remarks on the ‘*random walk on $\mathbb{Z}$ substitution*’, *i.e.* [BJS24, Ex. 8.3]. The substitution matrix (a band matrix where each line is $\dots 0010100\dots$) is just the transition matrix of the usual unbiased random walk on $\mathbb{Z}$, multiplied by 2. The substitution matrix is irreducible, periodic with period 2, and null recurrent. Although the matrix is not aperiodic, it can be shown that

$$(\lambda_i)_{i \in \mathbb{Z}} = \left(\dots \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \dots \right),$$

and that there exists a shift-invariant infinite σ -finite measure λ on X . In fact the whole aperiodicity hypothesis can be dispensed with (see [BJS24, Rem. 7.3]).

It is very important however to understand that this dynamical system is very different from the *usual random walk on $\mathbb{Z}$* . Indeed, both the path-space X and the measure λ are defined differently, and obey different constraints.

As it is very new, there is sadly a paucity of concrete examples (to our knowledge), and indeed the only other example of a substitution dynamical system in [BJS24] satisfying condition 2. of Theorem 7.18 is the already studied squared drunken man substitution (see [BJS24, Ex. 8.4]).

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