Descriptive set theory and the Becker-Kechris Theorem

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Abstract

In this informal note we provide some context and give two demonstrations of a theorem from Becker and Kechris on Polish realizations of Borel group actions; the original proof from [BK96] and the proof of a stronger version from [Hjo99]. The theorem is stated and proved after a part containing some prerequisites of descriptive set theory.

Contents

1 Introduction

This note aims to provide basic background on descriptive set theory, with the specific purpose of providing an (almost) 'self-contained' proof of the Becker-Kechris theorem on Polish realizations of Borel group actions ([BK96, Thm. 5.2.1]). Section 2 contains a lot of descriptive set theoretic results *in bulk*, and exists for the sole purpose of gathering them in one place, in a section of reasonable length. Most results in this section will be stated without proof, but precise references will be given (as much as possible). Section 3 still contains a lot of prelimirary results of descriptive set theory, but those ones are more specific, and tailored to our needs. We also start giving some proofs. Sometimes. Section 4 contains both the statement and two different proofs of the Becker-Kechris theorem, taken from [BK96, § 5.1], [Gao09, § 4.4] and [Hjo99].

The subjects covered in this note are very vast, and while our goal was to provide an accessible exposition, a lot of (seemingly arbitrary) choices had to be made. Maybe the most criticizable one is the fact that we chose to forgo the natural 'order of proofs' in favor of a more *thematic* one. We hope that any knowledgeable reader - who will notice that some results cannot be proved without some appearing later, for example - will forgive us.

A lot of references, more or less precise, are provided. We warmly recommend that any reader interested in those subjects consult them. We claim neither originality nor exhaustivity (not a shred) for anything contained in this note, which was written mostly as a support when learning the proofs.

2 Descriptive set theory cheat sheet

2.1 The first steps of descriptive set theory

Definition 2.1. A **Polish space** is a separable and completely metrizable topological space. In particular it is **Hausdorff** and **second countable**.

We have the following essential/useful facts about Polish spaces, given in no specific order. Recall that a G_δ set is a countable intersection of open sets. More on this will be discussed in Section 2.3.

Remark 2.2. There always exists a bounded (often by 1) metric which is compatible with the topology of a Polish space. Indeed, if d is a compatible metric, then $\min(1, d)$ is suitable, as it is equivalent to d. Moreover $\min(1, d)$ has the same Cauchy sequences as d, in particular every Polish space admits a compatible complete bounded metric.

Remark 2.3. By virtue of being second countable, Polish spaces satisfy the Lindelöf lemma: every open cover of the space admits a countable subcover.

Proposition 2.4 ([Kec95, Thm. 3.11])**.** *Let (X, τ) be a Polish space. Then $Z \subseteq X$ is Polish for the induced topology if and only if Z is G_δ in X. In particular a closed subspace of a Polish space is Polish.*

Proposition 2.5 ([Kec95, Prop. 3.3])**.** *Let $((X_n, \tau_n))_{n\in\mathbb{N}}$ be a sequence of Polish spaces. Both the product $\prod_{n\in\mathbb{N}} X_n$ and the sum (i.e. the disjoint union) $\bigsqcup_{n\in\mathbb{N}} X_n$ are Polish spaces, for their respective topologies. We recall also that the product topology on $\prod_{n\in\mathbb{N}} X_n$ has*

$$\left\{ \prod_{n\in\mathbb{N}} U_n \;\middle|\; U_n \in \tau_n \text{ and } U_n \neq X_n \text{ for finitely many } n \right\}$$

as a base, and that the open sets in the sum $\bigsqcup_{n\in\mathbb{N}} X_n$ are exactly the sets of the following form:

$$\left\{ U \subseteq \bigsqcup_{n\in\mathbb{N}} X_n \;\middle|\; \forall n \in \mathbb{N} : U \cap X_n \in \tau_n \right\}.$$

Here are a few topological spaces of particular importance in descriptive set theory.

Cantor space	$\mathcal{C} = \{0,1\}^{\mathbb{N}}$
Baire space	$\mathcal{N} = \mathbb{N}^{\mathbb{N}}$
Hilbert cube	$\mathbb{I}^{\mathbb{N}} = [0,1]^{\mathbb{N}}$

Theorem 2.6 ([Kec95, Thm. 4.14])**.** *Up to homeomorphism, the Polish spaces are exactly the G_δ subspaces of the Hilbert cube $\mathbb{I}^{\mathbb{N}}$.*

Theorem 2.7 ([Kec95, Cor. 6.5])**.** *A Polish space X is either finite, countable, or it contains a homeomorphic copy of the Cantor space $\mathcal{C}$ and thus has cardinality $2^{\aleph_0}$.*

Do note that in descriptive set theory, questions of cardinality are linked to the perfect set property (see [Kec95, § 6]), which we chose not to discuss here. The previous theorems are mainly here to give intuition about Polish spaces.

In a similar flavour, but for those who prefer the Baire space $\mathcal{N}$ rather than the Cantor space, we have the following. We recall that a subset A of a topological space is K_σ if it is a countable union of compact sets.

Theorem 2.8 (**Hurewicz**, see *e.g.* [Kec95, Thm. 7.10])**.** *Let X be a Polish space. Then X contains a closed subspace homeomorphic to $\mathcal{N}$ if and only if X is not K_σ.*

Polish spaces are the topological spaces that give rise to a widely used measurable structure via their Borel sets. We recall that the Borel σ-algebra $\mathcal{B}(X)$ of a topological space is the smallest σ-algebra containing the open sets, *i.e.* the σ-algebra generated by the topology of the space. We say that the σ-algebra is countably generated when the topology is second countable, so in particular it is the case for Polish spaces. In fact, one can say more: Borel sets coming from Polish topologies are classified up to cardinality of the underlying space. When this cardinality is $2^{\aleph_0}$, we call the resulting measurable spaces *standard Borel spaces.*

Definition 2.9. A **standard Borel space** X is an uncountable measurable space with a σ-algebra $\mathcal{B}(X)$ of subsets that are Borel for some Polish topology on X.

Proposition 2.10. *Let $((X_n, \mathcal{B}_n))_{n\in\mathbb{N}}$ be a sequence of standard Borel spaces. Both the product $\prod_{n\in\mathbb{N}} X_n$ and the sum $\bigsqcup_{n\in\mathbb{N}} X_n$ are standard Borel spaces, for their respective measurable structure. We recall also that the product σ-algebra on $\prod_{n\in\mathbb{N}} X_n$ is generated by sets of the form*

$$\left\{ \prod_{n\in\mathbb{N}} A_n \;\middle|\; A_n \in \mathcal{B}_n \text{ and } A_n \neq X_n \text{ for finitely many } n \right\},$$

(equivalently one can generate this σ-algebra by asking that $A_n \neq X_n$ for all but one n). Recall now that the Borel sets in the sum $\bigsqcup_{n\in\mathbb{N}} X_n$ are exactly the sets of the following form:

$$\left\{ A \in \bigsqcup_{n\in\mathbb{N}} X_n \;\middle|\; \forall n \in \mathbb{N} : A \cap X_n \in \mathcal{B}_n \right\}.$$

We quickly state an important fact. It is a consequence of a theorem that we will focus on in Section 3.3, and can also be deduced from Theorem 2.15.

Proposition 2.11 ([Kec95, Cor. 13.4])**.** *Let $(X, \mathcal{B})$ be a standard Borel space, and $Y \in \mathcal{B}$ be a Borel subset. Then $(Y, \mathcal{B}_{\restriction Y} = \{A \subseteq Y \mid A \in \mathcal{B}\})$ is also a standard Borel space.*

The following is the 'Borel version' of Theorem 2.7.

Theorem 2.12 ([Kec95, Thm. 13.6])**.** *Let X be Polish and $A \subseteq X$ be Borel. Then A is either finite, countable, or it contains a copy of a Cantor space. In particular, any standard Borel space has cardinality $2^{\aleph_0}$.*

As an important -and useful- example of a standard Borel space, we give the definition of the Effros Borel space.

Definition 2.13. Let (X, τ) be a topological space. We denote by $F(X)$ the set of closed subsets of X. We endow $F(X)$ with the σ-algebra $\mathcal{B}_{Eff}$ generated by the sets

$$\{F \in F(X) \mid F \cap U \neq \emptyset\},$$

for $U \in \tau$. The measurable space $(F(X), \mathcal{B}_{Eff})$ is called the **Effros Borel space** of X.

Theorem 2.14 (see *e.g.* [Kec95, Thm. 12.6]). *The Effros Borel space* $(F(X), \mathcal{B}_{Eff})$ *of a Polish space* X *is a standard Borel space.*

The terminology 'standard' is justified by the Borel isomorphism theorem (Theorem 2.17), which is proved by showing that any standard Borel space is Borel isomorphic to $\mathcal{C}$. The proof also uses the following theorem -or rather its corollary- which can be of independant interest. More about direct images of Borel sets will be discussed in Section 2.2.

Theorem 2.15 (**Lusin-Suslin**, see *e.g.* [Kec95, Thm. 15.1]). *Let* X *and* Y *be two Polish spaces, and let* $f : X \to Y$ *be a continuous map. Then for every Borel subset* A *of* X*, if* $f_{\restriction A}$ *is injective, then* $f(A)$ *is Borel.*

Corollary 2.16 (see *e.g.* [Kec95, Cor. 15.2]). *Let* X *and* Y *be two standard Borel spaces, and let* $f : X \to Y$ *be a Borel map. Then for every Borel subset* A *of* X*, if* $f_{\restriction A}$ *is injective, then* $f(A)$ *is Borel, and* f *is a Borel isomorphism between* A *and* $f(A)$*.*

Theorem 2.17 (**The Isomorphism Theorem**, [Kec95, Thm. 15.6]). *Any two standard Borel spaces* X *and* Y *are Borel isomorphic if and only if* $\mathrm{Card}(X) = \mathrm{Card}(Y)$*.*

In particular, an important consequence of Corollary 2.16 is the following:

Proposition 2.18 ([Kec95, Exercise. 15.4]). *Let* X *be a space, with two Polish topologies* τ *and* τ'*. If* $\tau \subseteq \mathcal{B}(\tau')$*, then* $\mathcal{B}(\tau) = \mathcal{B}(\tau')$*. In particular, a Polish refinement of a Polish topology yields the same Borel structure.*

2.2 Borel Hierarchy, and a glimpse of what lies beyond

We just saw that the Borel σ-algebra generated by the open sets of a Polish topology is 'flexible' in the sense of Theorem 2.17: they are all the same, for a given cardinality. Morally this flexibility can be explained by the possible 'complexity' of the Borel sets themselves: it is a popular opinion to consider non-Borel sets to be 'pathological', but even some Borel sets can be quite 'dysfunctional'. The Borel Hierarchy's aim is to classify Borel sets based on *how far they are* from the original topology of the space.

As usual anything in this section is classical and can be found in [Kec95, § 11.B, § 14 and § 22.A], and we also warmly recommend the *Introduction to Descriptive Set Theory* notes from Anush Tserunyan, available on her webpage.

Notation 2.19. We denote by ω_1 the **first uncountable ordinal**. For the rest of this section, α will denote an ordinal such that $1 \leqslant \alpha < \omega_1$. We also fix a metrizable space (X, τ) until the end of the section.

The Borel Hierarchy. We define the following three families of subsets of X:

$$\begin{cases} \Sigma^0_1(X,\tau) \coloneqq \tau = \{U \subseteq X \mid U \text{ is open}\}\,, \\ \Pi^0_1(X,\tau) \coloneqq \{X \setminus U \mid U \text{ is open}\} = \{F \subseteq X \mid F \text{ is closed}\}\,, \\ \Delta^0_1(X,\tau) \coloneqq \Sigma^0_1(X,\tau) \cap \Pi^0_1(X,\tau) = \{A \subseteq X \mid A \text{ is clopen}\}\,. \end{cases}$$

Note already that when there is no danger of confusion, for instance when there is only one topology involved, we will write Σ^0_1 instead of $\Sigma^0_1(X,\tau)$, and the same holds for all similar notations. We will use the notation $\Pi^0_1 = X \setminus \cdot\, \Sigma^0_1$ to signify that the sets in Π^0_1 are complements

of sets in Σ_1^0.
We then increase the lower indices by setting

$$\begin{cases} \Sigma_\alpha^0 \coloneqq \left\{ \bigcup_{n\in\mathbb{N}} A_n \;\middle|\; \forall\, n \in \mathbb{N} : A_n \in \Pi_{\alpha_n}^0, \text{ for some } \alpha_n < \alpha \right\}, \\ \Pi_\alpha^0 \coloneqq X \setminus \cdot\, \Sigma_\alpha^0, \\ \Delta_\alpha^0 \coloneqq \Sigma_\alpha^0 \cap \Pi_\alpha^0, \end{cases}$$

(by a transfinite induction). The classes Δ_α^0 are sometimes called the **ambiguous classes**.

Notation 2.20. For a collection $\mathcal{E}$ of subsets of X, we let

$$\begin{cases} \mathcal{E}_\sigma & \coloneqq \left\{ \bigcup_{n\in\mathbb{N}} A_n \;\middle|\; A_n \in \mathcal{E}, n \in \mathbb{N} \right\}, \\ \mathcal{E}_\delta & \coloneqq \left\{ \bigcap_{n\in\mathbb{N}} A_n \;\middle|\; A_n \in \mathcal{E}, n \in \mathbb{N} \right\}. \end{cases}$$

In particular, if we denote by (as is tradition) by $G(X)$ and $F(X)$ the families of open and closed subsets of X, respectively, we have

$$\begin{aligned} &\Sigma_1^0 = G(X), && \Pi_1^0 = F(X), \\ &\Sigma_2^0 = (F(X))_\sigma = F_\sigma(X), && \Pi_2^0 = (G(X))_\delta = G_\delta(X), \\ &\Sigma_3^0 = (G_\delta(X))_\sigma = G_{\delta\sigma}(X), && \Pi_3^0 = (F_\sigma(X))_\delta = F_{\sigma\delta}(X), \\ &\Sigma_4^0 = (F_{\sigma\delta}(X))_\sigma = F_{\sigma\delta\sigma}(X), && \Pi_4^0 = (G_{\delta\sigma}(X))_\delta = G_{\delta\sigma\delta}(X), \\ &\dots && \dots \end{aligned}$$

With these notations we have

$$\Sigma_{\alpha+1}^0(X,\tau) = (\Pi_\alpha^0(X,\tau))_\sigma \quad \text{and} \quad \Pi_{\alpha+1}^0(X,\tau) = (\Sigma_\alpha^0(X,\tau))_\delta.$$

We do not rigorously state the (non-trivial!) inclusion properties between these classes, as they are easy to read on Figure 1.

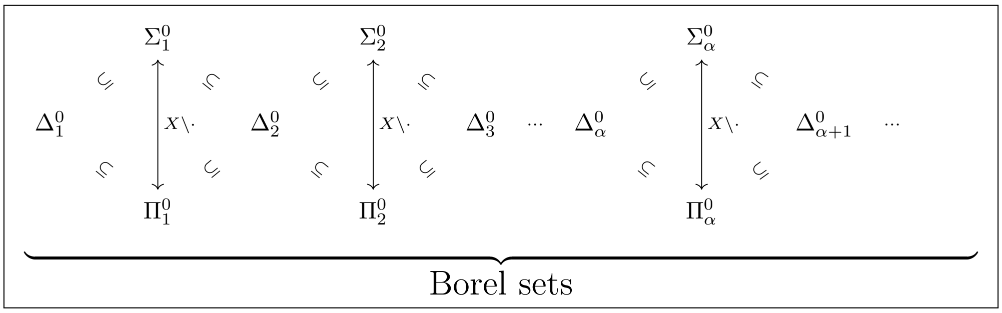

Figure 1: The Borel hierarchy.

We do however state the following closure properties of the classes of the Hierarchy, as well as the proposition justifying the terminology:

Proposition 2.21 ([Kec95, Prop. 22.1]). *The classes* Σ^0_α, Π^0_α *and* Δ^0_α *are closed under finite reunion, finite intersection, and continuous preimages. Moreover,* Σ^0_α *is closed under countable unions,* Π^0_α *is closed under countable intersection, and* Δ^0_α *is closed under complements.*

Proposition 2.22. *We have the following equality:*

$$\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha = \bigcup_{\alpha<\omega_1} \Delta^0_\alpha.$$

Finally, for uncountable Polish spaces, the classes are distinct.

Theorem 2.23 ([Kec95, Thm. 22.4]). *Let* X *be an uncountable Polish space. For any* $1 \leqslant \alpha < \omega_1$ *we have* $\Sigma^0_\alpha \neq \Pi^0_\alpha$*, and* $\Delta^0_\alpha \subsetneq \Sigma^0_\alpha \subsetneq \Delta^0_{\alpha+1}$ *as well as* $\Delta^0_\alpha \subsetneq \Pi^0_\alpha \subsetneq \Delta^0_{\alpha+1}$*. In other words, the inclusion in* Figure 1 *are all strict.*

In the rest of this section, we quickly define analytic sets, as well as co-analytic sets, and the more general class of projective sets. While one might stumble onto analytic sets ‘quite often’, as they are very natural sets to consider, co-analytic sets and projective sets, are more niche, and are presented here mostly for culture, and for the sake of having a (somewhat) complete picture in mind. People outside of pure descriptive set theory rarely have to work with those sets.

From the Lusin Novikov theorem (see [Kec95, Thm. 18.10]), we have the following strenghtening of Corollary 2.16.

Proposition 2.24 ([Kec95, Exercise. 18.14]). *Let* X *and* Y *be two standard Borel spaces, and let* $f : X \to Y$ *be a Borel countable-to-one function (i.e. the preimage of any point is at most countable). Then* $f(X)$ *is Borel.*

This result states that the class of Borel subsets of a standard Borel space is closed under countable-to-one Borel images, but this is not true for general Borel images. Actually, it is not even closed under continuous images, and even if we add a Polish topology, the class of Borel subsets of a Polish space is not closed under continuous images. Enter the analytic sets.

Definition 2.25. Let (X, τ) be a *Polish space.* A set $A \subseteq X$ is called **analytic** if there exists a Polish space Y and a continuous function $f : Y \to X$ with $f(Y) = A$. We denote by $\Sigma^1_1(X, \tau)$ the family of analytic subsets of X. Again, we will write Σ^1_1 if there is no risk of confusion.

It is possible to extend the definition: if X is a *standard Borel space* instead, and $A \subseteq X$, we say that A is **analytic** if there exists a Polish space Y and a Borel isomorphism $g : X \to Y$ such that A is analytic in Y. This is independant of the choice of Y and g.

Proposition 2.26 ([Kec95, Prop. 14.4]). *Let* X *and* Y *be two Polish spaces.*

1. *if* (A_n) *is a sequence of analytic sets, then* $\bigcup_{n\in\mathbb{N}} A_n$ *and* $\bigcap_{n\in\mathbb{N}} A_n$ *are analytic, i.e.* Σ^1_1 *is closed under countable unions and intersections;*

2. *if* $f : X \to Y$ *is Borel, then for* $A \subseteq X$ *analytic and* $B \subseteq Y$ *analytic,* $f(A)$ *and* $f^{-1}(B)$ *are analytic, i.e.* Σ^1_1 *is closed under Borel images and preimages.*

Borel sets are of course analytic, but in the converse true? It is actually not a trivial task to prove that the answer is negative.

Theorem 2.27 (**Suslin**, see *e.g.* [Kec95, Thm. 14.2] or [Sri98, Thm. 4.1.5]). *Let* X *be an uncountable Polish space. Then* $\mathcal{B}(X) \subsetneq \Sigma^1_1$*. In particular, this holds for a standard Borel space.*

We now define the co-analytic sets, and the first 'projective class' of sets, which, as we will see, is just the class of Borel sets themselves.

Definition 2.28. Let (X, τ) be a Polish space.

1. A set $A \subseteq X$ is called **co-analytic** if it is the complement of an analytic set, *i.e.* if $X \setminus A \in \Sigma_1^1$. Naturally, we denote by $\Pi_1^1(X, \tau) = \Pi_1^1$ the class of co-analytic sets.

2. The **bi-analytic** sets are the sets that are both anaytic and co-analytic. We denote by $\Delta_1^1(X, \tau) = \Delta_1^1$ their class, *i.e.* $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$.

The main tool to prove the important Theorem 2.30 is the following separation theorem of Lusin.

Theorem 2.29 (**The Lusin Separation Theorem**, see *e.g.* [Kec95, Thm. 14.7])**.** *Let X a standard Borel space, and let $A, B \in \Sigma_1^1$ be disjoint. Then there exists a Borel set C separating A and B, i.e. satisfying $A \subseteq C$ and $C \cap B = \emptyset$.*

Theorem 2.30 (**Suslin**, see *e.g.* [Kec95, Thm. 14.11])**.** *For any standard Borel space X, we have $\mathcal{B}(X) = \Delta_1^1$.*

We finish with the recursive definition of Projective sets, which is represented in Figure 2. By Theorem 2.30, we can see this Projective Hierarchy as an extension of the more well-known Borel Hierarchy. Again, we give very little details and refer the interested reader to [Kec95, § 37]. Notice that the lower indices are integers.

The Projective Hierarchy. Everything is defined in uncountable Polish spaces. The families Σ_1^1, Π_1^1 and Δ_1^1, are already defined, we then inductively define

$$\begin{cases} \Sigma_{n+1}^1 \coloneqq \left\{\mathrm{proj}_X(A) \mid X \text{ Polish}, A \subseteq X \times \mathcal{N}, A \in \Pi_n^1(X \times \mathcal{N})\right\}, \\ \Pi_{n+1}^1 \coloneqq X \setminus \cdot\, \Sigma_{n+1}^1 = \left\{X \setminus A \mid X \text{ Polish}, A \in \Sigma_{n+1}^1(X)\right\}, \\ \Delta_{n+1}^1 \coloneqq \Sigma_{n+1}^1 \cap \Pi_{n+1}^1. \end{cases}$$

This definition (for Σ_{n+1}^1) is maybe a bit confusing for those who are not well-versed in descriptive set theory (such as the author of this note), so we have the more natural following one:

$$\Sigma_{n+1}^1 = \left\{f(A) \mid A \in \Pi_n^1(Z), f : Z \to X \text{ continuous}, X, Z \text{ Polish}\right\},$$

which follows from the following proposition on closure properties.

Proposition 2.31 ([Kec95, Prop. 37.1 and Exercise 37.3])**.**

a) The Σ_n^1 are closed under Borel images and preimages, countable unions and intersections.

b) The Π_n^1 are closed under Borel preimages, countable unions and intersections.

c) The Δ_n^1 are closed under Borel preimages, complements, and countable unions. In particular, each Δ_n^1 is a σ-algebra, generalizing the Borel one.

The projective hierarchy is given in Figure 2.

We end this section by a statement affirming that the hierarchy is proper.

Theorem 2.32 ([Kec95, Thm. 37.7])**.** *For any uncountable Polish space X and any n, $\Delta_n^1 \subsetneq \Sigma_n^1 \subsetneq \Delta_{n+1}^1$, and idem for Π_n^1.*

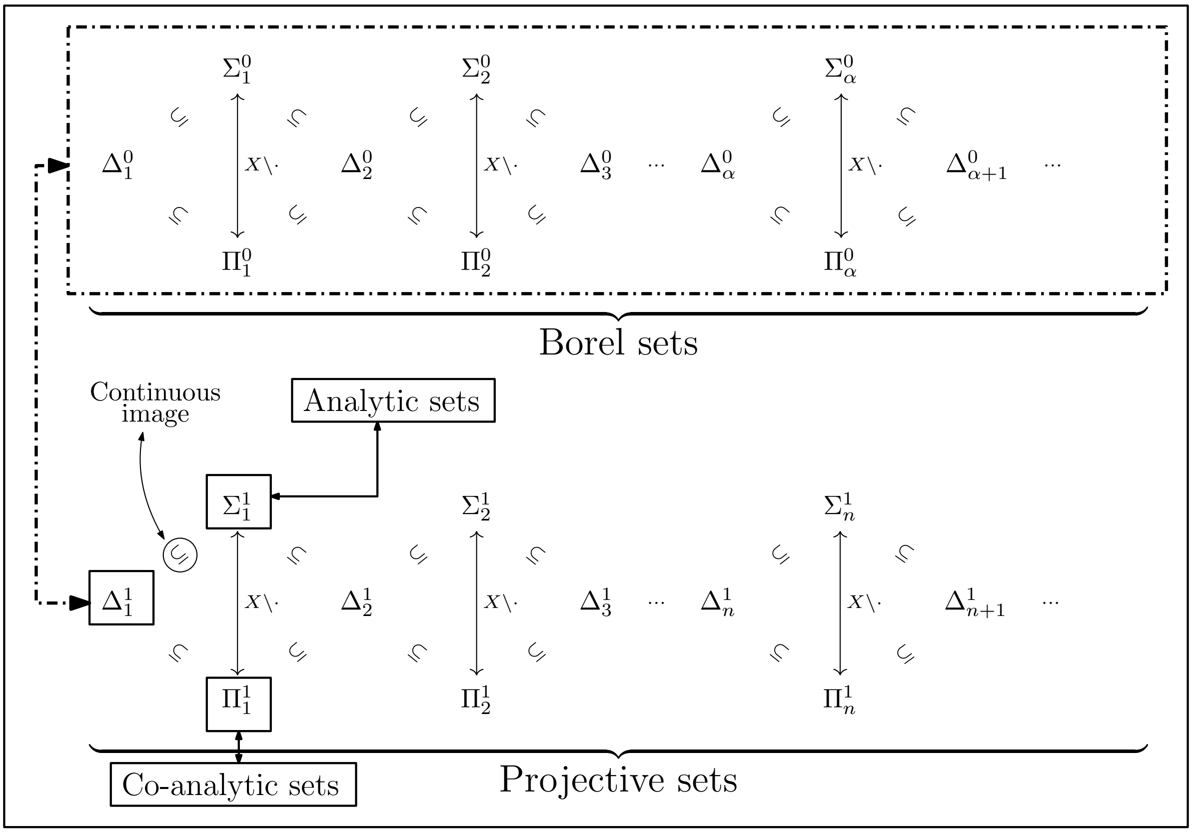

Figure 2: The Borel and Projective hierarchies.

2.3 Baire Category and Baire property

This section is dedicated to Baire category, the Baire Category Theorem, the Baire property BP (a.k.a. Baire measurability). We start with a few definitions, that bear a resemblance to analogous notions of measure theory (that may be more familiar to the ergodic theorist), that we will quicky go over in Section 2.6. Please bear with us until then.

For this section, we refer the reader that is eager to learn more (and to have proofs) to [Oxt71] (on top, of course, of the usual recommendations [Kec95, § 8] and Tserunyan's notes).

Definition 2.33. Let X be a topological space, along with a subset $A \subseteq X$.

- A is **nowhere dense** if its closure $\overline{A}$ has empty interior, *i.e.* if $\mathrm{Int}(\overline{A}) = \emptyset$, equivalently, if $X \setminus \overline{A}$ is dense, equivalently, if there is no open set in which A is dense. A is nowhere dense if and only if $\overline{A}$ is nowhere dense.
- A is **meager** (or of **first category**) if it is a countable union of nowhere dense sets, *i.e.* if there exists a sequence (A_n) of nowhere dense sets such that $A = \bigcup_{n\in\mathbb{N}} A_n$.
- A is of **second category** if it is non-meager.
- A is **comeager** if it is the complement of a meager set. It is also sometimes called **residual**.

Note that A is comeager if and only if it contains the intersection of a countable family of dense open sets.

Example 2.34. We give a few basic examples.

1. The Cantor space $\mathcal{C}$, seen as (it is homeomorphic to it) the triadic Cantor set, has empty interior, hence it is nowhere dense in $[0, 1]$. As an aside, Figure 3 is a representation of the first ten steps of the triadic Cantor space, with a radial construction, which is a bit less common than the usual 'linear' one. It is from the triadic Cantor set Wikipedia page.

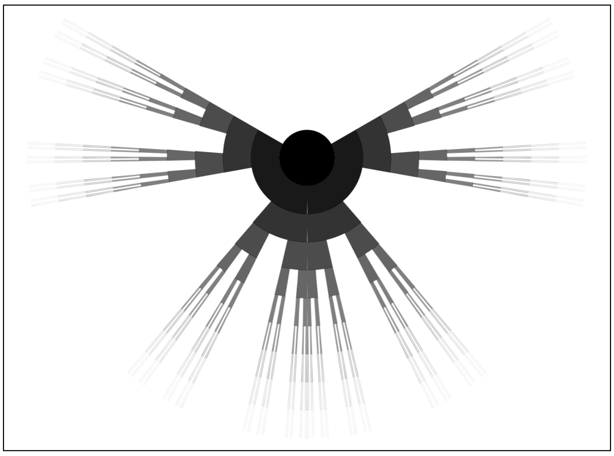

Figure 3: Ten steps of a radial representation of a triadic Cantor set construction.

2. A compact set is nowhere dense in $\mathcal{N}$ (recall in fact that $\mathcal{N}$ is, up to homeomorphism, the unique nonempty Polish zero-dimensional – *i.e.* having a topological base of clopen sets – for which all compact subsets have empty interior, see *e.g.* [Kec95, Thm. 7.7]), and thus a K_σ-set is meager in $\mathcal{N}$.

3. Any finite set is nowhere dense in $\mathbb{R}$, hence $\mathbb{Q}$ is meager in $\mathbb{R}$, and any interval of $\mathbb{R}$ is non-meager.

Remark 2.35. The meager sets represent 'small' sets, and are morally analogous (for the reader familiar with measure theory, more in Section 2.6) to sets of measure 0, a.k.a. null sets. Comeager sets, in contrast, are very big and are morally analogous to full measure sets, a.k.a. conull sets. In fact, the nowhere dense sets form an ideal, *i.e.* a collection of sets containing $\emptyset$ and closed under finite unions, as well as subsets. The meager sets form the corresponding σ-ideal, where we allow countable union instead. As pointed out by Kechris, being a σ-ideal is a notion of smallness, and null sets in a measure space also form a σ-ideal. We will define the corresponding σ-algebra later, which will be the Baire σ-algebra, containing both the meager sets and the open sets.

The following two basic properties follow from the definitions.

Proposition 2.36. *Let X be a topological space, along with a subset $A \subseteq X$.*

1. *A is meager $\Leftrightarrow$ A is contained in a countable union of closed nowhere dense sets.*
   In particular, every meager set is contained in a meager F_σ set.

2. *A is comeager $\Leftrightarrow$ A contains a countable intersection of dense open sets.*
   In particular, a dense G_δ set is comeager.

Before getting to the Baire σ-algebra, we give the definition of Baire spaces, and state the seminal Baire Category Theorem.

Definition 2.37. A topological space X is a **Baire space** if it satisfies the following equivalent conditions:

1. Every nonempty open set in X is non-meager.
2. Every meager set in X has empty interior.
3. Every comeager set in X is dense.
4. The intersection of countably many dense open sets in X is dense.
5. The union of countably many closed sets of empty interior in X has empty interior.

Proposition 2.38 ([Kec95, Prop. 8.3])**.** *Any open subset of a Baire space is a Baire space.*

In particular, in Baire spaces, a set is comeager if and only if it contains a dense G_δ set. In a sense, the Baire spaces are the 'well-behaved' spaces in which the 'opposite properties' of being meager and comeager are truly opposite.

Theorem 2.39 (**The Baire Category Theorem**, see *e.g.* [Kec95, Thm. 8.4])**.** *Every completely metrizable space is Baire, in particular every Polish space is Baire. Every locally compact Hausdorff space is Baire.*

Corollary 2.40. *In nonempty Polish spaces, dense meager sets are not G_δ. In particular, $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$.*

Proof. If a subset is dense G_δ, then it is comeager, and hence nonmeager. □

Remark 2.41. The Choquet game, a game of which a stronger version will be discussed in this note (see Section 3.2) can be used to check – amongst other things – if a space is Baire, see [Kec95, Thm. 8.11].

We can now give the definition of sets with the Baire property, which are the sets, that are 'meagerly' close to open sets. In other words, sets with the Baire property are 'almost-open'. As a warning, sets with the Baire property are not to be confused with (some definition of) Baire measurable sets (or sometimes Baire sets, not to be confused with Baire spaces, not to be confused with THE Baire space $\mathcal{N}$ ...), which are defined from a certain type of Baire functions, which we will not discuss here, although some authors still call them Baire-measurable. As it is very confusing, here we chose to stick to the name *Baire property* - BP -, even if, as we will see, they form a σ-algebra. For measurable functions with regards to this σ-algebra, we chose the (criticizable) name 'BP-measurable'.

Notation 2.42. Let X be a topological space. For any two subsets A, B of X, we write

$$A =^* B$$

to signify that A and B are equal modulo meager sets, *i.e.* that the symmetric difference $A \Delta B$ is meager. One can check that $=^*$ is an equivalence relation on $\mathcal{P}(X)$ that respects complementation and countable unions and intersections.

Definition 2.43. Let X be a topological space, and $A \subseteq X$. We say that A has the **Baire property**, that we will denote by **BP**, if $A =^* U$ for some open set $U \subseteq X$.

Proposition 2.44 ([Kec95, Prop. 8.22])**.** *For any topological space X, the class of sets with BP is a σ-algebra on X, and it is moreover the smallest σ-algebra containing all the open sets and all the meager sets. In particular, all open, closed, G_δ and F_σ sets have BP.*

Once again for the sole purpose of gaining some intuition, we state the following proposition.

Proposition 2.45 ([Kec95, Prop. 8.23])**.** *Let X be a topological space, and $A \subseteq X$. The following are equivalent:*

1. *A has BP;*
2. *$A = G \cup M$, where G is G_δ and M is meager;*
3. *$A = F \setminus M$, where F is F_σ and M is meager.*

Remark 2.46. Constructing sets without BP in $\mathbb{R}$ is not that easy. An example using the axiom of choice is given in [Kec95, Ex. 8.24].

We give the following useful property, and given that we will be needing it, we also give the proof. It is sometimes referred to as the *Baire alternative.*

Proposition 2.47 ([Kec95, Prop. 8.26])**.** *Let X be a topological space and suppose that $A \subseteq X$ has BP. Then either A is meager or there is a nonempty open set $U \subseteq X$ such that A is comeager in U. If X is a Baire space, exactly one of these alternatives holds.*

Proof. We write, by definition, $A \Delta U = M$, with U open and M meager. If A is not meager, necessarily $U \neq \emptyset$, and then A is comeager in U. If X is a Baire space, we just have to show that a meager set A can't be comeager in a nonempty open set U. This is clear, as if it was the case, both A and $U \setminus A$ would be meager, meaning that the open set U would be contained in two meager sets, contradicting 2. in Definition 2.37. □

We of course want to compare the newly obtained σ-algebra of sets with BP to the good ol' σ-algebra of Borel sets. We have the following.

Proposition 2.48 ([Kec95, Prop. 11.5])**.** *Every Borel set has BP, and every Borel function is BP-measurable. (We recall that a function $f : X \to Y$ between topological spaces is Borel if the preimage by f of any open set is Borel, and BP-measurable if the preimage of any open set has BP).*

The converse does not hold. In fact, the following theorem asserts that all analytic sets have BP, so any set in $\Sigma^1_1 \setminus \Delta^1_1$ has BP but is not Borel (such a set exists by Theorem 2.27).

Theorem 2.49 (**Lusin-Sierpiński**, see *e.g.* [Kec95, Thm. 21.6])**.** *Let X be a Polish space. All analytic sets have BP.*

Again the converse does not hold.

Proposition 2.50. *Let X be an uncountable Polish space. We have $\Sigma^1_1 \subsetneq \{\text{sets with BP}\}$.*

Proof. By Proposition 2.44, the complement of a set with BP has BP. Therefore, it suffices to find a set which is not analytic, but whose complement is analytic. In other words, we have to find a set in $\Pi^1_1 \setminus \Sigma^1_1$. By Theorem 2.27 such sets exist. □

We end this section with a interesting application related of BP related to Vitali sets. We give a quick definition, then the proof. Credits go to user Wojowu, from an answer to the question: 'Topological proof that a Vitali set is not Borel' on mathoverflow. More details can (most likely) be found in [Oxt71].

Definition 2.51. A **Vitali set** V is a subset of $[0, 1]$ such that, for each real number r, there exists exactly one $v \in V$ such that $v - r \in \mathbb{Q}$. One can construct a Vitali set by using the axiom of choice to choose a representant in $[0, 1]$ of each class of $\mathbb{R}/\mathbb{Q}$. In particular a Vitali set V is uncountable and the difference of two distinct elements of V is irrational.

Proposition 2.52. *Any Vitali set $V \subseteq [0, 1]$ does not have BP. In particular, it is not Borel.*

Proof. Assume that V has BP. By Proposition 2.47, either V is meager, or it is comeager in some open interval $U \subseteq [0,1]$. We first note that V cannot be meager: $[0,1]$ is contained in a countable union (over the rationals) of translates of V, and that would contradict 2. in Definition 2.37. On the other hand, V also cannot be comeager in U: if it were, for any $q \in \mathbb{Q}$, $U \cap (V+q)$ is contained in $U \setminus V$ (by definition of V), so by taking again the countable (over the rationals) union, we see that U is meager, hence V is meager, a contradiction. □

2.4 About Polish groups

As we have a particular interest for groups, we briefly present some of the most essential facts about Polish groups. Do note once again that this is a very rich theory, and we only scratch the surface here. On top of the usual recommendations, we refer to Julien Melleray's notes on descriptive - *group* - theory, which are available on his webpage, although in french, as well as [BK96, § 1]. This section is very disorderly, as there are many interesting results, and some choices had to be made as to what would be included. We hope any reader will forgive us.

Definition 2.53. A **topological group** is a group with a topology for which the applications $g \mapsto g^{-1}$ and $(g,h) \mapsto gh$ are continuous (for the topology on G and G^2 respectively). A **Polish group** is a topological group whose topology is Polish.

The following closure properties are basic.

Proposition 2.54 ([Gao09, Prop. 2.2.3])**.** *A countable product of Polish groups is a Polish group for the product topology.*

Proposition 2.55 ([Gao09, Prop. 2.2.1])**.** *Let G be a Polish group, and endow $H \leqslant G$ with the subspace topology. Then the following are equivalent:*

1. *H is Polish;*
2. *H is G_δ in G;*
3. *H is closed in G.*

As stated before, any Polish space admits a compatible complete metric, but as sad as it is, there is *a priori* no reason for this metric to be **left (or right)-invariant** (*i.e.* $d(g,g') = d(hg,hg')$ for any $g,g',h \in G$). On the other hand a left-invariant metric always exists for any metrizable topological group from the Birkhoff-Kakutani theorem (see *e.g.* [Kec95, Thm. 9.1]) but in general it is not complete. Polish groups that admit *compatible* complete *left-invariant* metrics are called **cli**. As an example, $\mathfrak{S}_\infty$ is not cli (see *e.g.* [Gao09, Ex. 2.2.7]).

We however have the following construction, which is similar in spirit to the notion of completion of a metric space, but in the context of topological groups, and uses Proposition 2.55.

Theorem 2.56 ([BK96, Thm. 1.1.2] and [BK96, Cor. 1.2.2])**.** *Let G be a topological group, with d a compatible left-invariant metric. Then the metric D defined on G by*

$$D(g,h) = d(g,h) + d(g^{-1},h^{-1})$$

is a compatible metric on G. Moreover, the multiplication on G extends in a unique manner to the completion $\widehat{G}$ of (G,D), which is then a topological group.
In particular, if G is Polish, then D is a compatible complete metric on G.

Proposition 2.57 ([Gao09, Prop. 2.2.6])**.** *Let G be a Polish group with a compatible left-invariant metric d. Then G is cli if and only if d is complete.*

We now turn our attention to quotients, which require a bit more care. We give the main steps, again with no details but with references.

Definition 2.58. Let H be a closed subgroup of a Polish group G, and let d be a left-invariant metric on G. We define a metric d^* on the set of right-cosets G/H of H by setting

$$\begin{aligned} d^*(Hg_1, Hg_2) &:= \inf \{d(k_1, k_2) \mid k_1 \in Hg_1, k_2 \in Hg_2\} \\ &= \inf \{d(g_1, k) \mid k \in Hg_2\} \qquad \text{(by left-invariance).} \end{aligned}$$

Lemma 2.59 ([Gao09, Lem. 2.2.8]). *With the same notations as before, d^* is a compatible metric (with the quotient topology) on G/H.*

The fact that our construction yields a Polish group is due to the following result of Sierpiński.

Theorem 2.60 (**Sierpiński**, see *e.g.* [Gao09, Thm. 2.2.9]). *Let X be a Polish space, Y a metrizable space, and $\pi : X \to Y$ a continuous and open surjection. Then Y is Polish.*

We thus have the following.

Proposition 2.61 ([BK96, Prop. 1.2.3], [Gao09, Thm. 2.2.10 and Exercise. 2.2.7]). *Let G be a Polish group, and let $H \leqslant G$ be a closed normal subgroup. Then G/H is a Polish space for the quotient topology. If moreover H is normal in G, then G/H is a Polish group and the compatible metric d^* is left-invariant.*

We give a bit of context about the next important fact we will mention: automatic continuity. Baire category results behave exceptionally well with Polish groups, and we can obtain very strong results with very little work. We start with the following, ensuring that Polish groups are 'big enough' to work with.

Proposition 2.62 ([Gao09, Prop. 2.3.1]). *Let G be a topological group. The following are equivalent, and are in particular satisfied for Polish groups:*

1. *G is a Baire space;*
2. *G is non-meager;*
3. *Every nonempty open subset of G is non-meager;*
4. *There exists a non-meager open subset of G;*
5. *There exists a non-meager subset of G with BP.*

Next is the very important Pettis Theorem:

Theorem 2.63 (**Pettis,** see *e.g.* [Gao09, Thm. 2.3.2]). *Let G be a topological group, and $A \subseteq G$ be non-meager with BP. Then the set $A^{-1}A = \left\{g^{-1}h \mid g, h \in A\right\}$ contains an open neighborhood of e_G.*

The following consequence of Theorem 2.63 is arguably its most useful one. It is the starting point of the study of *automatic continuity*, in which one aims to expand on the following result.

Proposition 2.64 ([BK96, Thm. 1.2.6]). *Let G and H be two Polish groups. Any BP-measurable homomorphism $\varphi : G \to H$ is continuous. If moreover $\varphi(G)$ is not meager, φ is also open.*

In particular, by Proposition 2.48 any Borel homomorphism between Polish groups is continuous. In fact, thanks to Theorem 2.49 we have the even more general following result: any analytic homomorphism (the preimage of an open set is analytic) is continuous. A direct consequence is the following:

Proposition 2.65 ([Gao09, Exercise. 2.3.7]). *Let G be a topological group, and τ_1 and τ_2 be two topologies on G producing the same Borel structure. Then $\tau_1 = \tau_2$.*

Next is another consequence of Pettis' result. We believe it is not mentioned enough, so as a sign of protest, we give a proof.

Proposition 2.66 ([Gao09, Exercise. 2.3.3]). *Every subgroup of a topological group G which has BP and is non-meager is clopen. In particular a closed non-meager subgroup of a topological group is clopen.*

Proof. Let $H \leqslant G$ have BP and be non-meager. By Theorem 2.63, $H^{-1}H = H$ contains a nonempty neighborhood U of e_G in G. It is then easy to see that H is open, as each $h \in H$ is contained in the open set $hU \subseteq H$.

The fact that H is closed is then a general fact: the left cosets of H are open (by virtue of G being a topological group), and their union is G, so H is closed as the union of all the cosets that are not H is open. □

Do note also that the second part of the previous proof yields the following easier result: any open subgroup of a topological group is clopen. Moreover, we get the following result, which is in a way the starting point of the study of the *small index property.*

Proposition 2.67 ([BK96, § 1.2]). *Let G be a Polish group and H a closed subgroup of G. Then G/H is at most countable if and only if H is clopen.*

We say that a Polish group G is *universal* if any Polish group is topologically isomorphic to a closed subgroup of G. Once again with culture as our main interest, we present two results of Uspenskij about universal Polish groups.

Theorem 2.68 (**Uspenskij**, see *e.g.* [Gao09, Thm. 2.5.2 and Thm. 2.5.3]). *The group* $\mathrm{Iso}(\mathbb{U})$ *of all isometries of the universal Urysohn space, and the group* $\mathrm{Homeo}(\mathbb{I}^{\mathbb{N}})$ *of all homeomorphisms of the Hilbert cube are both universal Polish groups.*

2.5 A bit of action

In this section we give a few important results about actions of some topological groups, which apply in particular for Polish groups. The most important result for us is indubitably Theorem 2.73, which we will use crucially in the second proof of Theorem 4.1. Once again this section will be very bare-bones, and we specifically refer to [BK96] for more on this rich topic.

Definition 2.69. Let G be a group, and X be a space. A **G-action**, or an **action of G on X**, is a map

$$\begin{array}{cccc} \alpha & : & G \times X & \longrightarrow & X \\ & & (g,x) & \longmapsto & g \cdot x \end{array}$$

satisfying $\alpha(e_G, x) = e_G \cdot x = x$ for any $x \in X$ and $\alpha(gh, x) = (gh) \cdot x = g \cdot (h \cdot x) = \alpha(g, \alpha(h, x))$ for any $g, h \in G$ and $x \in X$. We also say that X is a **G-space**. In particular, for each $g \in G$, the map $x \mapsto g \cdot x$ is a bijection of X, of inverse $x \mapsto g^{-1} \cdot x$ and the map sending g to $(x \mapsto g \cdot x)$ is a group homomorphism between G and the group of permutations of X.

If now G and X are topological, then the action α is **continuous** if it is continuous as a function from $G \times X$ to X, in which case we have a homomorphism of G into the group $\mathrm{Homeo}(X)$ of homeomorphisms of X. In this case we also say that X is a **topological G-space**, and if G and X are Polish, we call X a **Polish G-space**. We also say sometimes that α is **jointly continuous**, and this notion of continuity is not to be confused with the following weaker one.

Definition 2.70. Let G be a topological group, X a topological space, and α a G-action on X. The action α is **separately continuous** if for any $g \in G$ the map $\alpha_g : X \to X$ defined by $\alpha_g(x) = \alpha(g, x)$ is continuous and for any $x \in X$ the map $\alpha_x : G \to X$ defined by $\alpha_x(g) = \alpha(g, x)$ is continuous.

Of course joint continuity implies separate continuity and the converse is not true in general, but for actions of groups 'with Polish topologies', we have the following.

Theorem 2.71 ([Kec95, Thm. 9.14])**.** *Let G be a group with a topology that is metrizable and Baire and such that $h \mapsto gh$ is continuous for each $g \in G$. Let also G be acting on a metrizable space X. Then the action is jointly continuous if and only if it is separately continuous.*
In particular, by considering the left action of G on itself by multiplication, we have the following: if G is a group with a metrizable and Baire topology for which $g \mapsto g^{-1}$ is continuous and $(g, h) \mapsto gh$ is separately continuous, then G is a topological group.

Next is Miller's theorem, ensuring that some 'natural sets' arising from a Polish group action are well-behaved.

Theorem 2.72 (**Miller**, see *e.g.* [Gao09, Thm. 3.3.2])**.** *Let G be a Polish group acting on a standard Borel space X. Then for any $x \in X$ the stabilizer $G_x := \{g \in G \mid g \cdot x = x\}$ is closed in G, and the orbit $G \cdot x := \{g \cdot x \mid g \in G\}$ is Borel in X.*

Let us also quickly mention that Miller's theorem actually holds for *Borel G-spaces*, and that the 'every orbit is Borel' result for Polish G-spaces had already been proved by Ryll-Nardzewski (see *e.g.* [Gao09, Thm. 3.1.10]).

Next we give the statement of one of the most important 'realization' theorem for actions of Polish groups. In contrast to Theorem 4.1, the 'larger' space K is compact Polish, but the G-invariant set $\Phi(X)$ is 'just' Borel, while Theorem 4.1 allows us to directly 'put a Polish topology' on X.

Theorem 2.73 (**Becker-Kechris,** [BK96, Thm. 2.6.6])**.** *For any Polish group G, there exists a compact Polish space K along with a continuous G-action $\beta : G \times K \to K$, which are universal in the following sense: for any Borel G-action α on any standard Borel space X, there is a Borel injection $\Phi : X \to K$ which is G-equivariant: for any $g \in G$ and any $x \in X$:*

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

In other words, the Borel G-action α on X can be Borel embedded into the universal compact Polish G-space K.

For those who like commutative diagrams, Theorem 2.73 gives us the following one:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ K & \xrightarrow{\beta} & K \end{array}$$

2.6 A sprinkle of measure

Although this is not the subject matter of this note, one (the author) cannot help wanting to see things through a measured lens. Far from us the idea of redefining the basics of measure theory, our main goal is to link the *universally measurable* sets to the sets we already encountered. The main reference for this section (without surprise) is [Kec95, § 17].

Definition 2.74. Let X be a standard Borel space. A measure (finite or infinite) defined on $(X, \mathcal{B})$ is called a **Borel measure**.

All measures in this section will be Borel measures, and continuous, (or diffuse, or atomless, or nonatomic...), meaning that the measure of all singletons is 0.

Although we will not need it, we could not help but mention the following result, which is the measured analogue of Theorem 2.17. It justifies the terminologies *standard probability space* and *standard σ-finite space*, which we define thereafter.

Theorem 2.75 (**The measure Isomorphism Theorem**, [Kec95, Thm. 17.41]). *Let X be a standard Borel space, endowed with a continuous probability measure μ. Then (X, μ) is measure-isomorphic to $([0,1], \mathrm{Leb}_{\restriction [0,1]})$ (i.e. there exists a Borel isomorphism φ between X and $[0,1]$ such that the pushforward measure $\varphi_* \mu$ is equal to $\mathrm{Leb}_{\restriction [0,1]}$).*

Definition 2.76. We call a pair (X, μ) as in the statement of Theorem 2.75 a **standard probability space**, and thus they are all measure-isomorphic.

Corollary 2.77. *Let X be a standard Borel space, endowed with a continuous σ-finite measure λ. Then (X, λ) is measure-isomorphic to $(\mathbb{R}, \mathrm{Leb})$.*

Definition 2.78. We call a pair (X, λ) as in the statement of Corollary 2.77 a **standard σ-finite space**, and thus they are all measure-isomorphic.

We now get to our main goal. We start by defining the σ-algebras of sets that may not be Borel measurable, but which are close - up to measure 0 - to Borel measurable sets.

Definition 2.79. Fix a Borel measure λ on a standard Borel space $(X, \mathcal{B})$.

1. We say that $A \subseteq X$ is **λ-null** if there exists $B \in \mathcal{B}$ with $\lambda(B) = 0$ and $A \subseteq B$.
2. We denote by Null_λ the σ-ideal (containing $\emptyset$, closed under countable unions and subsets) of λ-null sets for λ.
3. The σ-algebra generated by $\mathcal{B} \cup \mathrm{Null}_\lambda$, comprised of sets of the form $A \cup N$ with $A \in \mathcal{B}$ and $N \in \mathrm{Null}_\lambda$ is denoted by Meas_λ, and its elements are called **Lebesgue-measurable** for the measure λ (which is identified to its natural completion defined by $\widehat{\lambda}(A \cup N) := \lambda(A)$).

Definition 2.80. A set $A \subseteq X$, where X is a standard Borel space, is called **universally measurable** if it is in Meas_λ for any σ-finite measure λ on X.

The following theorem justifies the existence of this section.

Theorem 2.81 (**Lusin**, see *e.g.* [Kec95, Thm. 21.10]). *Let X be a standard Borel space. Every analytic set in X is universally measurable.*

We can summarize the relations between sets of interest in Figure 4.

Establishing links between universally measurable sets and sets with BP seems like a difficult problem. Some implications may or may not have already been studied, but the author is not aware of any result. Any interested reader can consult [LNS10] for more (possibly some answers, as the author has not read through this paper...)

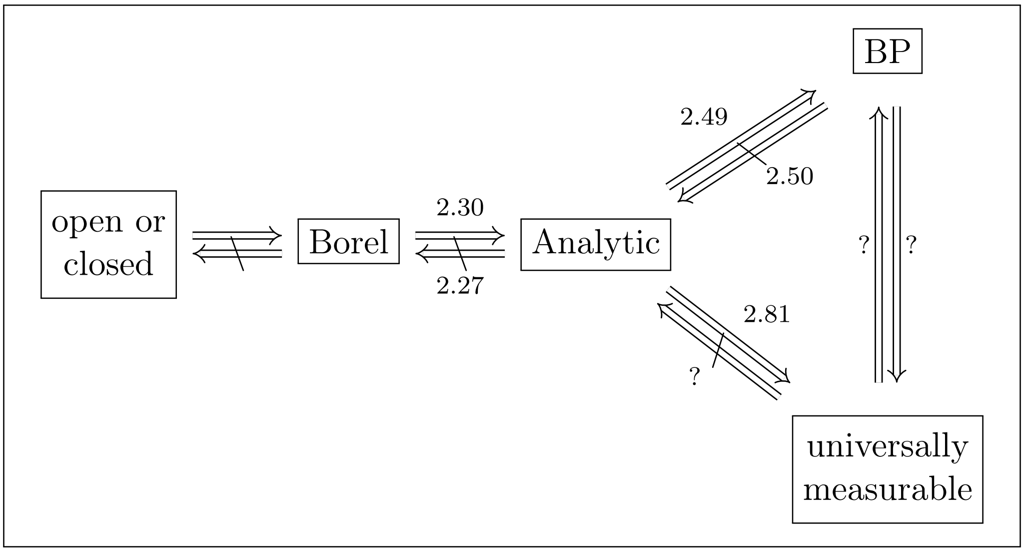

Figure 4: Some nice (non-)implications.

3 Preparing the proofs

3.1 Around the Vaught transforms

Everything from this section can be found in [BK96, § 5.1], [Gao09, § 3.2] and [Kec95, § 8.I and § 16.B]. We start by recalling the notations usually used in Baire category flavoured proofs: the category quantifiers. It is good to think of them as topological analogues of the notion of *conull sets* and *non-null sets*, respectively.

Notation 3.1. Let X be a topological space. We define the following **category quantifiers**:

$$\forall^* x \text{ means for comeager many } x,$$
$$\exists^* x \text{ means for non-meager many } x,$$

and we will sometimes write $\forall^* x \in X$ (resp. $\exists^* x \in X$) to indicate that we are quantifying on elements of X if there's more that one topological space involved. Those two notations mean that 'one can find a comeager (resp non-meager) set of elements satisfying...'.

For a subset $A \subseteq X$ and $x \in X$, the notation $A(x)$ means that $x \in A$, and as such we have the following:

$$\forall^* x\, A(x) \iff A \text{ is comeager in } X,$$
$$\exists^* x\, A(x) \iff A \text{ is non-meager in } X,$$

and similarly, if we quantify 'locally' on an open set $U \subseteq X$, we have

$$\forall^* x \in U\, A(x) \iff A \text{ is comeager in } U,$$
$$\exists^* x \in U\, A(x) \iff A \text{ is non-meager in } U.$$

We quickly give the following statement of the important theorem of category quantifiers, ensuring their 'commutativity', the Kuratowski-Ulam theorem.

Theorem 3.2 (**Kuratowski-Ulam**, see *e.g.* [Gao09, Thm. 3.2.1])**.** *Let X, Y be second countable spaces, and $A \subseteq X \times Y$ with BP. Then*

$$\forall^*(x, y)A(x, y) \iff \forall^* x \forall^* y A(x, y) \iff \forall^* y \forall^* x A(x, y).$$

We can now get to our main tool, the Vaught transform. To get an intuition on what it represents, we informally recall a few basic definitions. Whenever $G \curvearrowright X$ and $A \subseteq X$ (for any topological group G and space X), we define

$$\begin{aligned}[A] &= \text{`the smallest } G\text{-invariant set containing } A\text{'} = \{x \in X \mid \exists g \in G : g \cdot x \in A\}\,,\\ (A) &= \text{`the largest } G\text{-invariant set contained in } A\text{'} = \{x \in X \mid \forall g \in G : g \cdot x \in A\}\,.\end{aligned}$$

We call $[A]$ the **saturation** of A, (A) the **hull** of A, and we have $(A) \subseteq A \subseteq [A]$. We start by defining the 'global' Vaught transform, which generalizes the saturation and the hull by using category quantifiers instead.

Definition 3.3. Let G be a topological group acting on a space X. For any set $A \subseteq X$ we define the **global Vaught transforms**

$$\begin{aligned}A^{\Delta} &= \{x \in X \mid \exists^* g \in G : g \cdot x \in A\}\,,\\ A^{*} &= \{x \in X \mid \forall^* g \in G : g \cdot x \in A\}\,.\end{aligned}$$

We then localize these definitions to obtain the usual **local Vaught transforms**. For any nonempty open set $U \subseteq G$ we define

$$\begin{aligned}A^{\Delta U} &= \{x \in X \mid \exists^* g \in U : g \cdot x \in A\}\,,\\ A^{*U} &= \{x \in X \mid \forall^* g \in U : g \cdot x \in A\}\,.\end{aligned}$$

As we have just seen, the Vaught transforms are local generalizations of the inner and outer notions of saturations. As such, they satisfy a few basic properties, which we now recall.

Proposition 3.4 ([Gao09, Prop. 3.2.3],[Kec95, Prop. 16.3])**.** *Let G be a topological group acting on a space X. For any set $A \subseteq X$ the following hold.*

1. *Both A^{Δ} and A^* are invariant, and $(A) \subseteq A^* \subseteq A^{\Delta} \subseteq [A]$;*
2. *if G is Baire, then (A is G-invariant $\iff A = A^* \iff A = A^{\Delta}$).*

For localized Vaught transforms, the following version of item *2.* of the previous Proposition still holds. We do not however have equivalences (see *(3)* of Proposition 3.7).

Proposition 3.5. *Let G be a topological group, acting on a space X, and let $A \subseteq X$ be G-invariant. Then for any nonempty open set $U \subseteq G$ we have*

$$A = A^{\Delta U} = A^{*U}.$$

Proof. We prove the first equality, the second one is similar. Invariance clearly implies that $A \subseteq A^{\Delta U}$, as any element of U sends $x \in A$ to A. Conversely, if $x \in A^{\Delta U}$, for non-meager many $g \in U$ we have $g \cdot x \in A \Leftrightarrow x \in g^{-1} \cdot A = A$. □

Remark 3.6. It is often useful to work with symmetric open neighborhoods of the identity in G, and if $U \subseteq G$ is such a neighborhood, then for any $A \subseteq X$, we have $A^{\Delta U} \subseteq U \cdot A$.

Of course in practice, our acting group will be Polish, so we can say a bit more. Before stating the stability properties of Vaught transforms, we give a few more useful properties that are helpful when manipulating them. The following is widely used.

Proposition 3.7 ([Gao09, Prop. 3.2.5])**.** *Let G be a Polish group acting on a space X.*

(1) ***Monotonicity:*** *Consider subsets $A \subseteq B \subseteq X$ and open subsets $V \subseteq U \subseteq G$, we have*

$$\begin{array}{ccccccc} A^{*U} & \subseteq & A^{*V} & \subseteq & A^{\Delta V} & \subseteq & A^{\Delta U} \\ \cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\ B^{*U} & \subseteq & B^{*V} & \subseteq & B^{\Delta V} & \subseteq & B^{\Delta U} \end{array}$$

(2) ***De Morgan's laws:*** *For any subset $A \subseteq X$ and any nonempty open subset $U \subseteq G$ we have*

$$\begin{array}{rcl} X \setminus A^{\Delta U} & = & (X \setminus A)^{*U}, \\ X \setminus A^{*U} & = & (X \setminus A)^{\Delta U}. \end{array}$$

(3) *G-**invariance:** For any $A \subseteq X$, any nonempty open $U \subseteq G$ and any $g \in G$ we have*

$$\begin{array}{rcl} g \cdot x \in A^{\Delta V} & \iff & x \in A^{\Delta (Vg)}, \\ g \cdot x \in A^{*V} & \iff & x \in A^{*(Vg)}. \end{array}$$

We will be using the following two properties, which also give interesting insight on how to perceive Vaught transforms. We hope that they can provide intuition.

Proposition 3.8 ([Gao09, Prop. 3.2.5 and Prop. 3.2.6])**.** *Let G be a Polish group acting on a space X.*

(1) ***Countable union/intersection:*** *For subsets $A = \bigcup_{n\in\mathbb{N}} A_n \subseteq X$, $B = \bigcap_{n\in\mathbb{N}} B_n \subseteq X$ and nonempty open subsets $U = \bigcup_{m\in\mathbb{N}} U_m \subseteq G$ and $V = \bigcap_{m\in\mathbb{N}} V_m \subseteq G$ we have*

$$A^{\Delta U} = \bigcup_{n\in\mathbb{N}} A_n^{\Delta U} = \bigcup_{m\in\mathbb{N}} A^{\Delta U_m} = \bigcup_{n,m} A_n^{\Delta U_m},$$
$$B^{*V} = \bigcap_{n\in\mathbb{N}} B_n^{*V} = \bigcap_{m\in\mathbb{N}} B^{*V_m} = \bigcap_{n,m} B_n^{*V_m}.$$

(2) ***Δ-$*$ relations:*** *If in addition X is Polish and the G-action is continuous, then for any Borel subset $A \subseteq X$ and nonempty open subset $U \subseteq G$ we have*

$$\begin{array}{rcl} A^{\Delta U} & = & \bigcup \left\{ A^{*V} \mid V \subseteq U \text{ nonempty open} \right\}, \\ A^{*U} & = & \bigcap \left\{ A^{\Delta V} \mid V \subseteq U \text{ nonempty open} \right\}. \end{array}$$

Finally, the two following Vaught theorems are about the stability of the Vaught transforms with regards to the Borel hierarchy and analytic sets.

Theorem 3.9 ([Gao09, Thm. 3.2.7 and Thm. 3.2.9])**.** *Let G be a Polish group acting continuously on a Polish space X, and $A \subseteq X$. Consider a subset $A \subseteq X$ and a nonempty open subset $U \subseteq G$.*

- *If A is Σ^0_α (resp. Π^0_α) for some $1 \leqslant \alpha < \omega_1$, then $A^{\Delta U}$ and A^{*U} are Σ^0_α (resp. Π^0_α). In particular, the Vaught transform of a Borel set is Borel, and that of an open set is open.*
- *If A is analytic, then so are $A^{\Delta U}$ and A^{*U}.*

The first point can actually be strengthened, and still holds on Borel G-spaces.

Theorem 3.10 ([Gao09, Thm. 3.3.3])**.** *Let G be a Polish group acting in a Borel manner on a standard Borel space X. Let $A \subseteq X$ be a Borel subset and $U \subseteq G$ be nonempty and open. Then $A^{\Delta U}$ and A^{*U} are Borel.*

In the proofs of Theorem 4.1 we will be using families of the following form.

Notation 3.11. For any family $\mathcal{U}$ of nonempty open subsets of G and any family $\mathcal{A}$ of subsets of X, we define the following family of subsets of X:

$$\mathcal{A}^{\Delta \mathcal{U}} := \left\{ A^{\Delta U} \mid A \in \mathcal{A}, U \in \mathcal{U} \right\}.$$

3.2 Strong Choquet spaces

The content of this section is from [BK96, § 5.2], [Gao09, § 4.1] and [Kec95, § 8.D]. Here we present the strong Choquet game and the resulting strong Choquet spaces. The weaker notions of Choquet game and Choquet spaces also exist, but for the sake of brevity we will not present them. The interested reader can consult [Kec95, §8.C].

Definition 3.12. Given a nonempty topological space X, the **strong Choquet game** G^s_X is a two-player game defined by the following rules:

$$\begin{array}{lccccc} \text{I} & (x_0, U_0) & & (x_1, U_1) & & \\ \text{II} & & V_0 & & V_1 & \dots \end{array}$$

Players I and II take turns in playing nonempty open subsets of X, and in addition Player I is required to play a point $x_n \in X$, and their choices must satisfy the following rules:

1. the subsets (U_n), (V_n) must satisfy $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$;
2. Player I must play a point $x_n \in U_n$;
3. Player II must pick V_n such that $x_n \in V_n$.

The first player who violates one of these rules loses, if an infinite sequence of turns is played, Player II wins the game if $\bigcap_n V_n = \bigcap U_n \neq \emptyset$, and Player I wins otherwise. Moreover, as a degenerate case, if X is empty, we consider that Player I has no valid move and loses the game. It is possible that neither player has a winning strategy. A visual representation of the first two turns of a game is given in Figure 5.

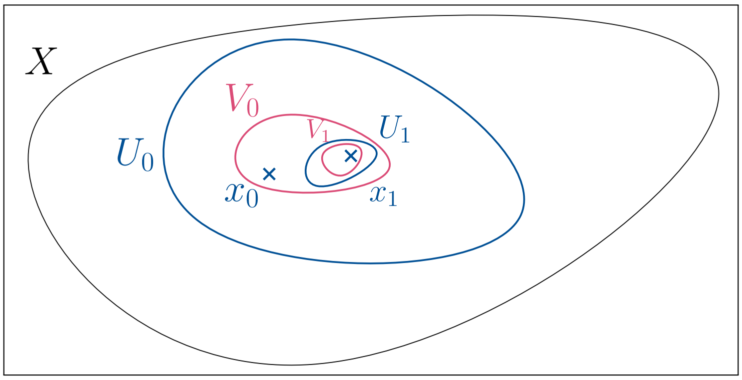

Figure 5: The first two turns of a strong Choquet game.

Definition 3.13. A topological space X is **strong Choquet** if Player II has a winning strategy in G^s_X.

It can be shown that any completely metrizable space or any locally compact Hausdorff space is strong Choquet. We are however interested in converse implications: strong Choquet is a condition that we will use to show that a space is Polish. Namely, we will use the following theorem.

Theorem 3.14 ([BK96, Cor. 5.2.4])**.** *A topological space X is Polish if and only if it is T_1, regular, second countable and strong Choquet.*

We recall that for a topological space, being T_1 means that for every pair of distinct points, each one has a neighborhood not containing the other point; being regular means that any closed subset F of X and any point x not contained in F have non-overlapping open neighborhoods; and finally being second countable means that the topology has a countable base.

As a note, T_1 implies being T_0 (which we did not define), so being T_1 and regular implies being T_0 and regular, which is equivalent to being T_3 (which we also did not define), and Theorem 3.14 is sometimes stated with this condition instead. As conventions around the property of being T_3 are blurry, we chose this formulation instead.

The condition of being regular has the following more practical reformulation, the proof of which is easy.

Lemma 3.15 ([Mun00, Lem. 31.1])**.** *A topological space X is regular if and only if for any point x of X and any neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.*

3.3 Changing the topology

This section gathers the last requirements for the proof of the Becker-Kechris theorem. Specifically, it deals with two technical results that help in making topology changes as 'smooth' as possible. The following result guarantees Polishness of topologies generated by countable unions.

Lemma 3.16 ([Gao09, Lem. 4.2.2 and Lem 4.3.2])**.** *Let X be a space, and $(\tau_n)_{n\in\mathbb{N}}$ a family of Polish topologies on X, such that $\bigcap_{n\in\mathbb{N}} \tau_n$ is Hausdorff.*

(1) The topology τ generated by $\bigcup_{n\in\mathbb{N}} \tau_n$ is Polish on X.

(2) If in addition G is a Polish group acting on X in a τ_n-continuous manner for each n, then $G \curvearrowright X$ is τ-continuous.

Proof. (1). We write $X_n := (X, \tau_n)$ and $Y := \prod_{n\in\mathbb{N}} X_n$ the product topological space. the diagonal D is defined by

$$D := \{(x_n) \in Y \mid x_n = x_{n+1} \ \forall n \in \mathbb{N}\}.$$

The fact that $\bigcap_{n\in\mathbb{N}} \tau_n$ is Hausdorff yields that D is closed in Y: indeed, if $(x_n) \in Y \setminus D$, there exists $n_1 < n_2$ such that $x_{n_1} \neq x_{n_2}$, and by hypothesis we can find U_{n_1} and U_{n_2} disjoint open neighborhoods (for any τ_n!) of x_{n_1} and x_{n_2} respectively, so

$$\left(\prod_{n=0}^{n_1-1} X_n\right) \times U_{n_1} \times \left(\prod_{n=n_1+1}^{n_2-1} X_n\right) \times U_{n_2} \times \left(\prod_{n=n_2+1}^{+\infty} X_n\right)$$

is a neighborhood of (x_n) in $Y \setminus D$, which is then open. Hence D is Polish, as a closed subset of a Polish space.

We now consider the embedding

$$\begin{array}{rccc} e & : & X & \longrightarrow & Y \\ & & x & \longmapsto & (x, x, x, \ldots). \end{array}$$

It is obviously a bijection from X to D. It is τ-continuous by definition of τ: the inverse image of any basic neighborhood of the form $\left(\prod_{n=0}^{k-1} X_n\right) \times U_k \times \left(\prod_{n=k+1}^{+\infty} X_n\right)$ in D is U_k, which is in $\tau_k \subseteq \tau$. We now prove that e is an open map. We let U be a subbasic open set in (X, τ) (*i.e.* an element of $\bigcup_{n\in\mathbb{N}} \tau_n$). By definition $U \in \tau_{n_0}$ for a certain n_0, we now observe that for any (x_n) in D, $(x_n) \in e(U)$ if and only if $x_{n_0} \in U$, so $e(U) \cap D$ is an open set of D.

This concludes the proof of *(1)*, as a continuous bijection which is an open map is an homeomorphism.

(2). From *(1)*, (X, τ) is homeomorphic to D, so it is enough to check that the (diagonal) G-action on D is continuous. This is immediate, as by hypothesis each $G \times (X, \tau_n) \to (X, \tau_n)$ is continuous, so in the product space everything is satisfied for basic neighborhoods. □

The main tool of this section is Theorem 3.18, and to prove it we will also need the following easy key lemma, which allows us to 'add closed sets' to our Polish topologies while retaining Polishness.

Lemma 3.17 ([Gao09, Lem. 4.2.1]). *Let (X, τ) be a Polish space, with F a closed subset of X. The topology τ_F generated by $\tau \cup \{F\}$ is Polish on X. Moreover, F is clopen in this topology.*

Proof. As both F and $X \setminus F$ are Polish subspaces of X, there exist compatible bounded complete metrics $d_F < 1$ and $d_{X \setminus F} < 1$ on F and $X \setminus F$ respectively. We define on X the following metric:

$$d(x, y) = \begin{cases} d_F(x, y) & \text{if } x, y \in F, \\ d_{X \setminus F}(x, y) & \text{if } x, y \in X \setminus F, \\ 1 & \text{otherwise,} \end{cases}$$

which is second countable (hence separable), complete, and compatible with τ_F. It is clear that F is clopen. □

We can now state and prove the main theorem 'topology change' theorem, which will allow us to iteratively define a convenient topology from the Vaught transforms in Section 4.

Theorem 3.18 ([Gao09, Thm. 4.2.3]). *Let (X, τ) be a Polish space, and let $\mathcal{A}$ be a countable collection of elements of $\Sigma^0_\alpha(X, \tau)$, for $1 \leqslant \alpha < \omega_1$. Then there exists a countable family $\mathcal{B}$ of elements of $\Sigma^0_\alpha(X, \tau)$ generating a Polish topology finer than τ and such that $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. The proof is by a transfinite induction on α. The starting case, for $\alpha = 1$ is trivial, as $\mathcal{B} = \mathcal{A} \cup \mathcal{U}$ works, for any countable base $\mathcal{U}$ of τ.

Successor case: Assume that the statement holds for $\alpha \geqslant 1$. We consider a countable collection $\mathcal{A} = (A_n)_{n \in \mathbb{N}} \subseteq \Sigma^0_{\alpha+1}(X, \tau)$. By definition of $\Sigma^0_{\alpha+1}(X, \tau)$ sets we write each A_n as $A_n = \bigcup_{m \in \mathbb{N}} B^m_n$, where each B^m_n is in $\Pi^0_\alpha(X, \tau)$. The collection $\mathcal{C} = \{X \setminus B^m_n \mid n, m \in \mathbb{N}\}$ is countable and included in $\Sigma^0_\alpha(X, \tau)$. Thus, by inductive hypothesis, we can find $\mathcal{D} \subseteq \Sigma^0_\alpha(X, \tau)$ which is countable, contains $\mathcal{C}$, and generates a Polish topology $\tau_{\mathcal{D}}$ finer than τ. Observe that by construction each set B^m_n is $\tau_{\mathcal{D}}$-closed, so Lemma 3.17 yields a Polish topology $\tau^{n,m}_{\mathcal{D}}$ refining $\tau_{\mathcal{D}}$. Moreover, $\bigcap_{n,m \in \mathbb{N}} \tau^{n,m}_{\mathcal{D}}$ is Hausdorff, as it is finer than $\tau_{\mathcal{D}}$, which is Hausdorff. Now Lemma 3.16 ensures us that the topology τ' generated by $\tau_{\mathcal{D}} \cup \{B^m_n \mid n, m \in \mathbb{N}\}$ remains Polish, and each A_n is τ'-open, as an union of subbasic sets, so τ' refines τ. We finally denote by $\mathcal{B}_{\mathcal{D}}$ a countable base for $\tau_{\mathcal{D}}$, and observe that $\mathcal{B}_{\mathcal{D}} \cup \{B^m_n \mid n, m \in \mathbb{N}\} \cup \mathcal{A}$ generates τ' (and contains $\mathcal{A}$ as required).

Limit ordinal: We now assume that α is a limit ordinal, and we consider a countable collection $\mathcal{A} = (A_n)_{n \in \mathbb{N}} \subseteq \Sigma^0_\alpha(X, \tau)$, and write each A_n as $A_n = \bigcup_{m \in \mathbb{N}} B^m_n$, where each B^m_n is in $\Pi^0_{\beta^m_n}(X, \tau)$ for some $\beta^m_n < \alpha$. We set $\mathcal{C} = \{B^m_n \mid n, m \in \mathbb{N}\}$. We choose a cofinal sequence (recall that $\alpha < \omega_1$) in α: $(\gamma_n)_{n \in \mathbb{N}}$ such that $\gamma_0 < \gamma_1 < \ldots < \alpha$. For each $n \in \mathbb{N}$ we define $\mathcal{C}_n :=$ $\left\{B \in \mathcal{C} \mid B \in Pi^0_{\gamma_n}(X, \tau)\right\}$, such that each $\mathcal{C}_n$ is a countable collection of $\Sigma^0_{\gamma_n+1}(X, \tau)$ sets. Since $\gamma_n + 1 < \alpha$ for any n, the inductive hypothesis guarantees that we can find a family $\mathcal{D}_n \supseteq \mathcal{C}_n$ such that the topology $\tau_{\mathcal{D}_n}$ is Polish and finer than τ. By Lemma 3.16 again, the topology τ' generated by $\bigcup_{n \in \mathbb{N}} \tau_{\mathcal{D}_n}$ is Polish. We conclude the proof by noticing that each A_n is τ'-open (it is clear by construction as $\bigcup \mathcal{D}_k \supseteq \mathcal{C}$) and that $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n \cup \mathcal{A}$ generates τ'. □

Remark 3.19. An important remark is that adjoining the complements of the sets of $\mathcal{A}$ to $\mathcal{A}$ (*i.e.* setting $\mathcal{A}' = \mathcal{A} \cup (X \setminus \cdot\, \mathcal{A})$ and applying Theorem 3.18 to $\mathcal{A}'$), it is possible to realize the elements of $\mathcal{A}$ as clopen sets in the finer topology given by Theorem 3.18.

4 The Becker-Kechris theorem

We can now state the main theorem of interest in this note: the theorem from Becker and Kechris on topological realization for Polish group actions.

Theorem 4.1 ([BK96, Thm. 5.2.1], see also [Gao09, 4.4.6])**.** *Let G be a Polish group acting in a Borel manner on a standard Borel space X. Then there exists a Polish topology τ on X which generates the Borel structure, and for which the G-action is continuous.*

4.1 First proof by Becker and Kechris

We present here the original proof by Becker and Kechris, and we will mainly follow Gao's presentation. For this whole section, we fix a Polish group G, a Borel G-action on a standard Borel space X, a base $\mathcal{U}$ for the topology on G and a collection $\mathcal{A}$ of analytic subsets of X (each statement will specify the conditions required on $\mathcal{A}$).

Before starting, we prove a preliminary technical lemma on Vaught transforms that will be useful at different steps of the proof. In addition to the Baire Category Theorem (see *e.g.* [Kec95, Thm. 8.4]), this proof (and some of the following) uses some results of descriptive set theory, for which we do not give a proof.

Lemma 4.2 ([Gao09, Lem. 4.4.1])**.** *Let $g \in G$, $x \in X$, $A \in \Sigma^1_1(X)$, and $W \subseteq G$ be open and nonempty. If $g \cdot x \in A^{\Delta W}$ then there exists nonempty open sets $U, V \in \mathcal{U}$ such that*

$$\begin{cases} x \in A^{\Delta U}, \\ g \in V, \\ UV^{-1} \subseteq W. \end{cases}$$

Proof. By G-invariance in Proposition 3.7, $g \cdot x \in A^{\Delta W}$ yields that $x \in A^{\Delta(Wg)}$. Thus, by definition, the set $H = \{h \in Wg \mid h \cdot x \in A\}$ is non-meager. Moreover, H is analytic as the preimage of the analytic set A by the Borel function $h \mapsto h \cdot x$ (see Proposition 2.26), so in particular it has BP (by Theorem 2.49). Therefore, by the Baire Category Theorem and the Baire alternative (Theorem 2.39 and Proposition 2.47), there exists $U_0 \in \mathcal{U}$ such that $U_0 \subseteq Wg$ and H is comeager in U_0. In other 'words', we have $x \in A^{*U_0}$.

Now, $U_0 g^{-1} \subseteq W$ so by continuity of the right multiplication by g^{-1} (and continuity of the inverse) there exists $U, V \in \mathcal{U}$ such that $U \subseteq U_0$, $g \in V$ and $UV^{-1} \subseteq W$. In particular we have $x \in A^{*U_0} \subseteq A^{*U} \subseteq A^{\Delta U}$ by monotonicity in Proposition 3.7. □

We now get to the proof of 4.1 itself. The main steps of the proof are as follows.

1. Define a topology τ on X via the Vaught transforms.

2. Show that the G-action is τ-continuous on X.

3. Prove that (X, τ) is T_1.

4. Prove that (X, τ) is regular.

5. Prove that (X, τ) is strong Choquet.

6. Conclude that (X, τ) is Polish and induces the original Borel structure of X.

Step 1. The desired topology will be the one generated by $\mathcal{A}^{\Delta \mathcal{U}}$ (see Notation 3.11), which we will denote by $\tau_{\mathcal{A}^{\Delta \mathcal{U}}}$. However, in order to obtain all the needed properties, $\mathcal{A}$ has to satisfy a few technical conditions, which will be detailed in the corresponding steps. The following iterated construction guarantees that $\mathcal{A}$ satisfies everything we want. It uses both Theorem 3.18 and the construction used in its proof.

We start by choosing a Polish topology $\widetilde{\tau}$ on X which generates its Borel structure, and we choose a countable base $\mathcal{A}_0$ for this topology. If $\mathcal{U}$ is a countable base for the topology of G, then for any $n \in \mathbb{N}$, we define

$$\mathcal{A}_{n+1} \supseteq \left\{X \setminus A^{\Delta U} \mid A \in \mathcal{A}_n, U \in \mathcal{U}\right\} \cup \mathcal{A}^{\Delta \mathcal{U}}$$

to be the countable family provided by Theorem 3.18. It generates a Polish topology on X, and all the elements of $\mathcal{A}_{n+1}$ are in $\Sigma^0_\alpha(X, \widetilde{\tau})$ for some $1 \leqslant \alpha < \omega_1$. Finally by Lemma 3.16, the topology $\tau_{\mathcal{A}}$ generated by $\mathcal{A} \coloneqq \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ is Polish. Note that $\mathcal{A}^{\Delta \mathcal{U}} \subseteq \mathcal{A}$, and that any element of $\mathcal{A}$ is still in some $\Sigma^0_\alpha(X, \widetilde{\tau})$, so in particular it is Borel.

The topology we are interested in is not $\tau_{\mathcal{A}}$, but $\tau_{\mathcal{A}^{\Delta \mathcal{U}}}$. By construction $\tau_{\mathcal{A}^{\Delta \mathcal{U}}}$ is second countable as soon as we ask that $\mathcal{U}$ is countable, (it has a countable subbasis, so it has a countable base). Finally, as we will see, it will satisfy the conditions of steps 3, 4 and 5.

Step 2. ([Gao09, Lem. 4.4.2]). The G-action on X is $\tau_{\mathcal{A}^{\Delta \mathcal{U}}}$-continuous.

Proof. We fix $g_0 \in G$, $x_0 \in X$, $A \in \mathcal{A}$ and $W \in \mathcal{U}$ nonempty open such that $g_0 \cdot x_0 \in A^{\Delta W}$. By Lemma 4.2 there exists nonempty open sets $U, V \in \mathcal{U}$ such that

$$\begin{cases} x_0 \in A^{\Delta U}, \\ g_0 \in V, \\ UV^{-1} \subseteq W. \end{cases}$$

The set $A^{\Delta U}$ is $\tau_{\mathcal{A}^{\Delta \mathcal{U}}}$-open, and we check that for all $g \in V$ and all $x \in A^{\Delta U}$, we have $g \cdot x \in A^{\Delta W}$. Indeed, $x \in A^{\Delta U}$ so by G-invariance and monotonicity in Proposition 3.7 $g \cdot x \in A^{\Delta (Ug^{-1})} \subseteq A^{\Delta (UV^{-1})} \subseteq A^{\Delta W}$. □

Step 3. ([Gao09, Lem. 4.4.3]). Let $\mathcal{B}$ be a base for a Polish topology $\tau_{\mathcal{B}}$ generating the Borel structure of X, and $\mathcal{A} \supseteq \mathcal{B}$. Then $\tau_{\mathcal{A}^{\Delta \mathcal{U}}}$ is T_1.

Proof. We fix $x \neq y \in X$, and our goal is to find $B \in \mathcal{B}$ and $U \in \mathcal{U}$ such that $x \in B^{\Delta U}$ and $y \notin B^{\Delta U}$.

We consider the following function:

$$\begin{array}{rccc} f & : & G & \longrightarrow & X \times X \\ & & g & \longmapsto & (g \cdot x, g \cdot y), \end{array}$$

which is a Borel function by assumption. By Proposition 2.48 it is BP-measurable, so by [Kec95, Thm. 8.38] and the Baire Category Theorem there exists a dense G_δ set $H \subseteq G$ on which f is continuous. We fix $g_0 \in H$. Since $g_0 \cdot x \neq g_0 \cdot y$, by virtue of $(X, \tau_{\mathcal{B}})$ being T_2 we can find two disjoint subsets $B, C \in \mathcal{B}$ such that $(g_0 \cdot x, g_0 \cdot y) \in B \times C$. Now by continuity of $f_{\restriction H}$ we can find an open set $U \subseteq G$ containing g_0 and such that $f(U \cap H) \subseteq B \times C$. By definition H is comeager in G, so $H \cap U$ is comeager in U, *i.e.* $x \in B^{*U}$ and $y \in C^{*U}$. Now by monotonicity and De Morgan's laws in Proposition 3.7 we have

$$x \in B^{*U} \subseteq B^{\Delta U} \subseteq (X \setminus C)^{\Delta U} = X \setminus C^{*U},$$

so $y \notin B^{\Delta U}$, concluding the proof. □

Step 4. ([Gao09, Lem. 4.4.4]). If $\mathcal{A}$ is such that for any $A \in \mathcal{A}$ and $U \in \mathcal{U}$ we have $X \setminus A^{\Delta U} \in \mathcal{A}$, then $\tau_{\mathcal{A}^{\Delta\mathcal{U}}}$ is regular.

Proof. We fix $A \in \mathcal{A}$, $W \subseteq \mathcal{U}$, and $x \in A^{\Delta W}$. By Lemma 3.15 it is enough to show that there exists $U \in \mathcal{U}$ such that $x \in A^{\Delta U} \subseteq \overline{A^{\Delta U}} \subseteq A^{\Delta W}$.

Since $e_G \cdot x = x$, by Lemma 4.2 there exists nonempty open sets $U_1, V_1 \in \mathcal{U}$ such that

$$\begin{cases} x \in A^{\Delta U_1}, \\ e_G \in V_1, \\ U_1 V_1^{-1} \subseteq W. \end{cases}$$

Applying Lemma 4.2 immediately again, to $e_G \cdot x = x \in A^{\Delta U_1}$ this time, we find nonempty open sets $U_2, V_2 \in \mathcal{U}$ such that

$$\begin{cases} x \in A^{\Delta U_2}, \\ e_G \in V_2, \\ U_2 V_2^{-1} \subseteq U_1. \end{cases}$$

We denote $U := U_2$ and claim that $A^{\Delta U}$ is the desired neighborhood. We fix $V \in \mathcal{U}$ such that $V \subseteq V_1^{-1} \cap V_2 \ni e_G$, and we set $F := (A^{\Delta U_1})^{*V}$. There are three things that we need to check.

(i) F is $\tau_{\mathcal{A}^{\Delta\mathcal{U}}}$-closed: By Proposition 3.7 again we have $X \setminus F = X \setminus (A^{\Delta U_1})^{*V} = (X \setminus A^{\Delta U_1})^{\Delta V}$ which is in $\tau_{\mathcal{A}^{\Delta\mathcal{U}}}$ by assumption.

(ii) $A^{\Delta U} \subseteq F$: Fix $y \in A^{\Delta U}$. For any $g \in V \subseteq V_2$, we have $U_2 g^{-1} \subseteq U_2 V_2^{-1} \subseteq U_1$ so $U \subseteq U_1 g$. Therefore, by monotonicity we have $\forall g \in V : y \in A^{\Delta U} \subseteq A^{\Delta(U_1 g)}$, which yields that $\forall g \in V : g \cdot y \in A^{\Delta U_1}$ by G-invariance. This easily implies $\forall^* g \in V : g \cdot y \in A^{\Delta U_1}$, hence $y \in (A^{\Delta U_1})^{*V} = F$ as desired.

(iii) $F \subseteq A^{\Delta W}$: Fix $y \in F$. Let g_0 be in V such that $g_0 \cdot y \in A^{\Delta U_1}$ (there are comeager many such g_0 by definition of F). Then by G-invariance y is in $A^{\Delta(U_1 g_0)}$, but $U_1 g_0 \subseteq U_1 V_1^{-1} \subseteq W$, so monotonicity concludes the proof. □

Step 5. ([Gao09, Lem. 4.4.5]). Let $\mathcal{U}$ be countable. Let $\mathcal{A}$ be a countable family of Borel subsets of X, containing X, such that $\mathcal{A}^{\Delta\mathcal{U}} \subseteq \mathcal{A}$ and such that the topology $\tau_{\mathcal{A}}$ generated by $\mathcal{A}$ is Polish. Then $\tau_{\mathcal{A}^{\Delta\mathcal{U}}}$ is strong Choquet.

Proof. In order to lighten the notations we write τ for $\tau_{\mathcal{A}^{\Delta\mathcal{U}}}$ in the rest of this proof. We denote by $\mathcal{B}$ the countable base of τ made of nonempty intersections of finitely many elements of $\mathcal{A}^{\Delta\mathcal{U}}$. By Polishness of $(X, \tau_{\mathcal{A}})$ and of G we also fix a $\tau_{\mathcal{A}}$-compatible complete metric d on X as well as a complete compatible metric d_G on G.

We consider the strong Choquet game $G^s_{(X,\tau)}$ with the following notations:

I	(x_0, M_0)		(x_1, M_1)		$\dots$
II		N_0		N_1	

First we describe a strategy for player II, and then we show that it is a winning strategy.

The τ-open sets $(N_n)_{n\in\mathbb{N}}$ are inductively defined as follows, along with sequences $(A_n)_{n\in\mathbb{N}}$ (with $A_{-1} = X$) of elements of $\mathcal{A}$ and $(U_n)_{n\in\mathbb{N}}$ (with $U_{-1} = G$) of elements of $\mathcal{U}$, so that they satisfy the following for any $n \in \mathbb{N}$:

$$\begin{cases} (1) & x_n \in N_n \subseteq \overline{N_n}^{\tau} \subseteq M_n, \\ (2) & N_n \subseteq A_n^{\Delta U_n}, \\ (3) & \operatorname{diam}_d(A_n) < \dfrac{1}{2^n} \text{ and } \overline{A_n}^{\tau_{\mathcal{A}}} \subseteq A_{n-1}, \\ (4) & \operatorname{diam}_{d_G}(U_n) < \dfrac{1}{2^n} \text{ and } \overline{U_n} \subseteq U_{n-1} \end{cases} \tag{$\star$}$$

We now prove that these sequences exist. Assume that the game has been played according to the rules up to the turn n (of Player II), and that each move of Player II has produced elements of the previously defined sequences satisfying the desired conditions. So far the game looks like this:

$$\begin{array}{llllllllll} \text{I} & (x_0, M_0) & & (x_1, M_1) & & \dots & (x_{n-1}, M_{n-1}) & & (x_n, M_n) \\ \text{II} & & N_0 & & N_1 & \dots & & N_{n-1} & \end{array}$$

and it is the turn of Player II. We show that N_n, A_n and U_n satisfying conditions $(\star)$ actually exist. First note that

$$x_n \subseteq M_n \subseteq N_{n-1} \subseteq A_{n-1}^{\Delta U_{n-1}}$$

by the induction hypothesis. By definition, this means that for non-meager many $g \in U_{n-1}$, $g \cdot x_n \in A_{n-1}$. We can just shrink A_{n-1} to get A_n. Indeed the fact that $\mathcal{A}$ is a base for $\tau_{\mathcal{A}}$, which is Polish on X, means that there exists $A_n \in \mathcal{A}$ satisfying (3) and such that for non-meager many $g \in U_{n-1}$, $g \cdot x_n \in A_n$ (at most countably many elements in $\mathcal{A}$ satisfy (3) and are contained in A_{n-1}, so at least one has work, otherwise it would contradict the non-meagerness).

We can shrank U_{n-1} to get $U_n \in \mathcal{U}$ satisfying (4) and such that there are non-meager many $g \in U_n$ satisfying $g \cdot x_n \in A_n$ with the same argument. Indeed sets of the form $A_i^{\Delta U_i}$ give us control over both G and X at the same time.

Hence $x_n \in A_n^{\Delta U_n}$. We then set

$$N := M_n \cap (A_n^{\Delta U_n}),$$

and since M is τ-open, by step 4 and Lemma 3.15 there exists $N_n \in \mathcal{B}$ such that $x_n \in N_n$ and $\overline{N_n}^{\tau} \subseteq N \subseteq M_n$. This guarantees that (1) is satisfied, and finally notice that (2) is trivially satisfied by definition of N. Notice also that condition (1) guarantees that playing N_n is a valid move.

The only remaining part is to show that the previously described strategy is winning for Player II. We use the same notations. By $(\star)$ for any n we have $x_n \in A_n^{\Delta U_n}$, *i.e.* there exists a non-meager set of g in U_n such that $g \cdot x_n \in A_n$. In particular there exists $g_n \in U_n$ and $y_n \in A_n$ such that $g_n \cdot x_n = y_n$. By $(\star)$ again (g_n) converges to some $g \in \bigcap_{n\in\mathbb{N}} U_n$ and (y_n) to some $y \in \bigcap_{n\in\mathbb{N}} A_n$. As the topology $\tau_{\mathcal{A}}$ refines τ, we also have $\lim_{\tau} y_n = y$. We now set $x := g^{-1} \cdot y$. By step 2 the G-action is τ-continuous, so the $x_n := g_n^{-1} \cdot y_n$ must τ-converge to x. We conclude the proof by showing that $x \in N_n$ for any $n \in \mathbb{N}$.

We fix $n \in \mathbb{N}$. For any $m > n$, $x_m \in N_m \subseteq \overline{N_m}^{\tau} \subseteq M_m \subseteq N_n$, and therefore $x = \lim_{m>n}^{\tau} x_m \in \overline{N_{n+1}}^{\tau} \subseteq N_n$, concluding the proof. □

Step 6. The proof is basically over, and it only remains to check that we have all we wanted.

The topology $\tau_{\mathcal{A} \Delta \mathcal{U}}$ is second countable by construction, is T_1 by Step 3, regular by Step 4 and strong Choquet by Step 5. Therefore it is Polish by Theorem 3.14.

By Theorem 3.10 and the construction of $\mathcal{A}$ in Step 1 (which guarantees that any element of $\mathcal{A}$ is Borel), any element in the subbase of $\tau_{\mathcal{A} \Delta \mathcal{U}}$ is Borel for the original structure of X. As noted in [BK96, Rem. 5.1.2], by Corollary 2.16 (used with the identity function) the Borel structure of $(X, \tau_{\mathcal{A} \Delta \mathcal{U}})$ can only be the original one.

This concludes the (first) proof of Theorem 4.1. ∎

4.2 Second proof by Hjorth

This section presents a second proof of Theorem 4.1 from Hjorth using a 'stronger' result (see [Hjo99, Thm. 2.2]), as well as the universal compact Polish G-space result (see Theorem 2.73).

This part is actually closer in spirit to what we did in Section 3.3, and as such Gao's book presents this proof before the one presented in Section 4.1, but here we opted for the chronological order. This proof is in two steps (Lemma 4.5 and Theorem 4.6). The main one consists in refining the topology by adding Vaught transforms of a closed set to it, while retaining both Polishness and continuity. The second step is mainly a transfinite induction generalizing the result of the first step.

For the first part we make use of the following Polish group, which is of interest in its own right. In the rest of this section, the notation $B(x, r)$ will represent an open ball of center x and radius r.

Definition 4.3. Let G be a topological group, endowed with a right-invariant compatible metric d which is bounded by 1. We denote by $\mathrm{Lip}^1(G, d)$ the set of 1-Lipschitz functions $f : G \to [0, 1]$, *i.e.* functions satisfying

$$|f(g) - f(h)| \leqslant d(g, h)$$

for all $g, h \in G$. Note that $\mathrm{Lip}^1(G, d)$ is nonempty as it contains $d(g, \cdot)$, for any $g \in G$.

We have the following basic properties for $\mathrm{Lip}^1(G, d)$:

Lemma 4.4 ([Hjo99, Lem.1.8 and Lem 1.10])**.** *Let G be a Polish group, endowed with a right-invariant compatible metric d which is bounded by* 1*. Under the topology of pointwise convergence, the space* $\mathrm{Lip}^1(G, d)$ *is compact, and the following map is a well-defined continuous G-action:*

$$\begin{array}{ccc} G \times \mathrm{Lip}^1(G, d) & \longrightarrow & \mathrm{Lip}^1(G, d) \\ (g, f) & \longmapsto & (g \cdot f : h \mapsto f(hg)) \,. \end{array}$$

Proof. As the topology of pointwise convergence on $\mathrm{Lip}^1(G, d)$ is the product topology on $[0, 1]^G$, compactness is ensured by the Tychonoff Theorem.

We next prove that $\mathrm{Lip}^1(G, d)$ is metrizable, which will yield Polishness (see *e.g.* [Kec95, Prop. 4.2]). We fix a dense countable set $Q = (q_n)_{n \in \mathbb{N}}$ in G. By continuity of the functions, we can identify $\mathrm{Lip}^1(G, d)$ with a closed subset of $[0, 1]^Q$, which can be endowed with the compatible metric defined by

$$d(f_1, f_2) = \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} |f_1(q_n) - f_2(q_n)|$$

for any two functions f_1, f_2 in $[0, 1]^Q$.

The only thing left to do is to prove the continuity of the G-action on $\mathrm{Lip}^1(G, d)$. We first check that the action is well-defined. For any $g_1, g_2, g, h, h' \in G$ and any $f \in \mathrm{Lip}^1(G, d)$ we have

$$(g_2 g_1 \cdot f)(h) = f(h g_2 g_1) = (g_1 \cdot f)(h g_2) = (g_2 \cdot (g_1 \cdot f))(h),$$

and by right-invariance of d we have

$$\left|f(hg) - f(h'g)\right| \leqslant d(hg, h'g) = d(h, h'),$$

which means that $(h \mapsto f(hg)) = g \cdot f$ is 1-Lipschitz. We can now get to the continuity of the action. We fix $h \in G$, as well as $B_d(h, \varepsilon)$, the open d-ball of center h and radius $\varepsilon > 0$. For g in a small enough neighborhood W of e_G, by continuity of the right multiplication hg is in $B_d(h, \varepsilon)$, so for any $f \in \mathrm{Lip}^1(G, d)$ we have

$$|(g \cdot f)(h) - f(h)| = |f(hg) - f(h)| \leqslant d(hg, h) < \varepsilon.$$

Therefore, if we define the open neighborhood

$$V(h, B_{[0,1]}(f_0(h), \varepsilon)) = \left\{f \in \mathrm{Lip}^1(G, d) \mid f(h) \in B_{[0,1]}(f_0(h), \varepsilon)\right\}$$

of some f_0 in $\mathrm{Lip}^1(G, d)$ comprised of the functions f satisfying $|f(h) - f_0(h)| < \varepsilon$ (recall that for open sets U in $[0.1]$, sets of the form $V(h, U)$ form a subbasis of the pointwise topology, see *e.g.* [Mun00, § 46]), then for any $g \in W$ and any $f \in V$ we have by the triangle inequality

$$|(g \cdot f)(h) - f_0(h)| \leqslant 2\varepsilon,$$

so $g \cdot f$ is in $V(h, B_{[0,1]}(f_0(h), 2\varepsilon))$. □

Lemma 4.5 ([Gao09, Lem. 4.3.1])**.** *Let G be a Polish group acting in a continuous manner on a Polish space (X, τ), and consider a closed set $F \subseteq X$. Then there exists a topology τ' on X such that the following hold:*

(1) τ' is Polish on X;

(2) $\tau' \supseteq \tau$, i.e. τ' refines τ;

(3) $\tau' \subseteq \Sigma^0_2(X, \tau)$;

(4) for any nonempty open set $U \subseteq G$, $F^{\Delta U} \in \tau'$;

(5) $G \curvearrowright (X, \tau')$ is continuous.

Proof. **Step 1:** We define the topology τ' on X and lay the groundwork.

We begin this proof by considering a right-invariant compatible metric d on G which is bounded by 1, and for each $x \in X$ we also define $f_x \in \mathrm{Lip}^1(G, d)$ by

$$f_x(g) = d(g, \{h \in G \mid h \cdot x \notin F\}) = \inf \{d(g, h) \mid h \cdot x \notin F\}.$$

We use the definition of the G-action on $\mathrm{Lip}^1(G, d)$ from Lemma 4.4, and endow it with its topology of pointwise convergence. We prove that the map $X \ni x \mapsto f_x \in \mathrm{Lip}^1(G, d)$ is G-equivariant, *i.e.* $f_{k \cdot x} = k \cdot f_x$ for any $k \in G$ and any $x \in X$. We fix $k \in G$ and $x \in X$. By right-invariance of d, for any $g \in G$ we have

$$\begin{aligned}
k \cdot f_x(g) &= f_x(gk) \\
&= \inf \{d(gk, h) \mid h \cdot x \notin F\} \\
&= \inf \left\{d(gk, hk^{-1}k) \mid hk^{-1}k \cdot x \notin F\right\} \\
&= \inf \left\{d(g, hk^{-1}) \mid hk^{-1}k \cdot x \notin F\right\} \\
&= \inf \left\{d(g, h') \mid h'k \cdot x \notin F\right\} \\
&= \inf \left\{d(g, h') \mid h' \cdot (k \cdot x) \notin F\right\} \\
&= f_{k \cdot x}(g).
\end{aligned}$$

The space $X \times \mathrm{Lip}^1(G,d)$ is endowed with the diagonal G-action defined by $g \cdot (x,f) \mapsto (g \cdot x, g \cdot f)$, which is continuous by hypothesis and by Lemma 4.4.

We define the following function

$$\begin{array}{rcccc} \varphi & : & X & \longrightarrow & X \times \mathrm{Lip}^1(G,d) \\ & & x & \longmapsto & (x, f_x). \end{array}$$

It is clearly injective, and by the previous argument, it is G-equivariant. We now prove that $\varphi(X)$ is a G_δ set in $X \times \mathrm{Lip}^1(G,d)$. To this end, we fix a dense countable set $G_0 \subseteq G$. For any $(x,f) \in X \times \mathrm{Lip}^1(G,d)$ we have

$$\begin{aligned} (x,f) \in \varphi(X) &\iff f = f_x \\ &\iff \forall g \in G_0 : f(g) = f_x(g) \\ &\iff \forall g \in G_0 : \forall q \in \mathbb{Q} : (f(g) < q \Leftrightarrow f_x(g) < q) \\ &\iff \forall g \in G_0 : \forall q \in \mathbb{Q} : (f(g) < q \Leftrightarrow \exists h \in B_d(g,q) \text{ such that } h \cdot x \notin F). \end{aligned}$$

Indeed the second and third equivalences are by density and continuity, and the fourth is by 1-Lipschitzness and by definition of f_x. In particular, for any $g \in G_0$, and any $q \in \mathbb{Q}$, the set $\{f \in \mathrm{Lip}^1(G,d) \mid f(g) < q\}$ is open and from the last formulation we get that $\{x \in X \mid f_x(g) < q\}$ is also open. Therefore, $\varphi(X)$ is G_δ as a countable intersection (over G_0 and $\mathbb{Q}$) of Boolean combinations of open subsets of $X \times \mathrm{Lip}^1(G,d)$.

We proved that $\varphi(X)$ is G_δ in the product $X \times \mathrm{Lip}^1(G,d)$, so in particular it is Polish by Proposition 2.4. We define on X the topology τ' by pulling back the product topology from $\varphi(X)$ along φ, *i.e.* we declare that $A \subseteq X$ is open if and only if $\varphi(A)$ is open in $X \times \mathrm{Lip}^1(G,d)$.

Step 2: We show that τ' is as desired.

Through the identification $\varphi : X \to \varphi(X)$ and from the previous argument (X, τ') is Polish, so *(1)* is satisfied, and the diagonal action on the G-invariant set $\varphi(X)$ is continuous, so by G-equivariance *(5)* is also satisfied.

By definition, φ is an open map, *i.e.* $\varphi(A)$ is open (in the product topology) for any τ-open set A. Indeed φ^{-1} is continuous as the projection on the first factor. From this we get that $\tau' \supseteq \tau$, so *(2)* is satisfied.

We check *(3)*: we fix A in τ', and show that A is in $\Sigma^0_2(X,\tau)$. Once again we fix a dense countable subset $G_0 \subseteq G$ and $\mathcal{B} = \{B(g,q) \subseteq G \mid g \in G_0 \text{ and } q \in \mathbb{Q}_{>0}\}$ a countable subbase of the topology on G. For all $q \in \mathbb{Q}_{>0}$, all $g \in G_0$ and all open sets $A \subseteq X$, sets of the form

$$A \times \{f \in \mathrm{Lip}^1(G,d) \mid f(g) < q\} \quad \text{or} \quad A \times \{f \in \mathrm{Lip}^1(G,d) \mid f(g) > q\}$$

form a subbase of the topology on $X \times \mathrm{Lip}^1(G,d)$. We check . We then have

$$\begin{aligned} &\varphi(x) \in A \times \{f \in \mathrm{Lip}^1(G,d) \mid f(g) < q\} \\ \iff &x \in A \text{ and } f_x(g) < q \\ \iff &x \in A \text{ and } \exists h \in B(g,q), h \cdot x \notin F, \end{aligned}$$

which means that $\varphi^{-1}(A \times \{f \in \mathrm{Lip}^1(G,d) \mid f(g) < q\})$ is τ-open. Now by definition of f_x we have

$$\begin{aligned} &\varphi(x) \in A \times \{f \in \mathrm{Lip}^1(G,d) \mid f(g) > q\} \\ \iff &x \in A \text{ and } f_x(g) > q \\ \iff &x \in A \text{ and } \exists q' \in \mathbb{Q}_{>0}, \exists V \in \mathcal{B}, (B(g, q+q') \subseteq V \text{ and } \forall h \in V : h \cdot x \in F), \end{aligned}$$

which is $\Sigma^0_2(X,\tau)$, as the condition on the second coordinate is a countable union (on $\mathbb{Q}_{>0}$ and $\mathcal{B}$) of $\Sigma^0_2(X,\tau)$ conditions.

We finally check *(4)*: we fix an nonempty open set $U \subseteq G$, and show that $F^{\Delta U}$ is τ'-open. Notice that $f_x(g) > 0 \implies g \cdot x \in F$ holds by definition. For the converse, by the Baire category Theorem if for any $W \subseteq U$ open we have $\forall^* g \in W : g \cdot x \in F$ then by continuity of the action and by density of the $g \in W$, $W \cdot x \in F$, so in particular $\forall g \in W : f_x(g) > 0$. Therefore by Proposition 3.8 the condition $x \in F^{\Delta U} = \bigcup \left\{ F^{*W} \mid \emptyset \neq W \subseteq U \text{ in a countable base} \right\}$ implies that $\exists^* g \in U : f_x(g) > 0$.

We then define

$$A := \{(x, f_x) \mid f_x(e_G) > 0\},$$

and by definition of the diagonal action we have

$$\begin{aligned} \varphi(x) \in A^{\Delta U} &\iff \exists^* g \in U, (g \cdot x = y \text{ and } g \cdot f_x = f_y, \text{ with } f_y(e_G) > 0) \\ &\iff \exists^* g \in U, f_{g \cdot x}(e_G) > 0 \\ &\iff \exists^* g \in U, f_x(g) > 0 \\ &\iff \exists^* g \in U, g \cdot x \in F \\ &\iff x \in F^{\Delta U}, \end{aligned}$$

where the penultimate equivalence holds by the previous argument. As the set A is open in the restriction of the product topology to $\varphi(X)$, then so is $A^{\Delta U}$ by Theorem 3.9, hence $F^{\Delta U}$ is τ'-open. □

For the next proof, if one wants to consult the references, we advise going directly for Hjorth's, as we believe there is a small mistake in Gao's exposition.

Theorem 4.6 (**Hjorth**, see [Hjo99, Thm. 2.2] or [Gao09, Thm. 4.3.3])**.** *Consider a Polish group G, a continuous G-action on a Polish space (X,τ), a countable collection $\mathcal{U}$ of nonempty open subsets of G, and a countable collection $\mathcal{A}$ of $\Sigma^0_\alpha(X,\tau)$ subsets of X, for some $1 \leqslant \alpha < \omega_1$. Then there exists a topology τ' on X such that the following hold:*

1. *τ' is Polish on X;*
2. *$\tau' \supseteq \tau$, i.e. τ' refines τ;*
3. *$\tau' \subseteq \Sigma^0_\alpha(X,\tau)$;*
4. *$\mathcal{A}^{\Delta\mathcal{U}} \subseteq \tau'$;*
5. *$G \curvearrowright (X,\tau')$ is continuous.*

Proof. Without loss of generality we can take $\mathcal{U}$ to be a countable base for the topology on G, comprised only of nonempty open sets. The proof is by a transfinite induction on α. The starting case, for $\alpha = 1$ is trivial, as $\tau = \tau'$ is satisfactory by Theorem 3.9.

Successor case: Assume that the statement holds for $\alpha \geqslant 1$. We enumerate $\mathcal{A} = (A_n)_{n\in\mathbb{N}} \subseteq \Sigma^0_{\alpha+1}(X,\tau)$. From the definition of $\Sigma^0_{\alpha+1}$-sets, we also let $(B_{n,m})_{n,m\in\mathbb{N}} \subseteq \Pi^0_\alpha(X,\tau)$ be such that for each n we have $A_n = \bigcup_{m\in\mathbb{N}} B_{n,m}$. We define the collection of complements $\mathcal{C} = \{X \setminus B_{n,m} \mid n,m \in \mathbb{N}\} \subseteq \Sigma^0_\alpha(X,\tau)$. It is countable, so by the inductive hypothesis, there exists τ_α satisfying the desired conditions with regards to $\mathcal{C}$, and in particular, by De Morgan's laws in Proposition 3.7, for any $U \in \mathcal{U}$ each $(X \setminus B_{n,m})^{\Delta U} = X \setminus B^{*U}_{n,m}$ is τ_α-open, *i.e.* each $B^{*U}_{n,m}$ is

τ_α-closed. Hence, for each n and m, we can apply Lemma 4.5: there exists a topology $\tau_{n,m}$ such that

(1) $\tau_{n,m}$ is Polish on X;

(2) $\tau_{n,m} \supseteq \tau_\alpha \supseteq \tau$;

(3) $\tau_{n,m} \subseteq \Sigma^0_2(X, \tau_\alpha) \subseteq \Sigma^0_{\alpha+1}(X, \tau)$;

(4) for any $U_0, U_1 \in \mathcal{U}$, $(B^{*U_0}_{n,m})^{\Delta U_1} \in \tau_{n,m}$;

(5) $G \curvearrowright (X, \tau_{n,m})$ is continuous.

We now prove that

$$A^{\Delta U}_n = \bigcup_{m\in\mathbb{N}} B^{\Delta U}_{n,m} = \bigcup \big\{(B^{*U_0}_{n,m})^{\Delta U_1} \mid m \in \mathbb{N}, U_0, U_1 \text{ nonempty open, } U_0U_1 \subseteq U\big\}. \qquad (\dagger)$$

The first equality is *(1)* of Proposition 3.8. For the second one, we notice by G-invariance in Proposition 3.7 that for any $m \in \mathbb{N}$ and any U_0, U_1 nonempty open, $U_0U_1 \subseteq U$ we have

$$(B^{\Delta U_0}_{n,m})^{\Delta U_1} = B^{\Delta U_0 U_1}_{n,m} \quad \text{and} \quad (B^{*U_0}_{n,m})^{*U_1} = B^{*U_0U_1}_{n,m}.$$

Therefore by monotonicity in Proposition 3.7 we have $B^{*U_0U_1}_{n,m} \subseteq B^{\Delta U}_{n,m}$, which yields ($\supseteq$) in the second equality of ($\dagger$), and if we write $B^{\Delta U}_{n,m} = \bigcup\big\{B^{*V}_{n,m} \mid V \subseteq U \text{ nonempty open}\big\}$ by *(2)* of Proposition 3.8, for any nonempty open $V \subseteq U$ such that $U_0U_1 \subseteq V$, we have $B^{*V}_{n,m} \subseteq B^{*U_0U_1}_{n,m} = (B^{*U_0}_{n,m})^{*U_1} \subseteq (B^{*U_0}_{n,m})^{\Delta V_1}$, yielding ($\subseteq$). We have thus proved that $\mathcal{A}^{\Delta\mathcal{U}} \subseteq \tau_{n,m}$.

We conclude by invoking Lemma 3.16, which guarantees everything we need from the topology τ' generated by $\bigcup_{n,m\in\mathbb{N}} \tau_{n,m}$.

Limit ordinal: We now assume that α is a limit ordinal. We keep the same notations, this time with $B_{n,m} \in \Pi^0_{\beta_{n,m}}(X, \tau)$, with $\beta_{n,m} < \alpha$. Then for any $n, m \in \mathbb{N}$, $B_{n,m} \in \Sigma^0_{\beta_{n,m}+1}(X, \tau)$ with $\beta_{n,m} + 1 < \alpha$. By the inductive hypothesis, for any n, m we can find a topology $\tau_{n,m}$ such that

(1') $\tau_{n,m}$ is Polish on X;

(2') $\tau_{n,m} \supseteq \tau$;

(3') $\tau_{n,m} \subseteq \Sigma^0_{\beta_{n,m}+1}(X, \tau)$;

(4') for any $U \in \mathcal{U}$, $B^{\Delta U}_{n,m} \in \tau_{n,m}$;

(5') $G \curvearrowright (X, \tau_{n,m})$ is continuous.

An application of Lemma 3.16 concludes, as the topology τ' generated by $\bigcup_{n,m\in\mathbb{N}} \tau_{n,m}$ is such that all the $A^{\Delta U}_n = \bigcup B^{\Delta U}_{n,m}$ (by Proposition 3.8) are τ'-open. □

Corollary 4.7 (**Hjorth**, see [Hjo99, Cor. 2.3] or [Gao09, Cor. 4.3.4])**.** *Let G be a Polish group, acting in a continuous manner on a Polish space (X, τ). Let (A_n) be a sequence of G-invariant Borel subsets of X. Then there exists a Polish topology τ' refining τ, making each of the A_n open, and such that the G-action on X is still τ'-continuous.*
Moreover, if each A_n is in $\Sigma^0_\alpha(X, \tau)$ (for some $1 \leqslant \alpha < \omega_1$), then $\tau' \subseteq \Sigma^0_\alpha(X, \tau)$.

Proof. Each A_n is G-invariant, so by Proposition 3.5, for any $U \subseteq G$ nonempty and open we have $A_n = A^{\Delta U}_n$. Applying Theorem 4.6 concludes. □

By using Theorem 2.73 to turn any Borel G-space into a (compact) Polish G-space, and then using Corollary 4.7 with the Borel invariant set $X \subseteq K$, we obtain Theorem 4.1 again. ■

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