Spatial models for boolean actions in the infinite measure-preserving setup

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Abstract

We show that up to a null set, every infinite measure-preserving action of a locally compact Polish group can be turned into a continuous measure-preserving action on a locally compact Polish space where the underlying measure is Radon. We also investigate the distinction between spatial and boolean actions in the infinite measure-preserving setup. We finally obtain a streamlined proof of a recent result of Avraham-Re'em and Roy: Lévy groups cannot admit nontrivial continuous measure-preserving actions on Polish spaces when the measure is locally finite.

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1   Introduction

The notion of flow is a convenient framework for understanding global solutions of differential equations. In full generality, a **flow** on a set X is an action $\alpha : \mathbb{R} \times X \to X$ of the real line $\mathbb{R}$ on X. The connection with differential equations (on say $\mathbb{R}^n$ for simplicity) is simply the following: a *global solution* of the differential equation associated to the vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ is a flow $\alpha : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$\frac{d}{dt} \left[\alpha(t,x) \right]_{\restriction t=0} = V(x).$$

In particular, for all $x \in \mathbb{R}^n$, the map $t \mapsto \alpha(t,x)$ is everywhere differentiable. Provided a global solution exists and is unique, the regularity of the corresponding flow usually reflects that of the vector field. In particular, if V is continuous, we cannot expect the flow to be something better than continuous in general.

Many flows arising as solutions of a differential equation preserve a natural σ-finite measure which comes from the geometry of the ambient manifold, thus allowing one to use ergodic-theoretic methods. A notable example is given by Hopf's study of the geodesic flow on hyperbolic surfaces [Hop71]. From the ergodic point of view, flows are identified up to **conjugacy**, and make sense even when the flow is only assumed to be a measurable map.

To be more precise, recall that a **standard Borel space** is an uncountable measurable space X endowed with a σ-algebra $\mathcal{B}(X)$ of Borel subsets coming from some Polish topology on X. All such spaces are isomorphic, see [Kec95, Thm. 15.6], justifying the terminology. A **Borel flow** on X is then an action $\alpha : \mathbb{R} \times X \to X$ which is a Borel map $\mathbb{R} \times X \to X$, meaning that it is measurable if we endow $\mathbb{R} \times X$ with the usual product σ-algebra of the Borel σ-algebra of $\mathbb{R}$ and that of X. Given a Borel σ-finite measure λ on X, a **measure-preserving flow** on (X, λ) is simply a Borel flow such that for all $t \in \mathbb{R}$, the (automatically Borel) bijection $\alpha(t, \cdot)$ satisfies $\lambda(A) = \lambda(\alpha(t,A))$ for all $A \subseteq X$ Borel. Finally, two flows α, β on respective σ-finite measured standard Borel spaces (X, λ) and (Y, η) are called **spatially isomorphic** when there is a measure-preserving Borel bijection $\Phi : X_0 \to Y_0$ between two conull Borel subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that for all $x \in X_0$ and all $t \in \mathbb{R}$ such that $\alpha(t,x) \in X_0$, we have

$$\Phi(\alpha(t,x)) = \beta(t, \Phi(x)).$$

Given that certain measure-preserving flows come from differential equations, it is natural to wonder whether conversely every measure-preserving flows is spatially isomorphic to a flow coming from a differential equation. Our main result shows that as far as the *topology* is concerned, every measure-preserving flow is isomorphic to a flow where the topology and the measure play nicely with each other, just as they do when they come from a differential equation. The key point is that the flow is continuous while the measure is *locally finite* (which implies that it is Radon since the ambient space is locally compact Polish, see Section 2.4). We state our result below in its most general form, replacing $\mathbb{R}$ by an arbitrary locally compact second-countable group, see Section 2.1 for relevant definitions.

Theorem A. *Let (X, λ) be a standard σ-finite space, let G be a locally compact Polish group and let $\alpha : G \times X \to X$ be a measure-preserving G-action on X. Then α is spatially isomorphic to a continuous measure-preserving G-action on a locally compact Polish space Y endowed with a Radon measure η.*

This result is new only when the measure λ is infinite. In order to explain why, we first recall the Becker-Kechris theorem, which holds in the even wider setup of Polish groups actions and does not require one to throw out null sets. Note that for locally compact Polish groups, item (1) below was first proven by Varadarajan, see [Var63, Thm. 3.2].

Theorem 1.1 ([BK96, Thm. 2.2.6 and Thm. 5.2.1]). *Let G be a Polish group and let $\alpha : G \times X \to X$ be a Borel G-action on a standard Borel space X.*

(1) There is a compact Polish space K endowed a continuous G-action $\beta : G \times K \to K$, and a Borel injection $\Phi : X \to K$ which is G-equivariant: for all $x \in X$ and all $g \in G$,

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Moreover, β and K can be chosen so as not to depend on α.

(2) There exists a Polish topology on X inducing its Borel structure such that α is continuous.

From the first item, we deduce that if λ is any Borel finite measure on X and $\alpha : G \curvearrowright (X, \lambda)$ is any measure-preserving action of a Polish group G, then the map $\Phi : X \to K$ is a conjugacy of α with the continuous measure-preserving action $\beta : G \curvearrowright (K, \Phi_*\lambda)$, where K is endowed with the pushforward measure $\Phi_*\lambda$. Since finite Borel measures on compact Polish spaces are automatically Radon, *when λ is finite, Theorem A holds for any measure-preserving action of any Polish group G.* So our Theorem A says something new only for infinite measures (in the above argument the pushforward measure $\Phi_*\lambda$ fails to be locally finite, being an infinite measure over a compact space). Actually, we have two ways of proving this result. The first uses the associated *boolean action* (more on that later), and the second one relies on the following measured version of the first item of the Becker-Kechris theorem.

Theorem B. *Let (X, λ) be a standard σ-finite space, let G be a locally compact Polish group and let $\alpha : G \times X \to X$ be a measure-preserving G-action on X. Then there is a continuous measure-preserving G-action on a locally compact Polish space Y and a Borel G-equivariant injection $\Phi : X \to Y$ such that the pushforward measure $\Phi_*\lambda$ is a Radon measure on Y.*

This result actually allows us to strengthen the conclusion of Theorem A: instead of just a full measure subset of X, the whole set X is taken into a continuous action with Radon measure. The main idea of the proof is to adapt Glasner, Tsirelson and Weiss' notion of G-continuous function (see [GTW05, Def. 2.1]) to the Borel setup, and to show that there is a countable set of integrable G-continuous bounded Borel functions that separates points. In order to prove the latter result, we use convolution.

Given Theorem A, the following question is very natural.

Question 1. Let G be a non locally compact Polish group. Let $\alpha : G \curvearrowright (X, \lambda)$ be a measure-preserving action on an infinite σ-finite standard measured space (X, λ), is α spatially isomorphic to a measure-preserving continuous G-action β on (Y, η) with Y locally compact Polish and η Radon?

One can be more conservative and only require Y to be Polish and η to be locally finite. There is however one obstruction to a positive answer: it may happen that for every $x \in X$ and every neighborhood of the identity $V \subseteq G$, the measurable set $\alpha(V, x)$ always has infinite measure (see Section 5.3 for an example when $G = \mathfrak{S}_\infty$ is the permutation group of the integers). Other than that, we do not see any obstruction and the question is wide open.

Motivated by item (2) from Theorem 1.1, we ask the following question.

Question 2. Let α be a measure-preserving action of a Polish group G on an infinite σ-finite standard measured space (X, λ). When can one endow X itself with a Polish topology compatible with its Borel structure so that the action α becomes continuous and the measure λ becomes locally finite?

This is open even when G is locally compact (note that Theorem A implies that the obstruction we just mentioned does not occur at all in this case, which is not clear at first sight).

We now move to the topic of boolean measure-preserving actions. These can be defined in many equivalent ways (see Section 2), but for this introduction the easiest is probably to describe them as continuous group homomorphisms $G \to \mathrm{Aut}(X, \lambda)$, where $\mathrm{Aut}(X, \lambda)$ is the group of all measure-preserving bijections of (X, λ) *identified up to measure zero*. A fundamental question is: which Boolean actions lift to genuine spatial actions?

In order to answer it, Glasner, Tsirelson and Weiss developed in the setup of a probability measure-preserving boolean action α the following notion which we already briefly mentioned: a function $f \in \mathrm{L}^\infty(X, \lambda)$ is G-**continuous** if $\|f - f \circ \alpha(g, \cdot)\|_\infty \to 0$ as $g \to e_G$. Here is a natural extension of the conclusion of [GW05, Thm. 1.7] to the case where λ is possibly infinite (but still σ-finite).

Theorem C (see Thm. 3.6). *Let G be a Polish group, and let $\alpha : G \to \mathrm{Aut}(X, \lambda)$ be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The following are equivalent:*

(1) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^1(X, \lambda)}^{\|\cdot\|_1} = \mathrm{L}^1(X, \lambda);$$

(2) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda);$$

(3) one can endow X with a locally compact Polish topology inducing its standard Borel space structure so that λ is Radon and α lifts to a continuous measure-preserving G-action on X.

Remark 1.2. We don't know if condition (3) can be changed so as to only require that X is Polish and λ is locally finite.

We use the above theorem to obtain natural infinite measure-preserving version of results of Glasner-Weiss [GW05].

Theorem D. *Let G be a Polish group, assume that G is either locally compact or non-archimedean. Then every boolean measure-preserving G-action on a standard σ-finite space (X, λ) can be lifted to a continuous measure-preserving action on X where X is endowed with a locally compact Polish topology inducing its standard Borel structure and λ is a Radon measure.*

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When G is locally compact, the result is actually not new and belongs to Mackey [Mac62]. However, our approach using convolution is very natural in the framework of G-continuous functions (see Theorem 4.3). Note that Mackey's theorem works in the more general setup of non-singular actions, but using the Maharam extension (which is an infinite measure-preserving boolean action), it is probably possible to deduce his general result from the above theorem. In the non-archimedean case, our proof is essentially the same as Glasner and Weiss', except that we use a simpler fixed point argument in L^2 relying on closed convex hulls (see Theorem 4.6). As we briefly mentioned earlier, Theorem D can be used to give another proof of Theorem A (see Theorem 4.4).

We also adapt the Glasner-Tsirelson-Weiss results on whirly actions (see [GTW05, Sec. 3]) to the infinite measure-presering setup, showing in Section 5.4 that the tautological boolean $\mathrm{Aut}(X, \lambda)$-action cannot be lifted to a Borel spatial action.

Finally, we obtain the following result on Lévy groups, analogous to the Glasner-Tsirelson-Weiss result that says that all their continuous probability measure-preserving actions on compact metric spaces are trivial [GTW05, Thm. 1.1].

Theorem E. *Every continuous measure-preserving spatial action of a Lévy group G on a Polish space X endowed with a σ-finite atomless locally finite measure λ is trivial, i.e. the set of fixed points $\{x \in X \mid \forall g \in G, g \cdot x = x\}$ is conull.*

While finishing our paper, we were informed by Emmanuel Roy that Nachi Avraham-Re'em and him had obtained the exact same result as above and put it on arXiv quite recently (see [ARR24, Thm. 5]). Our proof actually follows the same two steps as theirs. First, we construct a Poisson point process without using counting measures, by directly working in the Effros standard Borel space (see Theorem 6.3, which is a minimalistic version of Theorem 3 in their paper). And then we conclude the proof using the Glasner-Tsirelson-Weiss theorem: the natural action of the group on the space of closed subsets preserves the Poisson point process probability measure, so it has to be trivial, which implies the triviality of the action we started with.

Since our construction of the Poisson point process appears to be more elementary than Avraham-Re'em and Roy's, we decided to keep it in our paper, but the reader should definitely consult their work: they obtain a more precise description of the Poisson point process along with very nice applications of the above result to diffeomorphism groups, using the Maharam extension. They also have a complete description of the boolean probability measure-preserving actions one can obtain through this Poisson point process construction when starting from an infinite measure-preserving boolean action.

Finally, we don't know whether the conclusion of Theorem E holds for general infinite measure-preserving Borel actions, but recall that in general there are measure-preserving actions which cannot be spatially isomorphic to actions satisfying the hypothesis of the above theorem (see Section 5.3). Some related and intriguing open questions may be found in the last section of [ARR24].

2 Preliminaries

2.1 Spatial and boolean actions on standard σ-finite spaces

Recall that a **standard Borel space** is an uncountable measurable space X endowed with a σ-algebra $\mathcal{B}(X)$ of Borel subsets coming from some Polish topology on X. All such spaces are isomorphic, see [Kec95, Thm. 15.6], justifying the terminology.

Throughout the paper, (X, λ) will denote a standard Borel space X endowed with a nontrivial atomless σ-finite measure λ. Since rescaling the measure does not change the dynamics that we are interested in, there are only two fundamentally different cases:

- $\lambda(X) < +\infty$, in which case (X, λ) is isomorphic to $([0, \lambda(X)), \mathrm{Leb}_\restriction)$ where Leb is the Lebesgue measure (this is a direct consequence of [Kec95, Thm. 17.41]);

- $\lambda(X) = +\infty$, in which case (X, λ) is isomorphic to $(\mathbb{R}, \mathrm{Leb})$ (to see this, observe that by σ-finiteness, X can be cut into a countable partition (X_n) such that $\lambda(X_n) < +\infty$ for all n, then cut $\mathbb{R}$ into a countable partition consisting of half-open intervals I_n of length $\mathrm{Leb}(X_n)$ and apply the previous case).

Given a standard σ-finite space (X, λ), a **measure-preserving bijection** of (X, λ) is a Borel bijection $T : X \to X$ such that $T_*\lambda = \lambda$, that is, for all $A \subseteq X$ Borel, $\lambda(A) = \lambda(T^{-1}(A))$.

Definition 2.1. Let G be a Polish group, let (X, λ) be a standard σ-finite space. A **spatial** measure-preserving G-action on (X, λ) is a Borel action map $\alpha : G \times X \to X$ such that for all $g \in G$, the map $\alpha(g, \cdot) : X \to X$ defines a measure-preserving bijection of (X, λ).

Our main goal in this paper is to contrast this notion to that of a boolean action, which to our knowledge first appeared in a paper of Mackey [Mac62]. We will give in the next section several characterizations of this notion, one of which justifies the terminology and the connection with Mackey's definition. In order to motivate the definition we are giving here, notice that by definition the fact that α is an action map means that for all $g, h \in G$ and for *all* $x \in X$ and, we have

$$\alpha(gh, x) = \alpha(g, \alpha(h, x)).$$

Definition 2.2. Let G be a Polish group, let (X, λ) be a standard σ-finite space. A **boolean** measure-preserving G-action on (X, λ) is a Borel map $\alpha : G \times X \to X$ such that:

(i) for all $g, h \in G$, we have that for λ-*almost* all $x \in X$,

$$\alpha(gh, x) = \alpha(g, \alpha(h, x));$$

(ii) for all $g \in G$, the map $\alpha(g, \cdot) : X \to X$ is measure-preserving bijection of (X, λ). (in particular, using $g = h = e_G$ in condition (i), we deduce that $\alpha(e_G, x) = x$ for λ-almost all $x \in X$.)

Remark 2.3. By definition, Condition (i) means that for all $g, h \in G$ the following subset is conull:

$$X_{g,h} = \{x \in X : \alpha(gh, x) = \alpha(g, \alpha(h, x))\},$$

but $X_{g,h}$ does depend on g and h, in particular there does not need to be a conull subset $X_0 \subseteq X$ contained in all the $X_{g,h}$.

By definition, every measure-preserving spatial action is a measure-preserving boolean action. Spatial and boolean actions admit different natural notions of isomorphism.

Definition 2.4. Given measure-preserving spatial actions α on (X, λ) and β on (Y, η) of a Polish group G, a **spatial isomorphism** between them is a measure preserving bijection $\Phi : X \to Y$ and a conull subset $X_0 \subseteq X$ verifying: for all $g \in G$, and all x in X_0, we have

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Definition 2.5. Given measure-preserving boolean actions α on (X, λ) and β on (Y, η) of a Polish group G, a **boolean isomorphism** between them is a measure preserving bijection $\Phi : X \to Y$ such that for any g in G, there is a conull subset $X_g \subseteq X$ such that for all $x \in X_g$,

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Remark 2.6. Some authors choose to define spatial isomorphisms between measure-preserving action by imposing the conull subset X_0 to be G-invariant. Such a definition yields the same notion of being spatially isomorphic for actions of locally compact Polish groups, thanks to the following proposition.

Proposition 2.7. *Let G be a locally compact Polish group, and let α and β be two spatially isomorphic measure-preserving spatial G-actions on standard σ-finite spaces (X, λ) and (Y, η) respectively. There exist a measure-preserving bijection $\Phi : X \to Y$, and a G-invariant conull subset X' verifying: for all g in G and all x in X' we have*

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Proof. Denote by Φ the spatial isomorphism between α and β, and by Ψ its inverse. By Proposition B.5 of [Zim84], there exists a G-invariant Borel conull subset X_0 of X, and a Borel G-equivariant map

$$\widetilde{\Phi} : X_0 \longrightarrow Y,$$

such that $\widetilde{\Phi}$ coincides with Φ on a conull subset $X_1 \subseteq X_0$. It is however possible that $\widetilde{\Phi}$ is not injective. Consider then $\widetilde{\Psi}$ obtained by applying Prop B.5 of [Zim84] to Ψ. Similarly, there exists a G-invariant Borel conull set Y_0, and

$$\widetilde{\Psi} : Y_0 \longrightarrow X$$

veryfing $\widetilde{\Psi} = \Psi$ on a conull subset Y_1 in Y_0. Thus for all x in $X_1 \cap \widetilde{\Psi}(Y_1)$ we have $\widetilde{\Psi} \circ \widetilde{\Phi}(x) = x$, which ensures injectivity of $\widetilde{\Phi}$ on a conull set.

If we now let X' be the conull Borel set of all $x \in X_0$ such that $\widetilde{\Psi} \circ \widetilde{\Phi}(x) = x$, then by G-equivariance of $\widetilde{\Phi}$ and $\widetilde{\Psi}$, we have that X' is G-invariant as wanted. $\qquad \square$

Any spatial isomorphism between spatial actions is also a boolean isomorphism. Two natural problems arise from this for a fixed Polish group G

- the **realization problem**: given a boolean G-action, does it admit a *spatial realization* (or a *spatial model*), i.e. is it booleanly isomorphic to a spatial G-action?

- the **boolean to spatial isomorphism problem**: given a boolean isomorphism between spatial G-actions, is this isomorphism spatial?

When the answers to both theses questions are positive, note that the spatial realization of a boolean action is unique up to spatial isomorphism. Using the fact that any countable intersection of conull sets is conull, it is not hard to see that when G is countable discrete, all its boolean actions are booleanly isomorphic to spatial actions, and all boolean isomorphisms of G-actions are spatial. More generally, a fundamental theorem of Mackey (see [Mac62]) states that this is still true when the acting group G is locally compact Polish. The statement is the following.

Theorem 2.8 ([Mac62, Thm. 1 and Thm. 2]). *Let G be a locally compact Polish group.*

(1) Let $\alpha : G \times X \to X$ be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). There exists a standard σ-finite space (Y, η) and a spatial G-action β on (Y, η) such that α and β are booleanly isomorphic.

(2) Let α and β be two spatial measure-preserving G-actions on standard σ-finite spaces (X, λ) and (Y, η) respectively. Any boolean isomorphism Φ between α and β coincides with a spatial isomorphism on a G-invariant conull subset.

However, for non locally compact Polish groups, there might exist spatial actions which are booleanly isomorphic but not spatially. This is the case for $G = \mathfrak{S}_\infty$, and we will discuss this more thoroughly in Section 4.3.

Remark 2.9. Mackey's result holds more generally for *non-singular* actions. The same is true of the results of the next section (appropriately modified), but since our focus is on infinite measure-preserving action, we chose to remain in the measure-preserving setup throughout the whole paper, with a few exceptions.

2.2   Characterization of boolean actions

Following the introduction of [GTW05], we now provide other ways of understanding boolean actions, one of which will be used in our proof of Theorem A. We first need to describe the Polish group topology of the group of measure-preserving bijections.

Definition 2.10. Let (X, λ) be a standard σ-finite space. We denote by $\mathrm{Aut}(X, \lambda)$ the group of measure-preserving bijections, where two such bijections are identified if they coincide on a conull Borel subset of X.

The group $\mathrm{Aut}(X, \lambda)$ is endowed with **weak topology**, defined as follows: $T_n \to T$ if and only if for all $A \subseteq Y$ Borel subset of finite measure, one has $\lambda(T_n(A) \Delta T(A)) \to 0$.

In order to see that $\mathrm{Aut}(X, \lambda)$ is a Polish group, it is useful to recall the following definition.

Definition 2.11. The **measure algebra** $\mathrm{MAlg}_f(X, \lambda)$ is comprised of the Borel subsets of (X, λ) of finite measure, where two such subsets are identified if the measure of their symmetric difference is equal to zero. It is equipped with the metric d_λ, defined as follows :

$$d_\lambda(A, B) := \lambda(A \Delta B).$$

Proposition 2.12 ([LM22, Prop. 2.2]). $\mathrm{Aut}(X, \lambda)$ *is equal to the group of isometries of* $(\mathrm{MAlg}_f(X, \lambda), d_\lambda)$ *which fix $\emptyset$. As such, it is a Polish group when endowed with the weak topology.*

The following proposition tells us that boolean G-actions on (X, λ) and continuous group homomorphisms $G \to \operatorname{Aut}(X, \lambda)$ are basically the same thing. This result is proved in the introduction of [GTW05] in the finite measure case, and it extends almost verbatim to our context. However, their *near actions* have a small difference with our boolean actions: we require $\alpha(g, \cdot)$ to be a measure-preserving bijection while they only require that it is a measure-preserving map, and need an additional axiom ensuring that in the end that it is a bijection *up to a null set*. Having a more restrictive definition which requires $\alpha(g, \cdot)$ to be a bijection of the whole space, we will need to work a little harder in order to prove the corresponding lifting property (item (III) below).

Proposition 2.13. *Let G be a Polish group, and (X, λ) a standard σ-finite space. The following hold:*

(I) *Every boolean measure-preserving G-action α on (X, λ) yields a continuous group homomorphism $\pi_\alpha : G \to \operatorname{Aut}(X, \lambda)$.*

(II) *Given two boolean measure-preserving G-actions α, β, we have $\pi_\alpha = \pi_\beta$ if and only if the identity map is a boolean isomorphism between α and β.*

(III) *(lifting property) For every continuous group homomorphism $\pi : G \to \operatorname{Aut}(X, \lambda)$, there is a boolean measure-preserving G-action α such that $\pi = \pi_\alpha$. Furthermore, such a lift α is unique up to boolean isomorphism.*

Proof. (I): Given a boolean measure-preserving action α of G on (X, λ), we can consider the natural mapping

$$\begin{aligned} \pi_\alpha : G &\longrightarrow \operatorname{Aut}(X, \lambda) \\ g &\longmapsto \alpha(g, \cdot) \end{aligned}$$

which is well-defined by Condition (ii) from the definition of boolean action, and is a group homomorphism by Condition (i). Recall that any Borel group homorphism between Polish groups is automatically continuous by a result which dates back to Banach (see e.g. [Ros09, Thm. 2.2]). We are thus left with showing that π_α is a Borel map. By definition of the weak topology on $\operatorname{Aut}(X, \mu)$ it is enough to show that for every Borel finite measure subset A of X and every $\varepsilon > 0$, the set

$$\mathcal{B} := \{g \in G : \lambda(\pi_\alpha(g)(A) \triangle A) < \varepsilon\}$$

is Borel. By the definition of π_α, we can rewrite $\mathcal{B}$ as $\mathcal{B} = \{g \in G : \mu(\alpha(g, A) \triangle A) < \varepsilon\}$. Since the boolean action α is a Borel map, the subset $\Gamma := \{(g, x) \in X \times G : x \in \alpha(g, A)\}$ is Borel and hence $\Gamma_A := \Gamma \triangle (A \times G)$ is also Borel. By the Fubini-Tonelli theorem, this implies that the map

$$M : g \longmapsto \lambda(\{x \in X : x \in A \triangle \alpha(g, A)\})$$

which associates to $g \in G$ the measure of the horizontal section of Γ_A above g is also Borel. So we can conclude $\mathcal{B}$ is Borel by noting that $\mathcal{B} = M^{-1}([0, \varepsilon[)$, thus finishing the proof that π_α is Borel and hence continuous as desired.

(II): This is a straightforward consequence of the definition of $\operatorname{Aut}(X, \lambda)$, which identifies measure-preserving bijections which coincide on a conull set.

(III): Let us fix a continuous group homomorphism $\pi : G \to \operatorname{Aut}(X, \lambda)$. As explained in Section 2.1 we may as well assume that X is an interval of $\mathbb{R}$ and λ is the Lebesgue measure restricted to that interval. Denote by $\mathrm{L}^0(X, \lambda, X)$ the set of measurable functions from X to itself, up to equality λ-a.e. Its Borel structure is defined by requiring the functions

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$f \mapsto \lambda(A \cap f^{-1}(B))$ to be measurable, for any Borel subsets A and B of X with A of finite measure. This makes $\mathrm{L}^0(X, \lambda, X)$ a standard Borel space, and it is straightforward to check that $\mathrm{Aut}(X, \lambda)$ is a Borel subset of $\mathrm{L}^0(X, \lambda, X)$.

Following Glasner-Tsirelson-Weiss, we can now define a Borel way to lift any element in $\mathrm{L}^0(X, \lambda, X)$ to a genuine function $X \to X$. Define $V : \mathrm{L}^0(X, \lambda, X) \times X \longrightarrow X$ by

$$V(f, x) = \limsup_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(z) d\lambda(z).$$

Applying the functional version of Lebesgue's density theorem (see [Fre03, Thm. 223A.]), for any f in $\mathrm{Aut}(X, \lambda)$, the function $x \mapsto V(f, x)$ is in the same λ-a.e. equality equivalence class as f. Composing π with the restriction of V to $\mathrm{Aut}(X, \lambda)$, we obtain a Borel map $\alpha_0 : G \times X \to X$ verifying all the required conditions, except that $\alpha_0(g, \cdot) = V(\pi(g), \cdot)$ is a measure-preserving map $X \to X$ which is bijective *up to a null set*.

In order to correct this, let us first define $\beta(g)$ as the partial Borel map whose graph is the intersection of the graph of $\alpha_0(g, \cdot)$ with the inverse of the graph of $\alpha_0(g^{-1}, \cdot)$. In other words,

$$\beta(g)x = \alpha_0(g, x) \text{ if } \alpha_0(g^{-1}, \alpha_0(g, x)) = x,$$

otherwise $\beta(g)x$ is undefined.

Then $\beta(g)$ is a measure-preserving bijection between full measure subsets of X. We finally define $\alpha(g, \cdot)$ by $\alpha(g, x) = x$ if there is $n \in \mathbb{Z}$ such that $\beta(g)^n x$ is not defined, and otherwise $\alpha(g, x) = \beta(g)x$. By construction $\alpha(g, \cdot)$ is now a bijection of X itself, and it preserves the measure since it coincides with $\beta(g)$ on the full measure set of $x \in X$ such that $\beta(g)^n x$ is defined for all $n \in \mathbb{Z}$.

The proof that α is still a Borel map boils down to the fact that for every $n \in \mathbb{Z}$, the set of $(g, x) \in G \times X$ such that $\beta(g)^n x$ is not defined is Borel, which we leave to the reader. Since for every $g \in G$, the bijection $\alpha(g, \cdot)$ coincides with $\alpha_0(g, \cdot)$ on a full measure set, the map α now satisfies all the axioms of a boolean action with $\pi = \pi_\alpha$. Finally, the uniqueness of α is a direct consequence of (II). $\square$

By the above proposition, boolean measure-preserving G-actions are essentially the same thing as continuous group homomorphisms $G \to \mathrm{Aut}(X, \lambda)$. We can now justify the terminology properly: the space $\mathrm{MAlg}_f(X, \lambda)$ of finite measure Borel subsets up to measure zero can be equipped with the usual set theoretic operations Δ and $\bigcap$ so as to become a (non-unital) boolean ring. This boolean ring, when endowed additionally with the metric d_λ, has it automorphism group equal to $\mathrm{Aut}(X, \lambda)$ as a consequence of Proposition 2.12.

So continuous group homomorphisms $G \to \mathrm{Aut}(X, \lambda)$ are exactly continuous G-actions by automorphism on the metric boolean ring $(\mathrm{MAlg}_f(X, \lambda), \triangle, \cap, d_\lambda)$, and since these homomorphims are the same thing as boolean actions by the above proposition, the terminology is now fully justified.

As we will see in the next section, we can also view boolean actions as strongly continuous trace preserving actions on the von Neumann algebra $\mathrm{L}^\infty(X, \lambda)$.

2.3 Actions on function spaces

Given a standard σ-finite space (X, λ), we denote by $\mathrm{L}^0(X, \lambda)$ the space of measurable functions $X \to \mathbb{C}$, two such functions being identified if they coincide on a conull set. The following subspaces of functions are of importance to us:

- the space $\mathrm{L}^\infty(X, \lambda)$ of complex-valued bounded measurable functions on X, endowed with the essential supremum norm $\|\cdot\|_\infty$;

- the space $L^1(X, \lambda)$ of complex-valued integrable functions on X, endowed with the L^1 norm given by $\|f\|_1 = \int_X |f| \, d\lambda$;

- the space $L^2(X, \lambda)$ of complex-valued square integrable functions on X, endowed with the Hilbert space structure given by the scalar product $\langle f, g \rangle = \int_X \overline{f} g \, d\lambda$.

We have a natural left action of $\mathrm{Aut}(X, \lambda)$ on $L^0(X, \lambda)$ given by

$$T \cdot f(x) = f(T^{-1}(x)).$$

All the subspaces $L^\infty(X, \lambda)$, $L^2(X, \lambda)$ and $L^1(X, \lambda)$ are invariant under this action. Moreover, they are acted upon by isometries (for the last two, this uses the fact that elements of $\mathrm{Aut}(X, \lambda)$ preserves the measure, while for $L^\infty(X, \lambda)$ we only need that the measure class is preserved).

However, the $\mathrm{Aut}(X, \lambda)$-action on $L^\infty(X, \lambda)$ is not continuous (for the norm $\|\cdot\|_\infty$), while the $\mathrm{Aut}(X, \lambda)$-action on $L^2(X, \lambda)$ and $L^1(X, \lambda)$ are both continuous. Indeed, for any Borel subset A of (X, λ) of finite measure, A is sent by an element T of $\mathrm{Aut}(X, \lambda)$ to a subset $T(A)$ which is close to A in $(\mathrm{MAlg}_f(X, \lambda), d_\lambda)$ whenever T is close to id_X. This suffices to prove the continuity, as both actions are by isometries and linear combinations of finitely many characteristic functions of finite measure subsets are dense in $L^2(X, \lambda)$ and $L^1(X, \lambda)$ for their respective norms.

We denote by $\mathcal{B}(L^2(X, \lambda))$ the space of bounded operators on $L^2(X, \lambda)$, that is to say the bounded linear maps from $L^2(X, \lambda)$ to itself, and by M_f the operator of pointwise multiplication by a element f of $L^\infty(X, \lambda)$. We recall that the map

$$
\begin{array}{cccc}
M & : & L^\infty(X, \lambda) & \longrightarrow & \mathcal{B}(L^2(X, \lambda)) \\
 & & f & \longmapsto & M_f
\end{array}
$$

is an isometric $*$-isomorphism between $L^\infty(X, \lambda)$ (endowed with the algebra structure provided by pointwise multiplication) and a (von Neumann) subalgebra of $\mathcal{B}(L^2(X, \lambda))$, where $*$ is the involution taking $f \in L^\infty$ to its complex conjugate $x \mapsto \overline{f(x)}$. The map M allows us to endow $L^\infty(X, \lambda)$ with the **strong operator topology**: a net f_n converges to f if and only for all $\xi \in L^2(X, \lambda)$, $M_{f_n}\xi \to M_f\xi$ in L^2 norm.

Remark 2.14. Suppose λ is infinite and μ is a finite measure equivalent to λ. Let $f = \frac{d\lambda}{d\mu}$ be the Radon-Nikodym derivative, then the map $g \mapsto \sqrt{f}g$ induces a surjective isometry $L^2(X, \lambda) \to L^2(X, \mu)$ which commutes with the action by multiplication of $L^\infty(X, \mu) = L^\infty(X, \lambda)$. We conclude that strong convergence in $L^\infty(X, \mu)$ is an intrinsic notion which does not depend to the choice of a (possibly infinite) σ-finite measure in the class of μ.

We finally identify $\mathrm{Aut}(X, \lambda)$ to the following group.

Proposition 2.15. *Given any standard σ-finite space (X, λ), the group $\mathrm{Aut}(X, \lambda)$ naturally identifies to the group of $*$-automorphisms of $L^\infty(X, \lambda)$ which preserve the integral of elements of $L^\infty(X, \lambda) \cap L^1(X, \lambda)$.*

Proof. We have already observed at the beginning of this section that precomposition by the inverse allows to view every measure-preserving bijection T of (X, λ) as an integral-preserving automorphism α_T of $L^\infty(X, \lambda)$, namely $\alpha_T(f)(x) = f(T^{-1}(x))$. Observe that for every $A \subseteq X$ Borel, $\alpha_T(\chi_A) = \chi_{T(A)}$.

Conversely, every integral-preserving $*$-automorphims α of $L^\infty(X, \lambda)$ must take projections to projections. Since the projections which are integrable are naturally identified

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to characteristic functions of elements of $\mathrm{MAlg}_f(X, \lambda)$, every integral-preserving automorphism α of $\mathrm{L}^\infty(X, \lambda)$ defines a measure-preserving transformation T. Moreover, standard results on C^*-algebras yield that α is an isometry for $\|\cdot\|_\infty$ (see e.g. [Con00, Cor. 1.8]). Using the $\|\cdot\|_\infty$-density of the linear span of projections, we conclude that α is equal to α_T as wanted. $\qquad\square$

Having the above proposition and Proposition 2.13 in mind, we can view boolean actions on (X, λ) as integral-preserving actions by $*$-automorphisms[1] on $\mathrm{L}^\infty(X, \lambda)$ which are **continuous** in the following sense (derived from Proposition 2.12): whenever $g_n \to g$ and $A \in \mathrm{MAlg}_f(X, \lambda)$, we have $\int_X |g_n \cdot \chi_A - g \cdot \chi_A|\, d\lambda \to 0$. Boolean isomorphism then becomes the following.

Proposition 2.16. *Let α and β be two boolean G-actions on (X, λ) and (Y, η) respectively, both viewed as continuous actions by integral-preserving automorphisms of their respective L^∞ spaces. Then α and β are booleanly isomorphic if and only if there is an integral-preserving $*$-isomorphism $\rho : \mathrm{L}^\infty(X, \lambda) \to \mathrm{L}^\infty(Y, \eta)$ such that for all $f \in \mathrm{L}^\infty(X, \lambda)$, we have $\rho(\alpha(g, f)) = \beta(g, \rho(f))$.*

Proof. We may as well assume that $X = Y$ and $\lambda = \eta$, so that by the above proposition ρ can be lifted to a measure-preserving bijection $\Phi : X \to X$. Using the uniqueness of lifts of elements of $\mathrm{Aut}(X, \lambda)$ up to measure zero, it is then straightforward to check that Φ satisfies the required conditions to be a boolean isomorphism between the actions. $\qquad\square$

2.4 Locally finite measures

In this section, we introduce the main property of a Borel measure on a Polish space that we will be interested in, and connect it with the notion of a Radon measure when the ambient space is in addition locally compact.

Definition 2.17. Let λ be a Borel measure on a Polish space X. We say that λ is **locally finite** if every $x \in X$ admits an open neighborhood U such that $\lambda(U) < +\infty$.

Of course this property is only interesting for infinite measures. Let us first observe that it implies σ-finiteness.

Lemma 2.18. *Every locally finite measure λ on a Polish space X is σ-finite.*

Proof. By assumption, X is covered by finite measure open sets. Lindelöf's lemma grants us a countable subcover which witnesses the fact that λ is σ-finite. $\qquad\square$

Remark 2.19. A σ-finite measure on a standard Borel space needs not be locally finite. For instance, the counting measure on $\mathbb{Q}$ extends to a non locally finite σ-finite measure on $\mathbb{R}$. More interesting examples will be given later on.

We then recall a definition of Radon measures, following [Coh13, Sec. 7.2].

Definition 2.20. A measure λ on the Borel σ-algebra of a Hausdorff topological space X is called **Radon** when it verifies the following:

1. $\lambda(K) < +\infty$ for any compact subset $K \subseteq X$,

---

[1]This is not the right point of view if one wants to make sense of continuous actions on von Neumann algebras in general. In a more general setup, one conveniently uses the fact that any automorphism yields an isometry of the predual which completely determines it, thus getting a natural Polish topology on the automorphism group of any von Neumann algebra with separable predual.

2. for each open subset U of Y, we have $\lambda(U) = \sup\{\lambda(K) \mid K \subseteq U, \text{ and } K \text{ is compact}\}$ (inner regularity on open sets).

3. for each Borel subset A of X, we have $\lambda(A) = \inf\{\lambda(U) \mid A \subseteq U, \text{ and } U \text{ is open}\}$ (outer regularity on Borel sets).

Recall that a locally compact space is Polish if and only if it is second-countable (see [Kec95, Thm. 5.3]). In such spaces, local finiteness behaves particularly nicely.

Proposition 2.21. *Let X be a locally compact Polish space, let λ be a Borel measure on X. Then λ is Radon if and only if it is locally finite.*

Proof. Suppose λ is Radon, then since it is finite on compact sets and X is locally compact, we conclude that λ is locally finite.

Conversely, suppose λ is locally finite. Then by compactness every compact subset can be covered by finitely many finite measure open subsets, and hence has finite measure. The conclusion now follows from [Coh13, Prop. 7.2.3]. $\qquad\square$

Remark 2.22. As noted in [Coh13, Prop. 7.2.6], σ-finiteness implies inner regularity on all Borel sets, not only open sets. Putting together the previous proposition and Lemma 2.18, we see that locally finite Borel measures on locally compact Polish spaces are always inner regular on all Borel sets.

We finally quote the Riesz-Markov-Kakutani representation theorem, which will allow us to build Radon (equivalently locally finite in our context) measures on new spaces.

Theorem 2.23 (Riesz-Markov-Kakutani, see e.g. [Coh13, Thm. 7.2.8])**.** *Let Y be a locally compact Hausdorff space, and let Ψ be a positive linear functional on the space of complex-valued compactly supported continuous functions $C_c(Y)$. Then there exists a unique Radon measure η on Y such that*

$$\Psi(f) = \int_Y f d\eta$$

holds for each f in $C_c(Y)$.

2.5 Gelfand spaces in the non unital case

In order to build spatial actions, we will as in [GTW05] crucially use separable C^* algebras, but in our setup these will be non-unital. We thus start by recalling the Gelfand-Naimark theorem in the case of a non-unital commutative C^*-algebra.

Definition 2.24. The **spectrum** of a commutative C^*-algebra $\mathcal{A}$ is the space of **characters** on $\mathcal{A}$, that is to say the space of non-zero homomorphisms $\mathcal{A} \longrightarrow \mathbb{C}$. We denote by $\mathrm{Sp}(\mathcal{A})$ the spectrum of $\mathcal{A}$. It is locally compact and Hausdorff when equipped with the topology of pointwise convergence, and if $\mathcal{A}$ is unital, it is a compact space.

Proposition 2.25. *If a commutative C^*-algebra $\mathcal{A}$ is separable, then $\mathrm{Sp}(\mathcal{A})$ is a locally compact Polish space.*

Proof. By [Mur90, Rem. 4.4.1], $\mathrm{Sp}(\mathcal{A})$ is second-countable. Since every locally compact second-countable Hausdroff space is Polish [Kec95, Thm. 5.3], the conclusion follows. $\quad\square$

Theorem 2.26 (Gelfand-Naimark, see e.g. [Mur90, Thm. 2.1.10]). *Let $\mathcal{A}$ be a non-zero commutative C^*-algebra. The Gelfand representation*

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & C_0(\mathrm{Sp}(\mathcal{A})) \\ f & \longmapsto & \widehat{f} \end{array}$$

where $\widehat{f}(z) = z(f)$, between $\mathcal{A}$ and the space of continuous functions that vanish at infinity on $\mathrm{Sp}(\mathcal{A})$ is an isometric $*$-isomorphism.

We now give the proof of the following fact, which is probably well-known, but for which we have found no reference in the literature. Combined with Theorem 2.23, this will allow us to define a Radon measure on $\mathrm{Sp}(\mathcal{A})$. We are grateful to Georges Skandalis for pointing this fact out to us.

Proposition 2.27. *Let $\mathcal{A} = C_0(X)$, where X is a locally compact Polish space. Let also $\mathcal{I}$ be an ideal of $\mathcal{A}$, dense in $\mathcal{A}$ with regards to the sup norm $\|\cdot\|_\infty$. Then, $C_c(X) \subseteq \mathcal{I}$, where $C_c(X)$ denotes the space of compactly supported continuous functions on Y.*

Proof. Let us fix a non-zero function f in $C_c(X)$, supported in a compact subset K.

The Tietze extension theorem grants us a function g in $C_0(X)$ such that $g = 1$ on K. By density, we have a function $h \in \mathcal{I}$ such that $\|g - h\|_\infty < \frac{1}{2}$. In particular

$$\mathrm{supp}\, f \subseteq K \subseteq \left\{ x \in X \colon |h(x)| > \frac{1}{2} \right\}.$$

Applying the Tietze extension theorem again, we find $j \in C_0(Y)$ such that $j = \frac{1}{h}$ on K. We conclude the proof by noting that $hjf = f$ is in $\mathcal{I}$, as $\mathcal{I}$ is an ideal. $\qquad\square$

2.6 Density for C^* subalgebras of L^∞

In what follows, given a set of functions $\mathcal{F} \subseteq \mathrm{L}^\infty(X, \lambda)$, we denote by $(\mathcal{F})_1$ the intersection of $\mathcal{F}$ with the unit ball of $\mathrm{L}^\infty(X, \mu)$ for the norm $\|\cdot\|_\infty$.

Proposition 2.28. *Let λ be a σ-finite measure on a standard Borel space X. Let $\mathcal{G}$ be a unital C^*-subalgebra of $\mathrm{L}^\infty(X, \lambda)$, i.e. a $\|\cdot\|_\infty$-closed unital $*$-subalgebra of $\mathrm{L}^\infty(X, \lambda)$. The following four conditions are equivalent*

(1) $(\mathcal{G})_1 \cap \mathrm{L}^2(X, \lambda)$ is $\|\cdot\|_2$-dense in $(\mathrm{L}^\infty(X, \lambda))_1 \cap \mathrm{L}^2(X, \lambda)$;

(2) $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X, \lambda)$;

(3) the set of finite measure supported elements of $\mathcal{G}$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X, \lambda)$;

(4) $\mathcal{G} \cap \mathrm{L}^1(X, \lambda)$ is $\|\cdot\|_1$-dense in $\mathrm{L}^1(X, \lambda)$.

Moreover, these equivalent conditions imply:

(5) $\mathcal{G}$ is strongly dense in $\mathrm{L}^\infty(X, \lambda)$.

Finally, if λ is finite, (5) implies the four above conditions.

Proof. We clearly have that (3) implies (2). To see the converse, it suffices to show that finite measure supported elements of $\mathcal{G}$ are $\|\cdot\|_2$-dense in $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$.

To this end, let $f \in \mathcal{G} \cap \mathrm{L}^2(X, \lambda)$. For every $\delta > 0$ and $z \in \mathbb{C}$, let

$$p_\delta(z) = \begin{cases} 0 & \text{if } |z| \leqslant \delta, \\ z \times \frac{|z| - \delta}{|z|} & \text{otherwise.} \end{cases}$$

Since $\mathcal{G}$ is a C^*-algebra and p_δ is continuous, we have $p_\delta \circ f \in \mathcal{G}$. Moreover, $p_\delta \circ f$ has finite measure support because $f \in \mathrm{L}^2(X, \lambda)$, and the Lebesgue dominated convergence theorem ensures us that

$$\lim_{\delta \to 0} p_\delta \circ f = f$$

in the L^2-norm, so we conclude that the set of finite measure supported elements of $\mathcal{G}$ is $\|\cdot\|_2$-dense in $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$, which shows that (2) implies (3) as wanted.

Now since bounded elements are dense in $\mathrm{L}^2(X, \lambda)$, it is clear that (1) implies (2). Conversely, if $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$ is $\|\cdot\|_2$ dense in $\mathrm{L}^2(X, \lambda)$, let $f \in (\mathrm{L}^\infty(X, \lambda))_1 \cap \mathrm{L}^2(X, \lambda)$. Let $f_n \to f$ in $\|\cdot\|_2$ with $f_n \in \mathcal{G}$. Define the continuous function $q : \mathbb{C} \to \mathbb{C}$ by

$$q(z) = \begin{cases} z & \text{if } |z| \leqslant 1, \\ \frac{z}{|z|} & \text{otherwise.} \end{cases}$$

Since $\mathcal{G}$ is a C^*-algebra, for all $n \in \mathbb{N}$ we have $q \circ f_n \in (\mathcal{G})_1$. Moreover for all $x \in X$ we have $|q \circ f_n(x) - f(x)| \leqslant |f_n(x) - f(x)|$, so we also have $qf_n \to f$ in $\|\cdot\|_2$, thus finishing the proof that (2) implies (1). So conditions (1), (2) and (3) are all equivalent.

We now prove that (4) and (2) are equivalent using the square and root functions on complex moduli:

$$r(z) = r(|z| e^{i\theta_z}) := \sqrt{|z|} e^{i\theta_z}$$
$$s(z) = s(|z| e^{i\theta_z}) := |z|^2 e^{i\theta_z}.$$

Note that r and s are homeomorphisms of $\mathbb{C}$, inverse of each other. Moreover given $f \in \mathrm{L}^1(X, \lambda)$ the function $s \circ f$ is in $\mathrm{L}^2(X, \lambda)$ and vice-versa. Furthermore, if f is in $\mathcal{G}$, $r \circ f$ and $s \circ f$ remain in $\mathcal{G}$ as it is a C^* algebra. We prove that (4) implies (2).

Let then $g \in \mathrm{L}^2(X, \lambda)$, and let $f = s \circ g$ so that $g = r \circ f$ with $f \in \mathrm{L}^1(X, \lambda)$. By assumption, we have a sequence (f_n) converging to f for the L^1 norm, with $f \in \mathcal{G}$. By a classical theorem attributed to Riesz–Fischer (see e.g. [Bou04, IV §3 Thm. 3]), we can extract a subsequence (f_{n_k}) converging pointwise almost everywhere to f so that there exists $h \in \mathrm{L}^1(X, \lambda)$ verifying $|f_{n_k}| \leqslant h$ almost everywhere for any k. Thus, the sequence $(r \circ f_{n_k})$ is in L^2 and $r \circ f_{n_k} \to r \circ f$ pointwise almost everywhere. Moreover, we have $|r \circ f_{n_k}| \leqslant r \circ h \in \mathrm{L}^2(X, \lambda)$ for any k. By the L^2 version of the Lebesgue dominated convergence theorem, we have $r \circ f_{n_k} \to r \circ f$ in L^2 norm. In other words $g = r \circ f$ can be approximated for $\|\cdot\|_2$ by functions in $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$ as wanted, thus showing that (4) implies (2). The symmetric argument gives the reverse implication, so conditions (1), (2), (3) and (4) are all equivalent.

We now connect them to (5) by showing first that (1) implies strong density of $\mathcal{G}$ in $\mathrm{L}^\infty(X, \lambda)$. To this end, first note that $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^2(X, \lambda)$ is strongly dense in $\mathrm{L}^\infty(X, \lambda)$: if $f \in \mathrm{L}^\infty(X, \lambda)$ and (X_n) is an increasing sequence of subsets of finite measure subsets of X such that $X = \bigcup_{n \in \mathbb{N}} X_n$, then for all $\xi \in \mathrm{L}^2(X, \lambda)$

$$\|f\xi - f\mathbb{1}_{X_n}\xi\|_2^2 = \int_{X \setminus X_n} |f(x)\xi(x)|^2 \, d\lambda(x) \longrightarrow 0$$

by the Lebesgue dominated convergence theorem, so $f\mathbb{1}_{X_n} \to f$ strongly. Towards showing the desired implication, assume (1), it now suffices to strongly approximate any $f \in$

$L^\infty(X, \lambda) \cap L^2(X, \lambda)$ by a sequence of elements of $\mathcal{G}$. Replacing f by $f / \|f\|_\infty$ if need be, we may as well assume $f \in (L^\infty(X, \lambda))_1$. We then have $a_n \to f$ in $\|\cdot\|_2$ with $a_n \in (A)_1$. By density of step functions and the fact that $\|a_n\|_\infty \leqslant 1$, it suffices to show that if B is a finite measure subset, then $a_n \mathbb{1}_B \to f \mathbb{1}_B$ for $\|\cdot\|_2$, which is an immediate consequence of the fact that

$$\|a_n \mathbb{1}_B - f \mathbb{1}_B\|_2^2 = \int_B |a_n(x) - f_n(x)|^2 \, d\lambda(x) \leqslant \|a_n - f\|_2^2 \longrightarrow 0.$$

Finally, if λ is finite, let us show (5) implies (2). Take any generalized sequence (f_i) in $\mathcal{G}$ converging strongly to $f \in L^\infty(X, \lambda)$, then $f_i \mathbb{1}_X \to f \mathbb{1}_X$, which means exactly that $\|f_i - f\|_2 \to 0$. $\qquad\square$

Remark 2.29. Similar arguments show that we can add to the list of the first 4 equivalent conditions the L^1 versions of condition (1) and (3), namely

(1') $(\mathcal{G})_1 \cap L^1(X, \lambda)$ is $\|\cdot\|_1$-dense in $(L^\infty(X, \lambda))_1 \cap L^1(X, \lambda)$;

(3') the set of finite measure supported elements of $\mathcal{G}$ is $\|\cdot\|_1$-dense in $L^1(X, \lambda)$.

We did not include them in the statement for brevity.

As a concrete example of the above situation, we have the following well-known lemma, which will be useful in our characterization of spatial realizations of Boolean actions via Gelfand's theorem.

Lemma 2.30 (see [Coh13, Prop. 7.4.3]). *Let Y be a locally compact Hausdorff space equipped with a Radon measure η, and consider the C^*-algebra $C_0(Y)$ of continuous functions that vanish at infinity on Y. Then $C_0(Y) \cap L^2(Y, \eta)$ is $\|\cdot\|_2$-dense in $L^2(Y, \eta)$. In particular, the strong closure of $C_0(Y)$ in $\mathcal{B}(L^2(Y, \eta))$ is equal to $L^\infty(Y, \eta)$.* $\qquad\square$

3 G-continuity and spatial models

3.1 G-continuity

We recall the definition of G-continuity, which has notably been used in [KS11] and in [GW05] to discuss the existence of spatial models for boolean actions of Polish groups.

Definition 3.1. Let G be a Polish group, and let α be a measure-preserving boolean G-action on a standard σ-finite space (X, λ). We say that f in $L^\infty(X, \lambda)$ is G-**continuous** if $\left\| f - f \circ \alpha(g_n^{-1}, \cdot) \right\|_\infty \to 0$ whenever $g_n \to e_G$. The space of G-continuous functions will thereafter be denoted by $\mathcal{G}$.

Remark 3.2. Although G-continuity is defined for functions in $L^\infty(X, \lambda)$, the previous definition extends to non-essentially bounded functions. Indeed, only the difference of $f \circ \alpha(g_n^{-1}, \cdot)$ and f needs to be in $L^\infty(X, \lambda)$, when g is close enough to e_G.

The set $\mathcal{G}$ of G-continuous functions is easily seen to be a $*$-subalgebra of $L^\infty(X, \lambda)$. We will now see that it is actually a C^*-algebra, i.e. that it is $\|\cdot\|_\infty$-closed. This will be a direct consequence of the following well-known proposition.

Proposition 3.3. *Let (M, d) be a metric space on which a topological group G acts by isometries. Denote by M_G the set of $x \in M$ such that $d(g_n \cdot x, x) \to 0$ whenever $g_n \to e_G$. Then M_G is a closed subset of M, and it is the largest subset of M onto which G acts continuously.*

Proof. Let (x_k) be a sequence of elements of M_G converging to $x \in M$. Fix $\varepsilon > 0$. Towards showing $x \in M_G$, take a sequence (g_n) tending to e_G. By the triangle inequality, we have the following for all $k, n \in \mathbb{N}$:

$$d(x, g_n \cdot x) \leqslant d(x, x_k) + d(x_k, g_n \cdot x_k) + d(g_n \cdot x_k, g_n \cdot x).$$

Fix k large enough so that $d(x, x_k) < \varepsilon$. Since g_n is an isometry, we have $d(g_n \cdot x_k, g_n \cdot x) = d(x, x_k)$, and so both the first and third terms in the above sum are less than ε. For the second term, since x_k is a point of continuity for the G-action, it is smaller than ε for n big enough, which concludes the proof that X_G is closed.

Now by the definition of M_G, the restriction of the action of G on a set which intersects the complement of M_G cannot be continuous. We thus only need to show that the action on M_G is continuous. To this end, let $x_n \to x$ with $x_n, x \in M_G$ and let $g_n \to g$. Then

$$d(g_n \cdot x_n, g \cdot x) \leqslant d(g_n \cdot x_n, g \cdot x_n) + d(g_n \cdot x, g \cdot x)$$
$$= d(x_n, x) + d(g^{-1} g_n \cdot x, x).$$

Since $g^{-1} g_n \to e_G$ and $x \in M_G$, we have $d(g^{-1} g_n \cdot x, x) \to 0$, and since $x_n \to x$, we also have $d(x_n, x) \to 0$. So $d(g_n \cdot x_n, g \cdot x) \to 0$ as wanted, which finishes the proof. $\qquad\square$

Corollary 3.4. *Let $\pi : G \to \mathrm{Aut}(X, \lambda)$ be a boolean action. Then the space $\mathcal{G}$ of G-continuous functions is actually a C^*-subalgebra of $\mathrm{L}^\infty(X, \lambda)$ onto which G acts continuously.*

Proof. We have already observed that the G-action on $(M, d) = (\mathrm{L}^\infty(X, \lambda), \|\cdot\|_\infty)$ associated to π (given by $g \cdot f = f \circ \pi(g)^{-1}$) is by isometries, and by definition $\mathcal{G} = M_G$ so the conclusion directly follows from the previous proposition. $\qquad\square$

We finally observe that we have the following natural source of G-continuous functions, which by Gelfand's theorem can be viewed as the only source of G-continuous functions.

Lemma 3.5. *Let X be a locally compact Polish space, let $\alpha : G \times X \to X$ be a continuous G-action, consider the action on $C_0(X)$ by precomposition by the inverse, then every element of $C_0(X)$ is G-continuous, namely $\left\| f - f \circ \alpha(g^{-1}, \cdot) \right\|_\infty \to 0$ when $g \to e_G$, where $\|\cdot\|_\infty$ is the supremum norm.*

Proof. Let $f \in C_0(X)$, let $\varepsilon > 0$. Let K be a compact set such that for all $x \notin K$ we have $|f(x)| < \varepsilon$. By continuity of the action and of f, for every $x \in K$ there is a neighborhood U_x of x and a symmetric neighborhood $\mathcal{N}_x$ of e_G such that $|f(\alpha(g, x')) - f(x)| < \varepsilon$ for all $g \in \mathcal{N}_x$ and all $x' \in U_x$. Take a finite subcover $(U_{x_i})_{i=1}^n$ of K, let $\mathcal{N} = \bigcap_{i=1}^n \mathcal{N}_{x_i}$, then if $x \in U_{x_i}$, the triangle inequality yields

$$|f(\alpha(g, x)) - f(x)| \leqslant |f(\alpha(g, x)) - f(x_i)| + |f(x_i) - f(x)| < 2\varepsilon.$$

Now let $x \in X$ be arbitrary and take $g \in \mathcal{N}$, we have three cases to consider:

- if $x \in K$ then $|f(\alpha(g, x))) - f(x)| < 2\varepsilon$ by what we just did;

- if $x \notin K$ and $\alpha(g, x) \in K$ by symmetry of $\mathcal{N}$ $\left| f(\alpha(g^{-1}, \alpha(g, x)) - f(\alpha(g, x)) \right| < 2\varepsilon$ so $|f(x) - f(\alpha(g, x))| < 2\varepsilon$;

- if $x \notin K$ and $\alpha(g, x) \notin K$, then we both have $|f(x)| < \epsilon$ and $|f(\alpha(g, x))| < \varepsilon$, so again $|f(x) - f(\alpha(g, x))| < 2\varepsilon$.

We conclude that $\|f - f \circ \alpha(g, \cdot)\|_\infty < 2\varepsilon$ for all $g \in \mathcal{N}$, which finishes the proof that f is G-continuous. $\qquad\square$

3.2 Continuous Radon models for infinite measure-preserving boolean actions

Following the terminology of Glasner-Tsirelson-Weiss, a **continuous spatial model** for a boolean measure-preserving action α of a Polish group G on (X, λ) is a continuous G-action on a Polish measured space (Y, η) which is booleanly isomorphic to α. Moreover, if Y is a locally compact Polish space and η is Radon, we call the G-action on (Y, η) a **continuous Radon model**. We can now state and prove our analogue of the Glasner-Tsirelson-Weiss result in the context of possibly infinite measures.

Theorem 3.6. *Let G be a Polish group, and let α be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The following are equivalent:*

(1) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda),$$

(2) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^1(X, \lambda)}^{\|\cdot\|_1} = \mathrm{L}^1(X, \lambda),$$

(3) the action α admits a continuous Radon model.

Proof. In the whole argument, we systematically view boolean G-actions as continuous actions by integral preserving $*$-automorphisms on L^∞, as per Proposition 2.13 and 2.15. We also recall Proposition 2.16, which allows us to understand boolean isomorphism of action at the level of L^∞.

By Proposition 2.28, (1) and (2) are equivalent, since Proposition 3.4 ensures that $\mathcal{G}$ is a C^*-subalgebra of $\mathrm{L}^\infty(X, \lambda)$.

The implication (3)$\Rightarrow$(1) is a consequence of the fact that if β is the action on the continuous Radon model (Y, η) as in (3), then viewed as an action on $\mathrm{L}^\infty(Y, \eta)$, the restriction of β to $C_0(Y)$ is continuous. Since $C_0(Y)$ satisfies $\overline{C_0(Y) \cap \mathrm{L}^2(Y, \eta)}^{\|\cdot\|_2} \subseteq \mathrm{L}^\infty(Y, \eta)$, we obtain through the boolean isomorphism between α and β that $\mathcal{G}$ satisfies the desired density condition $\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda)$.

The converse (1)$\Rightarrow$(3) requires more work, as in [GTW05]. We fix a boolean measure-preserving G-action α, viewed as a continuous G-action by $*$-automorphisms on $\mathrm{L}^\infty(X, \lambda)$ which preserves the integral. We assume that its space $\mathcal{G}$ of G-continuous functions (which is a C^*-algebra by Corollary 3.4) satisfies condition (1).

Step 1. Choosing a suitable separable α-invariant C^*-subalgebra $\mathcal{A} \subseteq \mathcal{G}$. By Proposition 2.28, the set of finite measure-supported elements of $\mathcal{G}$ is $\|\cdot\|_2$ dense in $\mathcal{G}$. Since $\mathrm{L}^2(X, \lambda)$ is separable for the L^2 norm, we may fix a countable subset $\mathcal{D} \subseteq \mathcal{G}$ consisting of functions whose support has finite measure, such that $\mathcal{D}$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X, \lambda)$.

Let $\mathcal{E} = \alpha(G)\mathcal{D}$, then we claim that $\mathcal{E}$ is $\|\cdot\|_\infty$-separable: if Γ is a countable dense subset of G then by G-continuity the countable set $\alpha(\Gamma)\mathcal{D}$ is dense in $\mathcal{E}$. If we finally let $\mathcal{A}$ denote the C^*-algebra generated by $\mathcal{E}$, then $\mathcal{A}$ is still separable: the countable set of finite $\mathbb{Q}[i]$-linear combinations of finite products of elements of $\alpha(\Gamma)\mathcal{D} \cup (\alpha(\Gamma)\mathcal{D})^*$ is dense therein. By construction $\mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{A}$ is $\|\cdot\|_\infty$ separable. Moreover, it contains a $\|\cdot\|_2$ dense subset $\mathcal{D}$ of $\mathrm{L}^2(X, \lambda)$ consisting of functions whose support has finite measure.

Step 2. Building the space and the measure. By the theorem of Gelfand-Naimark (Theorem 2.26), the map

$$\rho \;:\; \mathcal{A} \;\longrightarrow\; C_0(\mathrm{Sp}(\mathcal{A}))$$

given by $\rho(a)(\chi) = \chi(a)$ for any χ in $\mathrm{Sp}(\mathcal{A})$, is an isometric $*$-isomorphism. Let $Y := \mathrm{Sp}\,\mathcal{A}$, then since $\mathcal{A}$ is separable we obtain from Proposition 2.25 that Y is locally compact Polish.

Now let

$$\mathcal{I} := \{a \in \mathcal{A} \colon \lambda(\mathrm{supp}\, a) < +\infty\}.$$

Then $\mathcal{I}$ is and ideal of $\mathcal{A}$ which is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X, \lambda)$ since its contains $\mathcal{D}$.

Functions in $\mathcal{I}$ are essentially bounded and have supports of finite measure, therefore they are integrable, so the measure λ defines a positive linear functional

$$
\begin{array}{rccc}
\Psi_\lambda & : & \mathcal{I} & \longrightarrow & \mathbb{C} \\
 & & f & \longmapsto & \int_X f d\lambda.
\end{array}
$$

The linear functional Ψ_λ can be transported through the Gelfand isomorphism, yielding the positive linear functional $\Psi_\lambda \circ \rho^{-1} : \rho(\mathcal{I}) \to \mathbb{C}$. Proposition 2.27 ensures that $C_c(Y) \subseteq \rho(\mathcal{I})$, where $C_c(Y)$ denotes the space of compactly supported continuous functions on Y. We then restrict $\Psi_\lambda \circ \rho^{-1}$ to $C_c(Y)$, and by Theorem 2.23 we obtain a unique Radon measure η on Y such that for all $f \in C_c(Y)$,

$$\int_Y f d\eta = \int_X \rho^{-1}(f) d\lambda.$$

Step 3. Extending ρ to the whole $\mathrm{L}^\infty(X, \lambda)$. We will now extend the definition of ρ to functions in $\mathrm{L}^\infty(X, \lambda)$. The following diagram summarises the situation and the notations that we will use throughout the rest of the proof.

$$
\begin{array}{ccc}
\mathrm{L}^2(X, \lambda) & \xrightarrow{\;\;\widetilde{\rho}\;\;} & \mathrm{L}^2(Y, \eta) \\
\big\uparrow{\scriptstyle \|\cdot\|_2\text{-cls}} & \scriptstyle \rho_{|\mathcal{I}} & \big\uparrow{\scriptstyle \|\cdot\|_2\text{-cls}} \\
\mathcal{I} & \xrightarrow{\hspace{1.5cm}} & \rho(\mathcal{I}) \\
\big\downarrow{\scriptstyle \|\cdot\|_\infty\text{-cls}} & & \big\downarrow{\scriptstyle \|\cdot\|_\infty\text{-cls}} \\
\alpha \;\circlearrowright\; \mathcal{A} & \xrightarrow{\;\;\rho\;\;} & C_0(Y) \;\circlearrowleft\; \beta \\
\big\downarrow{\scriptstyle \text{str-cls}} & & \big\downarrow{\scriptstyle \text{str-cls}} \\
\mathrm{L}^\infty(X, \lambda) & \xrightarrow{\;\;\overline{\rho}\;\;} & \mathrm{L}^\infty(Y, \eta) \\
\big\downarrow{\scriptstyle M} & & \big\downarrow{\scriptstyle M} \\
\mathcal{B}(\mathrm{L}^2(X, \lambda)) & \xrightarrow{\;\;\widehat{\rho}\;\;} & \mathcal{B}(\mathrm{L}^2(Y, \eta))
\end{array}
$$

By definition of η, for any f in $\mathcal{I}$ we have $\int_X f d\lambda = \int_Y \rho(f) d\eta$. In other words, $\rho_{|\mathcal{I}}$ preserves integrals, hence it takes the inner product of $\mathrm{L}^2(X, \lambda)$ to that of $\mathrm{L}^2(Y, \eta)$. Therefore ρ induces a surjective isometry $\widetilde{\rho}$ between the $\|\cdot\|_2$-closures of $\mathcal{I}$ and $\rho(\mathcal{I})$, which are $\mathrm{L}^2(X, \lambda)$ and $\mathrm{L}^2(Y, \eta)$ respectively, by density of $\mathcal{I}$ (this is where the fact that $\mathcal{D} \subseteq \mathcal{I}$ is crucially used) and by Proposition 2.30.

We then have a natural $*$-isomorphism $\widehat{\rho} : \mathcal{B}(\mathrm{L}^2(X, \lambda)) \to \mathcal{B}(\mathrm{L}^2(Y, \eta))$ given by the conjugacy by $\widetilde{\rho}$, namely for all $f \in \mathcal{B}(\mathrm{L}^2(X, \lambda))$

$$\widehat{\rho}(f) = \widetilde{\rho} f \widetilde{\rho}^{-1}$$

Being the conjugacy by a surjective isometry, $\widehat{\rho}$ must take the strong topology on $\mathcal{B}(\mathrm{L}^2(X,\lambda))$ to the strong topology on $\mathcal{B}(\mathrm{L}^2(Y,\eta))$.

Recall that we defined M as the multiplication embedding $f \in \mathrm{L}^\infty(X,\lambda) \mapsto M_f \in \mathcal{B}(\mathrm{L}^2(X,\lambda))$. We use the same notation for the multiplication embedding $M : \mathrm{L}^\infty(Y,\eta) \to \mathcal{B}(\mathrm{L}^2(Y,\eta))$. For any functions a in $\mathcal{A}$ and ξ in $\mathcal{I}$, we have

$$\rho(M_a\xi) = \rho(a\xi) = \rho(a)\rho(\xi) = M_{\rho(a)}\rho(\xi).$$

By density of $\mathcal{I}$ and by continuity of the multiplication, this still holds if we take ξ to be in $\mathrm{L}^2(X,\lambda)$, *i.e.* for any a in $\mathcal{A}$ and any ξ in $\mathrm{L}^2(X,\lambda)$, we have

$$\widetilde{\rho}(M_a\xi) = M_{\rho(a)}\widetilde{\rho}(\xi).$$

This is equivalent to saying that $\widetilde{\rho}M_a\widetilde{\rho}^{-1} = M_{\rho(a)}$, for any a in $\mathcal{A}$, which can be reformulated as $\widehat{\rho}(M_a) = M_{\rho(a)}$ by definition of $\widehat{\rho}$. Since $\mathcal{A} \cap \mathrm{L}^2(X,\lambda)$ is dense in $\mathrm{L}^2(X,\lambda)$, by Proposition 2.28 the strong-closure of $M(\mathcal{A})$ in $\mathcal{B}(\mathrm{L}^2(X,\lambda))$ is equal to $M(\mathrm{L}^\infty(X,\lambda))$. Moreover, by Lemma 2.30 the strong closure of $M(\rho(\mathcal{A})) = M(C_0(Y))$ in $\mathcal{B}(\mathrm{L}^2(Y,\eta))$ is equal to $M(\mathrm{L}^\infty(Y,\eta))$.

Therefore the isomorphism $\widehat{\rho}$ restricts to an isomorphism of von Neumann algebras between $M(\mathrm{L}^\infty(X,\lambda))$ and $M(\mathrm{L}^\infty(Y,\eta))$ sending the measure λ to η. As the homomorphisms M are isomorphisms on their images, defining $\overline{\rho}$ by

$$\overline{\rho}(f) = M^{-1}\widehat{\rho}M(f)$$

for any f in $\mathrm{L}^\infty(X,\lambda)$ allows us to extend ρ to $\overline{\rho} : \mathrm{L}^\infty(X,\lambda) \to \mathrm{L}^\infty(Y,\eta)$.

Step 4: Defining β and showing that it is the action we want. We now show that G acts on (Y,η) in a spatial manner, and that this action is booleanly isomorphic to the original boolean G-action on (X,λ).

Since $\overline{\rho}$ is an isomorphism $\mathrm{L}^\infty(X,\lambda) \to \mathrm{L}^\infty(Y,\eta)$, we already have our candidate boolean action β on $\mathrm{L}^\infty(Y,\eta)$, uniquely defined by letting, for all $f \in \mathrm{L}^\infty(X,\lambda)$,

$$\beta(g,\overline{\rho}(f)) = \overline{\rho}(\alpha(g,f)).$$

Now G acts on $\mathcal{A}$ via α, yielding a spatial action $\hat{\alpha}$ on $Y = \mathrm{Sp}\,\mathcal{A}$ defined by: for any $\chi \in Y$ and any $a \in \mathcal{A}$, $\hat{\alpha}(g,\chi)(a) = \chi(\alpha(g^{-1},a))$.

Since $\overline{\rho}$ extends ρ, we have in particular that for every $a \in \mathcal{A}$, $\beta(g,\rho(a)) = \rho(\alpha(g,a))$. So $\beta(g,\rho(a))(\chi) = \chi(\alpha(g,a))$ for any χ in Y. We can rewrite this as:

$$\beta(g,\rho(a))(\chi) = \chi(\alpha(g,a)) = \hat{\alpha}(g^{-1},\chi)(a) = \rho(a)(\hat{\alpha}(g^{-1},\chi))$$

In other words, when restricted to $\rho(\mathcal{A}) = C_0(Y)$, the action β coincides with the precomposition by the inverse associated to the spatial action $\hat{\alpha}$. By strong density of $M(C_0(Y))$ in $M(\mathrm{L}^\infty(Y,\eta))$ and strong continuity of $\overline{\rho}$, the same conclusion holds on $\mathrm{L}^\infty(Y,\eta)$, namely β coincides with the precomposition by the inverse associated to the spatial action $\hat{\alpha}$.

In other words β is the boolean action associated to $\hat{\alpha}$, and since β was integral preserving we have that $\hat{\alpha}$ is measure-preserving (this could also be checked directly using the Riesz-Markov-Kakutani theorem).

We finally need to check is that $\hat{\alpha}$ is continuous, using the fact that $\mathcal{A}$ consists solely of G-continuous functions. The topology on $\mathrm{Sp}(\mathcal{A})$ is the topology of pointwise convergence,

so we fix a in $\mathcal{A}$. Let (g_n) be a sequence of elements of G converging to e_G and let (χ_n) be converging to χ in $Y = \operatorname{Sp}\mathcal{A}$. We have

$$
\begin{aligned}
|\hat{\alpha}(g_n, \chi_n)(a) - \chi(a)| &= |\hat{a}(\hat{\alpha}(g_n, \chi_n)) - \chi(a)| \\
&\leqslant |\hat{a}(\hat{\alpha}(g_n, \chi_n)) - \hat{a}(\chi_n)| + |\hat{a}(\chi_n) - \hat{a}(\chi)| \\
&\leqslant \left\| \widehat{\alpha(g_n^{-1}, a)} - \hat{a} \right\|_{C_0(Y)} + |\chi_n(a) - \chi(a)| \\
&\leqslant \left\| \alpha(g_n^{-1}, a) - a \right\|_\infty + |\chi_n(a) - \chi(a)| .
\end{aligned}
$$

By convergence of (χ_n) the second term tends to zero as n grows, and the first term tends to zero as n grows by G-continuity of a. Therefore the G-action on Y is continuous as wanted. $\qquad\square$

4 Applications and examples

4.1 Construction of G-continuous functions when G is locally compact

In the whole section, we work with a fixed locally compact Polish group G, and denote by m its Haar measure. We start by recalling the definition of convolution between an essentially bounded function on a measure space and a function that is integrable with regards to the Haar measure of the group acting on said space.

Definition 4.1. Consider a boolean measure-preserving G-action α on (X, λ) as in Definition 2.2. Let $f \in \mathrm{L}^\infty(X, \lambda)$ and $\delta \in \mathrm{L}^1(G, m)$. For any x in X, the **convolution product** $\delta * f : X \to \mathbb{C}$ is defined by: for all $x \in X$,

$$
\delta * f(x) = \int_G \delta(g) f(\alpha(g^{-1}, x)) dm(g).
$$

Using for δ an approximation of the identity, we will see that G-continuous functions are always dense for boolean measure-preserving actions. We start by proving the following well-known lemma.

Lemma 4.2. *Consider a locally compact Polish group G, and a boolean measure-preserving action of G on a standard σ-finite space (X, λ). Let $\delta : G \to \mathbb{R}_+$ be continuous of integral 1 and compactly supported. Then, for any function f in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^1(X, \lambda)$, $\delta * f$ is in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^1(X, \lambda)$ with $\|\delta * f\|_1 \leqslant \|f\|_1$, and is G-continuous.*

Proof. We first check that $\delta * f$ is in $\mathrm{L}^1(X, \lambda)$, as it is clear that it is in $\mathrm{L}^\infty(X, \lambda)$. Using Fubini's theorem and the fact that δ takes nonnegative values we have

$$
\begin{aligned}
\int_X |\delta * f(x)| \, d\lambda(x) &= \int_X \left| \int_G \delta(g) f(g^{-1}x) dm(g) \right| d\lambda(x) \\
&\leqslant \int_X \int_G \left| \delta(g) f(g^{-1}x) \right| dm(g) d\lambda(x) \\
&= \int_G \delta(g) \int_X \left| f(g^{-1}x) \right| d\lambda(x) dm(g) \\
&= \|f\|_1 ,
\end{aligned}
$$

the last equality being a consequence of the fact that $\int_G \delta(g) dm(g) = 1$ and g preserves the measure. In particular we have $\delta * f \in \mathrm{L}^1(X, \lambda)$ and moreover $\|\delta * f\|_1 \leqslant \|f\|_1$.

We finally prove that it is G-continuous. Let us take $\varepsilon > 0$. For any h in G we have

$$\delta * f(hx) = \int_G \delta(g) f(g^{-1}hx) dm(g) = \int_G \delta(hg) f(g^{-1}x) dm(g).$$

Therefore, we have

$$|\delta * f(x) - \delta * f(hx)| \leqslant \int_G |\delta(g) - \delta(hg)| \left| f(g^{-1}x) \right| dm(g). \tag{1}$$

By Lemma 3.5 for the G-action on itself by left translation, we can fix a neighborhood $\mathcal{N}$ of e_G such that for all $h \in \mathcal{N}$ and $g \in G$, $|\delta(g) - \delta(hg)| < \varepsilon$. Also, note that if $K = \operatorname{supp}\delta$, then $\delta(g) - \delta(hg) = 0$ for all $g \notin K \cup h^{-1}K$, which has measure at most $2m(K)$ since m is a left Haar measure. It follows that

$$\int_G |\delta(g) - \delta(hg)| \, dm(g) < 2\varepsilon \times m(K)$$

Therefore we can control the left hand side of Equation (1) by the L^∞ norm of f:

$$|\delta * f(x) - \delta * f(hx)| < 2\varepsilon \times m(K) \times \|f\|_\infty$$

which concludes the proof. $\qquad\square$

Theorem 4.3. *Let G be a locally compact Polish group, and consider a boolean measure-preserving G-action on a standard σ-finite space (X, λ). Then the action admits a continuous spatial model on a locally compact Polish space (Y, η), where η is a Radon measure.*

Proof. We want to apply condition (2) from Theorem 3.6 in order to conclude.

Since bounded functions are dense in $L^1(X, \lambda)$, we fix some $f \in L^\infty(X, \lambda) \cap L^1(X, \lambda)$, some $\varepsilon > 0$, and our aim is to find a G-continuous function which is ε-close to f for $\|\cdot\|_1$.

For any $\delta \in L^1(G, m)$ compactly supported of integral 1 taking nonnegative values, we know from Lemma 4.2 that $\delta * f$ is in $L^\infty(X, \lambda) \cap L^1(X, \lambda)$ and is G-continuous. By Fubini's Theorem, we also have

$$
\begin{aligned}
\|f - \delta * f\|_1 &= \int_X \left| f(x) - \int_G \delta(g) f(\alpha(g^{-1}, x)) dm(g) \right| d\lambda(x) \\
&= \int_X \left| \int_G \delta(g) f(x) dm(g) - \int_G \delta(g) f(\alpha(g^{-1}, x)) dm(g) \right| d\lambda(x) \\
&\leqslant \int_X \int_G \delta(g) \left| f(x) - f(\alpha(g^{-1}, x)) \right| dm(g) d\lambda(x) \\
&= \int_G \delta(g) \int_X \left| f(x) - f(\alpha(g^{-1}, x)) \right| d\lambda(x) dm(g) \\
&= \int_G \delta(g) \| f - f \circ \alpha(g^{-1}, \cdot) \|_1 dm(g)
\end{aligned}
$$

But as noted in Section 2.3, the G-action on $L^1(X, \lambda)$ is continuous. Therefore there exists a neighborhood $\mathcal{N}_\varepsilon$ of e_G such that $\| f - f \circ \alpha(g^{-1}, \cdot) \|_1 < \varepsilon$ for any g in $\mathcal{N}_\varepsilon$. Taking now δ as above whose support is moreover contained in $\mathcal{N}_\varepsilon$, we obtain $\|f - \delta * f\|_1 < \varepsilon$, which concludes the proof. $\qquad\square$

Combining Mackey's theorem (Theorem 2.8), and Theorem 4.3, we obtain the following.

Theorem 4.4. *Let G be a locally compact Polish group and (X, λ) be a standard σ-finite space. Let also α be a measure-preserving G-action on (X, λ). Then α is spatially isomorphic to a continuous measure-preserving G-action on a locally compact Polish space Y endowed with a Radon measure η.*

Proof. By Theorem 4.3 the boolean action associated with α admits a Polish model β on (Y, η), where Y is locally compact and Polish, and η is a Radon measure. By Mackey's theorem (item (2) of Theorem 2.8), since α and β are booleanly isomorphic, they are in fact spatially isomorphic. $\square$

In section 5.2 we give another way to obtain Theorem 4.4 without relying on the use of Mackey's theorem. By working directly with measurable bounded functions and the genuine supremum norm instead of equivalence classes of functions up to measure zero, we explicitly construct the desired spatial isomorphism.

4.2 Actions of non-archimedean groups

We denote by $\mathfrak{S}_\infty$ the group of all permutations of $\mathbb{N}$. Recall that a **non-archimedean** group is a topological group admitting a basis of neighborhoods of the identity made of open subgroups. Any such Polish group is isomorphic to a closed subgroup of $\mathfrak{S}_\infty$ (see [BK96, Thm. 1.5.1]).

We give a similar result as the one obtained for locally compact Polish groups (Theorem 4.3), but this time for non-archimedean groups. The following proof is inspired by the one from [GW05, Thm. 2.3].

Proposition 4.5. *Let G be a Polish non-archimedean group, and consider α a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The set of G-continuous functions with regards to this action is dense in $\mathrm{L}^2(X, \lambda)$.*

Proof. Consider an element f of $\mathrm{L}^2(X, \lambda)$, and fix $\varepsilon > 0$. As G acts continuously on $\mathrm{L}^2(X, \lambda)$ and as G is non-archimedean, there exists an open subgroup H such that for any h in H we have $\left\| f - f \circ \alpha(h^{-1}, \cdot) \right\|_2 < \varepsilon$. Consider now

$$\mathcal{C} = \overline{\mathrm{Conv}\{(f \circ \alpha(h^{-1}, \cdot)) \mid h \in H\}}^{\|\cdot\|_2},$$

the L^2-closure of the convex hull of the orbit of f under the H-action. Note that $\mathcal{C}$ is contained it the closed ball centered in f and of radius ε.

By the Projection onto a closed convex set theorem (see e.g. [Rud87, Thm. 4.10.]), there is a unique element of minimal norm ξ in $\mathcal{C}$, satisfying in particular $\|f - \xi\|_2 \leqslant \varepsilon$. H acts by unitaries on $\mathcal{C}$, it preserves the norm and therefore ξ is fixed by the H-action. In particular, ξ is G-continuous, although it might be unbounded.

To conclude the proof, we need only remember that $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^2(X, \lambda)$ is dense in $\mathrm{L}^2(X, \lambda)$. If ξ is not in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^2(X, \lambda)$, we approach it with a function $\widehat{\xi}$ in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^2(X, \lambda)$, ε-close to ξ for $\|\cdot\|_2$. As ξ is fixed by H, it is G-continuous, so $\widehat{\xi}$ is also G-continuous. The function $\widehat{\xi}$ is the one we sought. $\square$

Theorem 4.6. *Let G be a Polish non-archimedean group, and consider a boolean measure-preserving G-action on a standard σ-finite space (X, λ). Then the action admits a continuous Radon model.*

Proof. By Proposition 4.5, the G-continuous functions are dense in $\mathrm{L}^2(X, \lambda)$, and we can then apply Theorem 3.6 to conclude the proof. $\square$

By this theorem it is always possible to find a continuous Radon model for a boolean measure-preserving action of $\mathfrak{S}_\infty$. However, it is not always possible to turn the boolean isomorphism into a spatial one. Section 5.3 gives a counter-example.

Remark 4.7. It is a result of Nessonov that for every boolean *non-singular* $\mathfrak{S}_\infty$-action α on (X, λ), there is an equivalent σ-finite measure $\lambda' \in [\lambda]$ such that α becomes measure-preserving [Nes20, Thm. 1.1]. So our result yields that every boolean non-singular $\mathfrak{S}_\infty$-action admits a continuous Radon model.

4.3 An ergodic boolean action with two distincts spatial models

In this section we present an example of an ergodic boolean action with two distincts spatial models on probability spaces, which was explained to us by Todor Tsankov. The action is that of $\mathfrak{S}_\infty$ on the countable product of the unit interval endowed with the Lebesgue measure $\left([0,1]^\mathbb{N}, \lambda^{\otimes\mathbb{N}}\right)$. The first spatial action α is the natural action of $\mathfrak{S}_\infty$ by permutation of the coordinates. For the second action β, we first define the space $\mathrm{LO}(\mathbb{N})$ of linear orderings of $\mathbb{N}$:

$$\mathrm{LO}(\mathbb{N}) = \left\{x \subseteq \mathbb{N}^2 \mid x \text{ is a linear order on } \mathbb{N}\right\},$$

and for any element x (that we will denote in a more usual way by $n <_x m \iff (n, m) \in x$) of $\mathrm{LO}(\mathbb{N})$, the $\mathfrak{S}_\infty$-action β is defined by

$$(n, m) \in \beta(\sigma, x) \iff n(\sigma \cdot <_x)m \iff \sigma^{-1}(n) <_x \sigma^{-1}(m)$$

for any σ in $\mathfrak{S}_\infty$. We define on $\mathrm{LO}(\mathbb{N})$ the uniform measure μ_u by

$$\mu_u\left([k_1 <_x k_2 <_x \ldots <_x k_n]\right) = \frac{1}{n!}$$

for any distinct $k_1, \ldots, k_n$ in $\mathbb{N}$, where

$$[k_1 <_x k_2 <_x \ldots <_x k_n] = \{x \in \mathrm{LO}(\mathbb{N}) \mid k_1 <_x k_2 <_x \ldots <_x k_n\}.$$

This probability measure is invariant under the $\mathfrak{S}_\infty$-action β, and do note that it is the only such measure by construction. Do recall also that, as $\mathrm{LO}(\mathbb{N})$ is a compact space as a closed subset of $\{0,1\}^{\mathbb{N}^2}$, $\mathfrak{S}_\infty$ is not extremely amenable. Indeed, $\mathfrak{S}_\infty$ contains transpositions, and in particular it cannot preserve a linear ordering of $\mathbb{N}$, hence the previous assertion.

The correspondence between the two actions is defined as follows:

$$\begin{array}{rccc} \Phi & : & [0,1]^\mathbb{N} & \longrightarrow & \mathrm{LO}(\mathbb{N}) \\ & & (x_i)_{i\in\mathbb{N}} & \longmapsto & <_x, \end{array}$$

where $<_x$ is defined by $n <_x m \iff x_n < x_m$ (the order $<$ on the right-hand side being the natural order on $[0,1]$). Note that for $\lambda^{\otimes\mathbb{N}}$-almost all x in $[0,1]^\mathbb{N}$, the x_n are distinct, so Φ is well defined. We then show that the images by Φ of two distinct elements are isomorphic as orderings of $\mathbb{N}$.

Definition 4.8. An order $<$ on a set A is **dense on** A if for any two elements n, m in A such that $n < m$, there exists a third element k in A such that $n < k < m$.

Definition 4.9. An order $<$ on a set A is **unbounded on** A if for any element k in A, there exists two elements n and m in A such that $n < k < m$.

Proposition 4.10. *An order $<$ in $\mathrm{LO}(\mathbb{N})$ is dense and unbounded on $\mathbb{N}$, μ_u-a.e.*

Proof. Fix n, m and k in $\mathbb{N}$. We have

$$\mathbb{P}_{\mu_u}\left([n < k < m] \mid [n < m]\right) = \frac{1}{3},$$

therefore by Borel-Cantelli's Lemma, as n and m vary, $n < k < m$ happens infinitely often, and thus $<$ is unbounded μ_u-a.e. The same argument used with k varying instead also ensures that $<$ is dense μ_u-a.e. which concludes the proof. $\qquad\square$

The following back-and-forth argument is well-known and classical, but we give it for the sake of completeness.

Proposition 4.11. *Any two dense and unbounded orders on infinite countable sets are isomorphic, i.e. there exists an order preserving bijection between the respective spaces.*

Proof. Let $A = \{a_i \mid i \in \mathbb{N}\}$ and $B = \{b_i \mid i \in \mathbb{N}\}$ be two countable infinite sets, and let $<_A$ and $<_B$ be two dense, unbounded linear orderings on A and B respectively. We inductively construct an embedding $\varphi : (A, <_A) \hookrightarrow (B, <_B)$.
We arbitrarily send a_0 to b_0. We then send a_1 to $\varphi(a_1)$ such that $(b_0, \varphi(a_1))$ is ordered like (a_0, a_1). At rank n, a_n is sent to $\varphi(a_n)$, an element that respects the order of $(a_0, \dots, a_n)$, which necessarily exists because $<_B$ is dense and unbounded.
In other words, for any finite subset F of $(A, <_A)$, and for any a in $A \setminus F$, it is possible at any rank to extend the existing embedding $F \hookrightarrow (B, <_B)$ into $F \cup \{a\} \hookrightarrow (B, <_B)$. By doing the same construction with $(B, <_B)$, and given that $<_B$ is also dense and unbounded, it is possible to define by induction

$$\varphi_n : F_n \subseteq (A, <_A) \hookrightarrow (B, <_B)$$

such that F_n is a finite subset of A, and such that for any n we have

$$\begin{cases} \{a_0, \dots, a_n\} \subseteq F_{2n} \\ \{b_0, \dots, b_n\} \subseteq \varphi_{2n+1}(F_{2n+1}) \end{cases}$$

The limit function of the φ_n is the order-preserving bijection we sought. $\qquad\square$

Proposition 4.11 ensures that μ_u-a.e. there is only one $\mathfrak{S}_\infty$-orbit on $\mathrm{LO}(\mathbb{N})$. On the other hand, the natural $\mathfrak{S}_\infty$-action on $\left([0,1]^{\mathbb{N}}, \lambda^{\otimes \mathbb{N}}\right)$ only admits orbits of measure 0. We then have the following proposition.

Proposition 4.12. *The application $\Phi : \left([0,1]^{\mathbb{N}}, \lambda^{\otimes \mathbb{N}}\right) \longrightarrow (\mathrm{LO}(\mathbb{N}), \mu_u)$, sending (x_i) to $<_x$, where $<_x$ is defined by $n <_x m \iff x_n < x_m$, is $\mathfrak{S}_\infty$-equivariant, measure-preserving and $\lambda^{\otimes \mathbb{N}}$-a.e. injective.*

Proof. Checking the $\mathfrak{S}_\infty$-equivariance is straightforward, and so is checking the fact that Φ is measure-preserving. For the $\lambda^{\otimes \mathbb{N}}$-a.e. injectivity we first fix q in $\mathbb{Q} \cap [0,1]$. The Law of Large Numbers ensures that for $\lambda^{\otimes \mathbb{N}}$-almost any sequence (x_i) we have

$$\lim_{n \to +\infty} \frac{|\{i \in \{0, \dots, n-1\} : x_i < q\}|}{n} = q. \tag{2}$$

Taking a countable intersection of conull sets, we can therefore find a conull subset X_0 of $[0,1]^{\mathbb{N}}$ such that whenever (x_i) is in X_0, Equation (2) holds for any q in $\mathbb{Q} \cap [0,1]$. A

straightforward density argument now shows that Equation (2) holds for any q in $[0, 1]$ and any $(x_i) \in X_0$.

Let us conclude by showing that $\Phi_{\restriction X_0}$ is injective: take distinct elements $x = (x_i)$ and $y = (y_i)$ in X_0, there is an index k such that $x_k \neq y_k$. Then taking $q = x_k$ and $q = y_k$ for (x_i) and (y_i) respectively in (2), we obtain that

$$\lim_{n \to +\infty} \frac{|\{i \in \{0, \dots, n-1\} \colon x_i < x_k\}|}{n} \neq \lim_{n \to +\infty} \frac{|\{i \in \{0, \dots, n-1\} \colon y_i < y_k\}|}{n}.$$

This implies that $<_x \neq <_y$, as their respective proportions of integers smaller than the integer k are different. $\qquad\square$

The previous proposition gives an example of a negative answer to the boolean to spatial isomorphism problem for the case of the non-locally compact group $\mathfrak{S}_\infty$, for actions on probability spaces:

Proposition 4.13. *Let $G = \mathfrak{S}_\infty$. The G-action α on $\left([0,1]^{\mathbb{N}}, \lambda^{\otimes \mathbb{N}}\right)$ and the G-action β on $(\mathrm{LO}(\mathbb{N}), \mu_u)$ are booleanly isomorphic, but not spatially isomorphic.*

Proof. We define Φ as in Proposition 4.12. This application is (well-defined and) injective on a conull subset $X_0 \subseteq [0,1]^{\mathbb{N}}$. As it is also G-equivariant, setting $X_g = X_0 \cap g \cdot X_0$ for any g in G defines a boolean isomorphism between α and β. They are not spatially isomorphic however, since α only has orbits of measure 0, but Proposition 4.11 ensures that β has a conull orbit. $\qquad\square$

5   Spatial actions and continuous Radon models

In this section, we are interested in a related question on Borel G-actions on infinite measured spaces: given such an action, it is natural to ask whether it can be embedded G-equivariantly into a continuous G-action on a locally compact Polish space so that the pushforward measure is a Radon measure. We will first give an abstract characterization of this, then show that it is always the case for locally compact groups, and give a simple example where this fails for $G = \mathfrak{S}_\infty$.

5.1   Characterization

In this section, we will work in the space $\mathcal{L}^\infty(X)$ which we define as the C^*algebra of all bounded Borel functions $X \to \mathbb{C}$, endowed with the norm of uniform convergence $\|f\|_\infty = \sup_{x \in X} |f(x)|$ (no essential supremum here!). Observe that if G acts on X, the notion of G-continuous element of $\mathcal{L}^\infty(X)$ still makes sense for the norm $\|\cdot\|_\infty$ that we just defined. We also denote by $\mathcal{L}^1(X, \lambda)$ the space of integrable complex-valued functions on X.

Theorem 5.1. *Let G be a Polish group, suppose $\alpha : G \curvearrowright (X, \lambda)$ is a Borel action by measure-preserving bijections on a standard σ-finite space (X, λ). The following are equivalent:*

(i) *There is a continuous G-action β on a Polish locally compact space Y and a Borel injective map $\Phi : X \to Y$ such that $\Phi_* \lambda$ is Radon and $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$ for all $x \in X$ and all $g \in G$;*

(ii) *The algebra of G-continuous functions in $\mathcal{L}^\infty(X, \lambda) \cap \mathcal{L}^1(X)$ contains a countable subset which separates points.*

Proof. Let us begin by the easier implication (i)⟹(ii), and assume we have $\alpha : G \times X \to X$ Borel preserving a σ-finite measure λ, $\beta : G \times Y \to Y$ continuous and $\Phi : X \to Y$ injective such that for all $x \in X$, $\Phi(\alpha(g,x)) = \beta(g, \Phi(x))$.

By compactness, every compactly supported continuous function on Y is G-continuous with respect to the action β. Since Y is locally compact second-countable, it admits a compatible proper metric d. Let $\{y_n : n \in \mathbb{N}\}$ be a countable dense subset of Y, and define for each $n \in \mathbb{N}$ a continuous compactly supported function f_n by

$$f_n(x) = d(y_n, x)$$

The density of (y_n) implies that this family of G-continuous functions separates the points of Y. It then follows from the injectivity of Φ that $(f_n \circ \Phi)_{n \in \mathbb{N}}$ separates the points of X, and by the equivariance of Φ it consists of G-continuous functions with respect to α as wanted.

We now prove the converse implication (ii)⟹(i). Let us fix a countable set of functions $\mathcal{D}$ that separates points. We then consider Γ a countable dense subgroup of G and give a similar argument to the one in Step 1 of the proof of Theorem 3.6. By G-continuity the $\|\cdot\|_\infty$-closure of $\mathcal{E}_\Gamma := \{f \circ \alpha(\Gamma, \cdot) \mid f \in \mathcal{D}\}$ is equal to $\mathcal{E}_G := \{f \circ \alpha(G, \cdot) \mid f \in \mathcal{D}\}$. We denote by $\mathcal{A}$ the C^*-algebra generated by $\mathcal{E}_G$. The countable set of finite $\mathbb{Q}[i]$-linear combinations of finite products of elements of $\mathcal{E}_\Gamma \cup (\mathcal{E}_\Gamma)^*$ is dense in $\mathcal{A}$. By construction $\mathcal{A}$ is then separable, as the $\|\cdot\|_\infty$-closure of a countable set (of integrable functions). Moreover every function in $\mathcal{A}$ is G-continuous and $\mathcal{A}$ separates the points of X.

Step 1. Building the space and the measure. By theorem 2.26, we have the following isometric $*$-isomorphism:

$$\rho \ : \ \begin{array}{ccc} \mathcal{A} & \longrightarrow & C_0(\mathrm{Sp}(\mathcal{A})) \\ a & \longmapsto & \widehat{a} \end{array}$$

with $\widehat{a}$ being the evaluation on a. By separability of $\mathcal{A}$ and Proposition 2.25, the space $Y := \mathrm{Sp}(\mathcal{A})$ is locally compact Polish.

This time, we consider

$$\mathcal{I} = \mathcal{A} \cap \mathcal{L}^1(X, \lambda).$$

It is immediate to check that $\mathcal{I}$ is an ideal of $\mathcal{A}$, and it contains $\mathcal{D}$ by construction. Since $\mathcal{I}$ consists of integrable functions, λ defines a positive linear functional

$$\Psi_\lambda \ : \ \begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \int_X f d\lambda. \end{array}$$

We then use ρ to get a positive linear functional $\Psi_\lambda \circ \rho^{-1} : \rho(\mathcal{I}) \to \mathbb{C}$. The ideal $\mathcal{I}$ is $\|\cdot\|_\infty$-dense in $\mathcal{A}$, so by Proposition 2.27 $C_c(Y) \subseteq \rho(I)$, where $C_c(Y)$ denotes the space of compactly supported continuous functions on Y. By Theorem 2.23, restricting $\Psi_\lambda \circ \rho^{-1}$ to $C_c(Y)$ gives a unique Radon measure η on Y such that for all $f \in C_c(Y)$,

$$\int_Y f d\eta = \int_X \rho^{-1}(f) d\lambda. \tag{3}$$

Step 2. Borel embedding of X into Y. Consider the following map

$$\Phi \ : \ \begin{array}{ccc} X & \longrightarrow & Y \\ x & \longmapsto & \mathrm{ev}_x \end{array}$$

where $\mathrm{ev}_x(a) = a(x)$. Since the algebra $\mathcal{A}$ separates the points of X, this map is injective. Let us prove that it is also Borel. Sets of the form

$$\{\mathrm{ev}_x \colon |\mathrm{ev}_x(a) - \mathrm{ev}_{x_0}(a)| < \varepsilon\},$$

where $x_0 \in X$, $a \in \mathcal{A}$ and $\varepsilon > 0$ form a subbasis for the topology of pointwise convergence on $\mathrm{Im}(\Phi)$. The preimage by Φ of such a set is

$$\{x \in X \colon |a(x) - a(x_0)| < \varepsilon\}$$

which is Borel since a is a Borel map. Therefore Φ is Borel and injective.

We define the G-action $\widetilde{\alpha}$ on $\mathcal{L}^{\infty}(X)$, which is the precomposition by the inverse: $\widetilde{\alpha}(g, a)(x) = a(\alpha(g^{-1}, x))$. This allow us to define the desired G-action β on on $Y = \mathrm{Sp}(\mathcal{A})$ by

$$\beta(g, y)(a) := y(\widetilde{\alpha}(g^{-1}, a)).$$

We easily check that it is a left action on Y: for any character $y \in Y$ and any two elements g and h in G we have

$$\begin{aligned}
\beta(g, \beta(h, y))(a) &= \beta(h, y)(\widetilde{\alpha}(g^{-1}, a)) \\
&= y(\widetilde{\alpha}(h^{-1}, \widetilde{\alpha}(g^{-1}, a))) \\
&= y(\widetilde{\alpha}((gh)^{-1}, a)) \\
&= \beta(gh, y)(a).
\end{aligned}$$

Moreover, for $y = \mathrm{ev}_x$ we have $\beta(g, y)(a) = \mathrm{ev}_x(\widetilde{\alpha}(g^{-1}, a)) = a(\alpha(g, x)) = \mathrm{ev}_{\alpha(g,x)}(a)$, that is to say that, Φ is G-equivariant: $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$.

In order to conclude, we have to check that β is continuous and preserves η. The fact that it is measure-preserving is a direct consequence of the uniqueness in Theorem 2.23. Indeed, $\mathcal{I}$ is G-invariant and for any function f in $\mathcal{I}$,

$$\int_X \widetilde{\alpha}(g, f)(x) d\lambda(x) = \int_X f(\alpha(g^{-1}, x)) d\lambda(x) = \int_X f(x) d\lambda(x).$$

This means that G preserves the integral of elements of $\mathcal{I}$, and by (3) this implies that it preserves the integral of elements of $C_c(Y)$ (with regards to η). By uniqueness, the pushforward of η by the action of any element of G is equal to η. The G-action β is therefore measure-preserving.

Finally, the continuity of β is obtained in the exact same way as in Step 4 of the proof of Theorem 3.6. $\qquad\square$

Remark 5.2. It is tempting to try to add the following third condition to the above theorem:

(iii) *There is a continuous G-action on a Polish space Y and a Borel injective map $\Phi :$ $X \to Y$ such that $\Phi_* \lambda$ is locally finite and $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$ for all $x \in X$ and all $g \in G$.*

Clearly *(i)* implies *(iii)*, but we don't know if the converse is true, although we suspect it is not. As we will see in section 5.3, the only examples of actions not satisfying *(i)* that we have actually fail *(iii)*.

5.2 The case of locally compact groups

As in the previous section, we use the space $\mathcal{L}^\infty(X)$ of of all bounded Borel functions $X \to \mathbb{C}$, endowed with the norm of uniform convergence $\|f\|_\infty = \sup_{x \in X} |f(x)|$. We also use convolution as defined in Section 4.1, noting that the proof of Lemma 4.2 yields the following statement for everywhere defined functions.

Lemma 5.3. *Consider a locally compact Polish group G, and a spatial measure-preserving action of G on a standard σ-finite space (X, λ). Let $\delta : G \to \mathbb{R}_+$ be continuous of integral 1 and compactly supported. Then, for any $f \in \mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$, the function $\delta * f$ is in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ and is G-continuous.* $\square$

We can now prove Theorem D after first recalling its statement.

Theorem 5.4. *Let G be a locally compact Polish group, and denote by α a spatial measure-preserving G-action on a standard σ-finite space (X, λ). Then there exists a locally compact Polish space Y endowed with a continuous action β and a Borel injection $\theta : X \to Y$ such that the pushforward measure $\theta_* \lambda$ is Radon and for all $x \in X$ and all $g \in G$,*

$$\theta(\alpha(g, x)) = \beta(g, \theta(x)).$$

Proof. By Item (1) of Theorem 1.1 we may as well assume that X is compact and α is continuous (this is also a consequence of an earlier and easier result of Varadarajan, see [Var63, Thm. 3.2]). We also denote by d a compatible metric on X and fix a sequence (x_n) enumerating a dense subset of X. By Theorem 5.1, it is sufficient to prove that there exists a countable set $\mathcal{D}$ of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ that separates points. We will construct this set $\mathcal{D}$ using convolution with regard to the Haar measure m on G.

We begin by choosing for every $\varepsilon > 0$ an open neighborhood $\mathcal{N}_\varepsilon$ of e_G in G such that for all $g \in \mathcal{N}_\varepsilon$ and all $x \in X$ we have $d(\alpha(g^{-1}, x), x) < \varepsilon$. Such a neighborhood exists by continuity of α and compactness of X. We then fix a continuous function $\delta_\varepsilon : G \to \mathbb{R}_+$ of integral 1 with a compact support included in $\mathcal{N}_\varepsilon$.

Let us fix $n \in \mathbb{N}$ and $\varepsilon > 0$, our first aim is to define countably many integrable functions separating the points of the open ball $B(x_n, \varepsilon)$ from the points in $X \setminus B(x_n, 3\varepsilon)$. By the choice of δ_ε, for any $x \in B(x_n, \varepsilon)$ and any $y \in X \setminus B(x_n, 3\varepsilon)$, we have

$$\begin{cases} \delta_\varepsilon * \mathbb{1}_{B(x_n, 2\varepsilon)}(x) = \int_G \delta_\varepsilon(g) \mathbb{1}_{B(x_n, 2\varepsilon)}(g^{-1}x) dm(g) = \int_G \delta_\varepsilon(g) dm(g) = 1 \\ \delta_\varepsilon * \mathbb{1}_{B(x_n, 2\varepsilon)}(y) = \int_G \delta_\varepsilon(g) \mathbb{1}_{B(x_n, 2\epsilon)}(g^{-1}y) dm(g) = 0 \end{cases} \tag{4}$$

However the above function $\delta_\varepsilon * \mathbb{1}_{B(x_n, 2\varepsilon)}$ could very well fail to be integrable. We therefore set $X = \sqcup_{k \in \mathbb{N}} X_k$, with $\lambda(X_k) = 1$ for any k, and define

$$A_{n,k,\varepsilon} = X_k \cap B(x_n, 2\varepsilon),$$

so that $\mathbb{1}_{A_{n,k,\varepsilon}} \in \mathcal{L}^1(X, \lambda)$, for any positive integer k, and hence $\delta_\varepsilon * \mathbb{1}_{A_{n,k,\varepsilon}} \in \mathcal{L}^1(X, \lambda)$ for all $\varepsilon > 0$ by Lemma 5.3.

In order to obtain a countable set of functions, we enumerate $\mathbb{Q}_{>0}$ as $\mathbb{Q}_{>0} = \{\varepsilon_i : i \in \mathbb{N}\}$. We can finally define our countable set of functions:

$$\mathcal{D} := \left\{ \delta_{\varepsilon_i} * \left(\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \right) : i, n, N \in \mathbb{N} \right\}.$$

By Lemma 5.3 we have $\mathcal{D} \subseteq \mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ and $\mathcal{D}$ only contains G-continuous functions.

We now prove that $\mathcal{D}$ separates the points of X. To this end, let x and y be in X such that for all $f \in \mathcal{D}$ we have $f(x) = f(y)$. Fix i and n in $\mathbb{N}$. We have $B(x_n, 2\varepsilon_i) = \bigsqcup_{k \in \mathbb{N}} A_{n,k,\varepsilon_i}$, hence we have the pointwise convergence

$$\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \underset{N \to +\infty}{\longrightarrow} \mathbb{1}_{B(x_n, 2\varepsilon_i)}.$$

Therefore by dominated convergence (using δ_{ε_i} as a dominating function) we have:

$$\delta_{\varepsilon_i} * \left(\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \right)(x) = \int_G \delta_{\varepsilon_i}(g) \left(\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \right)(g^{-1}x) dm(g)$$

$$\underset{N \to \infty}{\longrightarrow} \int_G \delta_{\varepsilon_i}(g) \mathbb{1}_{B(x_n, 2\varepsilon_i)}(g^{-1}x) dm(g) = \delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}(x).$$

The same holds for y, and thus we have

$$\delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}(x) = \delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}(y).$$

By (4), the function $\delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}$ takes the value 1 on $B(x_n, \varepsilon_i)$ and the value 0 on $X \setminus B(x_n, 3\varepsilon_i)$, so the previous equality ensures that for any $i, n \in \mathbb{N}$, we can never have simultaneously $d(x_n, x) < \varepsilon_i$ and $d(x_n, y) > 3\varepsilon_i$. By density of (x_n) and the fact that (ε_i) takes arbitrarily small values, we have $x = y$, which concludes the proof. $\qquad \square$

5.3 A counterexample when $G = \mathfrak{S}_\infty$

Let us consider $G = \mathfrak{S}_\infty$ in this section. The action α that we consider is the $\mathfrak{S}_\infty$-action on $X = \{0,1\}^\mathbb{N}$ by permutation of the coordinates. Let us also fix a sequence (p_n) of weights, with $p_n \in]0,1[$ for all n in $\mathbb{N}$. We also require that $p_n \neq p_m$ for $n \neq m$, and that $p_n \to \frac{1}{2}$. Consider the measures μ_n defined by :

$$\mu_n := (p_n \delta_1 + (1 - p_n) \delta_0)^{\otimes \mathbb{N}},$$

and define the measure λ on X by

$$\lambda := \sum_{n \in \mathbb{N}} \mu_n.$$

Let X_∞ be the subspace of X consisting of sequences that take infinitely many times the value 0, and infinitely many times the value 1. First note that $\mathfrak{S}_\infty$ acts transitively on X_∞. We have $\lambda(X \setminus X_\infty) = 0$, as for any n in $\mathbb{N}$ we have $\mu_n(X \setminus X_\infty) = 0$, and thus, up to a null set, we can restrict to X_∞.

Each μ_n is a probability measure on X, so λ is an infinite measure. We now verify that λ is σ-finite. Indeed, the strong law of large numbers ensures us that if we define a family (X_n) of Borel sets by

$$X_n := \left\{ x \in X : \frac{|\{k \in \{0, \ldots, m-1\} : x_k = 1\}|}{m} \underset{m \to +\infty}{\longrightarrow} p_n \right\},$$

that is to say that X_n is the set of the sequences of X with a proportion p_n of 1, then for all n in $\mathbb{N}$ we have $\mu_n(X_n) = 1$. We thus have

$$\lambda \left(X \setminus \bigsqcup_{n \in \mathbb{N}} X_n \right) = 0.$$

The condition $p_n \neq p_m$, for $n \neq m$, ensures us that the X_n are disjoint sets, and therefore λ is σ-finite.

The following lemma provides an obstruction to having a continuous locally finite model.

Lemma 5.5. *Let $\mathcal{N}$ be a neighborhood of the identity in $\mathfrak{S}_\infty$, let $x \in X_\infty$. Then we have $\lambda(\alpha(\mathcal{N}, x)) = +\infty$.*

Proof. For every n let H_n denote the open subgroup of $\mathfrak{S}_\infty$ given by

$$H_n = \{\sigma \in \mathfrak{S}_\infty \colon \forall i \in \{0, \cdots n\},\, \sigma(i) = i\}$$

Then the sequence (H_n) is a neighborhood basis of the identity in $\mathfrak{S}_\infty$, so we can find some H_N contained in our neighborhood $\mathcal{N}$. For any x_0 in X_∞, we have

$$\alpha(H_N, x) = \{y \in X_\infty \colon \forall i \in \{0, \ldots, N\},\, y_i = x_i\}$$

so since $p_n \to \frac{1}{2}$ we have

$$\mu_n(\alpha(H_N, x)) \underset{n \to +\infty}{\longrightarrow} \left(\frac{1}{2}\right)^{N+1}.$$

Therefore,

$$\lambda(\alpha(H_N, x)) = \sum_{n \in \mathbb{N}} \mu_n(\alpha(H_N, x)) = +\infty,$$

which concludes the proof. $\qquad\qquad\square$

Remark 5.6. By continuity of the action, for any x in X_∞, for any neighborhood $\mathcal{V}$ of x there exists N in $\mathbb{N}$ such that $\alpha(H_N, x) \subseteq \mathcal{V}$. So by Lemma 5.5 and the density of X_∞, all nonempty open subsets of X have infinite measure.

We have all the tools to show that Theorem 5.4 can fail badly for non locally compact Polish groups.

Proposition 5.7. *The spatial $\mathfrak{S}_\infty$-action α on (X, λ) defined above cannot be spatially isomorphic to a continuous action on a Polish space Y endowed with a locally finite measure η.*

Proof. Suppose that we have a continuous action β on (Y, η) with η locally finite and Y Polish, and that $\Phi : X_0 \to Y$ is a spatial isomorphism between α and β. Let us take some $x \in X_0$, then there is an open set U containing y such that $\eta(U) < \infty$. By continuity of β, there exists a neighborhood of the identity $\mathcal{N} \subseteq G$ satisfying $\beta(\mathcal{N}, \Phi(x)) \subseteq U$. However by Lemma 5.5, $\lambda(\alpha(\mathcal{N}, x)) = +\infty$. In particular, $\lambda(\alpha(\mathcal{N}, x) \cap X_0) = +\infty$. By the equivariance condition satisfied by Φ, the set $\beta(\mathcal{N}, \Phi(x))$ contains the infinite measure set $\Phi(\alpha(\mathcal{N}, x) \cap X_0)$, so it has infinite measure, contradicting the fact that it is contained in the finite measure set U. $\qquad\square$

We now define a second $\mathfrak{S}_\infty$-action β as the action on $\bigsqcup_n (\{0,1\}^\mathbb{N}, \mu_n)$ by permutation of the coordinates, but this time in different distinct copies of $\{0,1\}^\mathbb{N}$. This will yield an interesting infinite measure-preserving example of non spatial boolean isomorphism (see Section 4.3 for Tsankov's example in the finite measure case).

Proposition 5.8. *The $\mathfrak{S}_\infty$-action α on $(\{0,1\}^\mathbb{N}, \lambda)$ and the $\mathfrak{S}_\infty$-action β on $\bigsqcup_n (\{0,1\}^\mathbb{N}, \mu_n)$ are booleanly isomorphic, but not spatially isomorphic.*

Proof. As X_n denotes the μ_n-conull set of sequences with a proportion p_n of coordinates equal to 1, it is possible to send X_n to a copy of itself in the n-th copy of $\{0,1\}^{\mathbb{N}}$ by defining

$$\Phi : \bigsqcup_{n \in \mathbb{N}} X_n \subseteq \{0,1\}^{\mathbb{N}} \longrightarrow \bigsqcup_{n \in \mathbb{N}} X_n \subseteq \bigsqcup_{n \in \mathbb{N}} \{0,1\}^{\mathbb{N}}.$$

For a fixed permutation σ, λ-almost any sequence x in $X_\sigma = \bigsqcup_n X_n$ satisfies $x \in X_k \Rightarrow \alpha(\sigma, x) \in X_k$, and therefore $\Phi(\alpha(\sigma, x)) = \beta(\sigma, \Phi(x)) \in \Phi(X_k)$. The actions α and β are not spatially isomorphic however, as α only has one orbit in $\{0,1\}^{\mathbb{N}}$, but β cannot send an element of $\Phi(X_n)$ to $\Phi(X_m)$ whenever $n \neq m$. $\qquad\square$

Remark 5.9. More generally, by Proposition 5.7, α cannot admit a continuous Radon model, so any boolean isomorphism between α and another continuous action on a space endowed with a Radon measure cannot be a spatial isomorphism.

Remark 5.10. Similar ideas work for instance when $G = \text{Aut}(\mathbb{Q}, <)$, identifying $\mathbb{N}$ to $\mathbb{Q}$ and replacing X_∞ by the set of all sequences $(x_q) \in \{0,1\}^{\mathbb{Q}}$ such that the set $\{q \in \mathbb{Q} \colon x_q = 1\}$ is both dense and codense in $\mathbb{Q}$ (meaning that for all rationals $q_1 < q_2$ there are q and q' in the interval (q_1, q_2) such that $x_q = 1$ and $x_{q'} = 0$). Indeed, this new set has μ_n measure 1, and a back-and-forth argument shows that it consists of a single $\text{Aut}(\mathbb{Q}, <)$-orbit. It would be nice to identify more precisely the class of non-archimedean groups for which there exists a spatial infinite measure-preserving action which cannot be spatially isomorphic to a continuous action on a locally finite Polish measured space.

5.4 The natural action of $\text{Aut}(X, \lambda)$ admits no spatial model

Whirly actions were introduced in the probability measure-preserving context by Glasner, Tsirelson and Weiss, who showed such actions cannot have spatial models. In this section we record the natural extension of this definition to the infinite measure-preserving setup, and show that whirly actions admit no spatial models. As an example, we prove that the natural boolean action of $\text{Aut}(X, \lambda)$ is whirly and hence does not admit a spatial model.

Definition 5.11 ([GTW05, Sec. 3])**.** A boolean measure-preserving action α of a Polish group G on (X, λ) is **whirly** if for all sets A and B of positive measure, for every neighborhood $\mathcal{N}$ of e_G, there exists a element g in $\mathcal{N}$ such that $\lambda(A \cap \alpha(g, B)) > 0$.

Proposition 5.12. *Let $G = \text{Aut}(X, \lambda)$. The natural boolean action of G on (X, λ) is whirly.*

Proof. Let A and B be two Borel subsets of X of positive measure, and fix $\varepsilon > 0$. We consider $A_0 \subseteq A$ and $B_0 \subseteq B$ two disjoint Borel subsets, with $\lambda(A_0) = \lambda(B_0) < \varepsilon/2$. We define a element S of $\text{Aut}(X, \lambda)$ as follows:

$$\begin{cases} S_{\upharpoonright X \setminus (A_0 \cup B_0)} = \text{id}_{X \setminus (A_0 \cup B_0)} \\ S(A_0) = B_0 \\ S(B_0) = A_0, \end{cases}$$

and see that for ε small, S is close to the identity on X, for the weak topology. This concludes the proof, as $\lambda(A \cap SB) > 0$. $\qquad\square$

Let us now explain the link between whirly actions and G-continuity. We have the following infinite measure version of [GTW05, Prop. 3.3]. The proof is actually the same, but we provide it for the reader's convenience.

Proposition 5.13. *Consider a boolean measure preserving action α of a Polish group G on (X, λ). If the action is whirly, then all G-continuous functions are constant. Moreover, such an action cannot admit a spatial model.*

Proof. We first explain why the existence of G-continuous non-constant functions taking values in $\mathbb{R}$ contradicts the fact that the action is whirly. .

Let f be a G-continuous non-constant function taking values in $\mathbb{R}$. There exists $a < b$ in $\mathbb{R}$ such that $A := f^{-1}(]-\infty, a[)$ and $B := f^{-1}(]b, +\infty[)$ have positive measure. By G-continuity, there exists a neighborhood $\mathcal{N}$ of e_G such that any g in $\mathcal{N}$ satisfies $\left\| f \circ \alpha(g_n^{-1}, \cdot) - f \right\|_\infty < b - a$. This implies that $\lambda(A \cap \alpha(g, B)) = 0$, contradicting the fact that the action is whirly.

Now for a complex valued function, one simply has to consider the real and imaginary parts, as they are G-continuous whenever f is. Assuming that f is non-constant implies that at least one or the other is non-constant, thus implying that the action cannot be whirly.

For the second part of the statement, suppose the action has a spatial model on (Y, η). By The Becker-Kechris theorem (Theorem 1.1), it is possible to assume that the G-action on Y is continuous, with Y compact Polish. The support of η is closed, thus compact, call it K. Since G acts continuously on Y, any function in $C_c(K)$ is G-continuous by Lemma 3.5, and thus any non-constant function in $C_c(K)$ gives a non-constant G-continuous function, a contradiction. $\qquad\square$

Combining Propositions 5.12 and 5.13, we obtain the following.

Proposition 5.14. *Let (X, λ) be a standard σ-finite space. The natural boolean action of $\mathrm{Aut}(X, \lambda)$ on (X, λ) cannot admit a spatial model.*

Remark 5.15. The previous discussion holds in the non-singular setup as well, considering the tautological boolean non-singular action on (X, λ) of the group $\mathrm{Aut}(X, [\lambda])$ of all *non-singular* bijections (identified up to equality on a conull set). The action is whirly in the exact same sense as above, and the proof of Proposition 5.13 adapts in a straightforward manner to non-singular whirly actions.

6 Poisson point processes and Lévy groups

6.1 Construction of the Poisson point process

Let X be a Polish space, we can endow its space of closed subsets $\mathcal{F}(X)$ with the lower Vietoris topology (also called lower semi-finite topology in [Mic51]), which is the second-countable T_0 topology generated by declaring, for every $U \subseteq X$ open, that the set

$$\mathcal{V}_U = \{F \in \mathcal{F}(X) \colon F \cap U \neq \emptyset\}$$

is open. The associated Borel σ-algebra is called the Effros σ-algebra, and it turns $\mathcal{F}(X)$ into a standard Borel space, see for instance [Che21, Cor. 9.3].

Lemma 6.1. *Let $U \subseteq X$. The function which takes $F \in \mathcal{F}(X)$ to $|U \cap F|$ is lower semi-continuous if we endow $\mathcal{F}(X)$ with the lower Vietoris topology, in particular it is Borel.*

Proof. Suppose $|U \cap F_0| \geqslant k$, then we find $U_1, ..., U_k$ disjoint opens subsets of U such that $F_0 \cap U_i$ is non empty, and this will be true of any $F \in \bigcap_{i=1}^k \mathcal{V}_{U_i}$, hence the result. $\qquad\square$

Remark 6.2. In particular, the set of all $F \in \mathcal{F}(X)$ such that $F \cap U$ is finite is Borel.

Let us now recall that the Poisson law of intensity $\lambda > 0$ is the probability measure Pois^λ on $\mathbb{N}$ given by

$$\mathrm{Pois}^\lambda(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$$

and that its expected value is equal to λ.

We now define *the* **Poisson point process** of intensity $\kappa > 0$ on any Polish space endowed with an atomless locally finite Borel measure as the probability measure whose existence and uniqueness are granted by the following elementary version of [ARR24, Thm. 3].

Theorem 6.3. *Let λ be a locally finite atomless Borel measure on a Polish space X, let $\kappa > 0$. There is a unique probability measure $\mathbb{P}^X_{\kappa,\lambda}$ on $\mathcal{F}(X)$ such that for all $U \subseteq X$ open of finite measure, if the random variable F has law $\mathbb{P}^X_{\kappa,\lambda}$, then the random variable $|F \cap U|$ follows the Poisson law of parameter $\kappa\lambda(U)$.*

Remark 6.4. The theorem does not yield what is usually called a Poisson point process per se (see [ARR24, Def. 1.1]). In order to have one, we would need to further check that for all $A \subseteq X$ Borel, the map $F \in \mathcal{F}(X) \mapsto |A \cap F|$ is $\mathbb{P}^X_{\kappa,\lambda}$-Lebesgue measurable, and that it follows a Poisson law of parameter $\kappa\lambda(A)$. Since we don't need this stronger property, we do not prove it and refer the reader to the proof of [ARR24, Thm. 3].

Proof. In order to see the uniqueness, we first remark that closed subsets of the form $\mathcal{C}_U = \{F \in \mathcal{F}(X)\colon F \cap U = \emptyset\} = \mathcal{F}(X) \setminus \mathcal{V}_U$ form a π-system (because $\mathcal{C}_{U_1} \cap \mathcal{C}_{U_2} = \mathcal{C}_{U_1 \cup U_2}$) which generates the σ-algebra. But by construction $\mathcal{C}_U$ has probability $e^{-\kappa\lambda(U)}$ when $\lambda(U)$ is finite, otherwise it has probability 0, which proves uniqueness by the monotone class theorem.

For the existence, let us first prove it when λ is finite. In this case, we may as well restrict ourselves to defining the measure on the subset $\mathcal{P}_f(X)$ of finite subsets of X, which is Borel by Lemma 6.1.

The map $\Phi : \mathbb{N} \times X^{\mathbb{N}} \to \mathcal{P}_f(X)$ which takes $(k, (x_n)_{n \geqslant 0})$ to $\{x_n\colon n < k\}$ is easily seen to be continuous if we put on $\mathcal{P}_f(X)$ the topology induced by the lower Vietoris topology on $\mathcal{F}(X)$, in particular it is Borel. Renormalize λ to a probability measure $\widetilde{\lambda} = \frac{\lambda}{\lambda(X)}$. Endow $\mathbb{N} \times X^{\mathbb{N}}$ with the probability measure $\mu = \mathrm{Pois}^{\kappa\lambda(X)} \otimes \widetilde{\lambda}^{\otimes\mathbb{N}}$. We claim that the pushforward measure $\Phi_*\mu$ is the desired Poisson Point Process.

In order to see this, first note that since λ is atomless, μ-almost all $(k, (x_n))$ satisfy that $x_n \neq x_m$ whenever $n \neq m$. It follows that given $U \subseteq X$ open and $l \in \mathbb{N}$, if for $k \geqslant l$ we denote by $\mathcal{P}_l(\{0, \ldots, k-1\})$ the set of subsets of $\{0, \ldots, k-1\}$ of cardinality l, the event $|F \cap U| = l$ has probability

$$\Phi_*\mu(|F \cap U| = l) = e^{-\kappa\lambda(X)} \sum_{k \geqslant l} \left(\frac{\lambda(X)^k}{k!} \sum_{K \in \mathcal{P}_l(\{0,\ldots,k-1\})} \left(\frac{\kappa\lambda(U)}{\lambda(X)} \right)^l \left(\frac{\kappa\lambda(X \setminus U)}{\lambda(X)} \right)^{k-l} \right)$$

Letting $x = \lambda(X)$ and $u = \lambda(U)$, we rewrite this as

$$e^{-\kappa x} \sum_{k \geqslant l} \kappa^k \frac{1}{k!} \binom{k}{l} u^l (x-u)^{k-l} = e^{-\kappa x} \sum_{k \geqslant l} \frac{\kappa^k u^l (x-u)^{k-l}}{l!(k-l)!}$$

$$= e^{-\kappa x} \frac{(\kappa u)^l}{l!} e^{\kappa(x-u)}$$

$$= e^{-\kappa u} \frac{(\kappa u)^l}{l!}.$$

Since $u = \lambda(U)$, our probability measure $\Phi_*\mu$ has the required distribution and hence the existence is proven when λ is a finite measure.

Let us now deal with the case where λ is infinite. By local finiteness write $X = \bigcup_n V_n$ where each V_n is open of positive finite measure and satisfies $V_n \subseteq V_{n+1}$. Each V_n is Polish for the induced topology, and if we denote by λ_n the restriction of λ to V_n, what we have done so far grants us a unique Borel probability measure $\mathbb{P}^{V_n}_{\kappa,\lambda}$ on $\mathcal{P}_f(V_n)$ such that for all $U \subseteq V_n$ open, if F has law $\mathbb{P}^{V_n}_{\kappa,\lambda}$ then the random variable $|F \cap U|$ has law $\mathrm{Pois}^{\kappa\lambda(U)}$.

For $n \in \mathbb{N}$, consider the continuous hence Borel map $\pi_n : \mathcal{P}_f(V_{n+1}) \to \mathcal{P}_f(V_n)$ which maps F to $F \cap V_n$. Form the projective limit

$$Y = \varprojlim \mathcal{P}_f(V_n) = \left\{ (F_n) \in \prod_{n\in\mathbb{N}} \mathcal{P}_f(V_n) \colon \forall n \in \mathbb{N},\ F_n = F_{n+1} \cap V_n \right\},$$

which is a Borel subset of the standard Borel space $\prod_n \mathcal{P}_f(V_n)$ by [Par05, Thm. V.2.5].

For every $n \in \mathbb{N}$, it is a straightforward consequence of uniqueness that $\pi_{n*}\mathbb{P}^{V_{n+1}}_{\kappa,\lambda} = \mathbb{P}^{V_n}_{\kappa,\lambda}$, so by Kolmogorov's consistency theorem (see e.g. [Par05, Thm. V.3.2]) we have a unique Borel probability measure μ on Y such that for all $n \in \mathbb{N}$, $p_{n*}\mu = \mathbb{P}^{V_n}_{\kappa,\lambda}$, where $p_n : Y \to \mathcal{P}_f(V_n)$ is the projection on coordinate n.

Let us finally consider the map $\Psi : Y \to \mathcal{F}(X)$ which takes (F_n) to the increasing union $\bigcup_n F_n$. First note that Ψ is well defined since $\bigcup_n F_n$ has finite intersection in each V_n, and the V_n's exhaust X so $\bigcup_n F_n$ is a discrete subset of X without accumulation points, hence closed (and countable). Next, Ψ is a Borel (actually continuous) map since $\bigcup_n F_n \cap U \neq \emptyset$ if and only if there exists n such that $F_n \cap (V_n \cap U) \neq \emptyset$. We claim that $\mathbb{P}^X_{\kappa,\lambda} := \Psi_*\mu$ is the desired Poisson Point Process. Indeed, if U is an arbitrary open subset of X of finite measure, we have that $|F \cap U| = k$ if and only if there exists N such that for all $n \geqslant N$, $|F \cap (U \cap V_n)| = k$. For a fixed N, the event that for all $n \geqslant N$, $|F \cap (U \cap V_n)| = k$ is the intersection of a decreasing sequence of events which has probability

$$\mathbb{P}^X_{\kappa,\lambda}\left(\forall n \geqslant N,\ |F \cap (U \cap V_n)| = k\right) = \lim_{n\to+\infty} e^{-\kappa\lambda(U\cap V_n)} \frac{(\kappa\lambda(U \cap V_n))^k}{k!} = e^{-\kappa\lambda(U)} \frac{(\kappa\lambda(U))^k}{k!}.$$

It follows that the event $|F \cap U| = k$ also has probability $e^{-\kappa\lambda(U)} \frac{(\kappa\lambda(U))^k}{k!}$ as wanted. $\qquad\square$

Remark 6.5. The proof of uniqueness is actually a version of Rényi's theorem (see for instance [LP17, Thm. 6.10]), namely it shows that $\mathbb{P}^X_{\kappa,\lambda}$ is uniquely defined by the fact that for every finite measure open subset $U \subseteq X$, we have

$$\mathbb{P}^X_{\kappa,\lambda}(F \cap U = \emptyset) = e^{-\kappa\lambda(U)}.$$

Remark 6.6. Since the measure is locally finite, for any $\kappa > 0$, $\mathbb{P}^X_{\kappa,\lambda}$-almost every $F \in \mathcal{F}(X)$ is locally finite, that is to say that every point of F admits a neighborhood whose intersection with F is a singleton.

Proposition 6.7. *Let X be a Polish space endowed with a σ-finite atomless locally finite measure λ, let $\kappa > 0$, let U_1 and U_2 be two disjoint open subsets of X of finite measure. Suppose F has law $\mathbb{P}^X_{\kappa,\lambda}$, then the random variables $|F \cap U_1|$ and $|F \cap U_2|$ are independent.*

Proof. Consider, for $i \in \{1, 2\}$, the map $p_i : \mathcal{F}(X) \to \mathcal{F}(U_i)$ which takes F to $F \cap U_i$. By the defining properties of the Poisson point process $p_{i*}\mathbb{P}^X_{\kappa,\lambda} = \mathbb{P}^{U_i}_{\kappa,\lambda}$, and we need to check that $(p_1 \times p_2)_*\mathbb{P}^X_{\kappa,\lambda} = \mathbb{P}^{U_1}_{\kappa,\lambda} \otimes \mathbb{P}^{U_2}_{\kappa,\lambda}$. By the monotone class theorem, it suffices to show

these two measures coincide on the π-system formed of sets of the form $\mathcal{C}_{W_1} \times \mathcal{C}_{W_2}$, where $\mathcal{C}_{W_i} = \{F \in \mathcal{F}(U_i) \colon F \cap W_i = \emptyset\}$, and W_i is an open subset of U_i.

But by definition $(p_1 \times p_2)^{-1}(\mathcal{C}_{W_1} \times \mathcal{C}_{W_2}) = \mathcal{C}_{W_1 \sqcup W_2} = \{F \in \mathcal{F}(X) \colon F \cap (W_1 \sqcup W_2) = \emptyset\}$, which has measure

$$
\begin{aligned}
\mathbb{P}^X_{\kappa,\lambda}(\mathcal{C}_{W_1 \sqcup W_2}) &= e^{-\kappa\lambda(W_1 \sqcup W_2)} \\
&= e^{-\kappa\lambda(W_1)} e^{-\kappa\lambda(W_2)} \\
&= \mathbb{P}^{U_1}_{\kappa,\lambda}(\mathcal{C}_{W_1}) \mathbb{P}^{U_2}_{\kappa,\lambda}(\mathcal{C}_{W_2}),
\end{aligned}
$$

so we are done. $\qquad\square$

6.2 Application to actions of Lévy groups

Definition 6.8. A sequence (X_n, d_n, m_n) of metric spaces with probability measure is a **Lévy family** if it satisfies the following condition: if (A_n) is a sequence of measurable subsets of (X_n) such that $\liminf m_n(A_n) > 0$, then for any $\varepsilon > 0$, $\lim_n m_n((A_n)_\varepsilon) = 1$, where $(A_n)_\varepsilon := \{x \in X_n \colon d_n(x, A_n) < \varepsilon\}$.

A Polish group G is a **Lévy group** if there is a sequence of compact subgroups (K_n) with $K_n \subseteq K_{n+1}$, such that $\bigcup_n K_n$ is dense in G, and such that the (K_n, m_n), where m_n is the normalized Haar measure, is a Lévy family when equipped with a right-invariant compatible metric d on G (the choice of such a d does not matter).

We refer the reader to [Pes06] for more about Lévy groups. Glasner, Tsirelson and Weiss proved the following fundamental result.

Theorem 6.9 ([GTW05, Thm. 1.1])**.** *Every measure-preserving spatial action of a Lévy group G on a possibly atomic standard probability space (X, μ) is trivial, i.e. the set of fixed points $\{x \in X \colon \forall g \in G, g \cdot x = x\}$ is conull.*

We obtain a analogous result for continuous actions of Lévy groups on Polish spaces with a infinite atomless σ-finite measure λ, under the assumption that the measure is locally finite. This result was first obtained by Avraham-Re'em and Roy, see [ARR24, Thm. 5].

Theorem 6.10. *Every continuous measure-preserving spatial action of a Lévy group G on a Polish space X endowed with a σ-finite atomless locally finite measure λ is trivial, i.e. the set of fixed points $\{x \in X \mid \forall g \in G, g \cdot x = x\}$ is conull.*

Proof. Suppose by contradiction that we have an action as above on (X, λ), but that the action is not trivial. Consider the standard Borel space $\mathcal{F}(X)$ endowed with the Poisson Point Process on (X, λ) of intensity κ, that we denoted by $\mathbb{P}^X_{\kappa,\lambda}$ Then $(\mathcal{F}(X), \mathbb{P}^X_{\kappa,\lambda})$ is a standard probability space, and we now show that G acts spatially on it.

Since every element of G defines a homeomorphism of X, it sends any closed subset of X to a closed subset of X, and in particular G acts on $\mathcal{F}(X)$. Since $g \cdot \mathcal{V}_U = \mathcal{V}_{g^{-1}U}$ and G acts by homeomorphism on X, we have that G acts by homeomorphisms on $\mathcal{F}(X)$ for the lower Vietoris topology.

Let us show that this action is continuous for the lower Vietoris topology. Given $U \subseteq X$ open, we need to show that the set of couples (g, F) such that $g \cdot F \cap U \neq \emptyset$ is open. So take (g_0, F_0) such that $g_0 \cdot F_0 \cap U \neq \emptyset$, then $F_0 \cap g_0^{-1} \cdot U \neq \emptyset$, then let $x_0 \in F_0 \cap g_0^{-1} \cdot U$. In particular $g_0 \cdot x \in U$, so by continuity of the action of G on X, there is a neighborhood V of g_0 and W of x_0 such that $V \cdot W \subseteq U$. Consider the neighborhood $V \times \mathcal{V}_W$ of (g_0, F_0),

then any element (g, F) in the neighborhood $V \times \mathcal{V}_W$ of (g_0, F_0) satisfies $F \cap W \neq \emptyset$, and hence $g \cdot F \cap g \cdot W \neq \emptyset$. Since $g \in v$ we have $g \cdot W \subseteq U$, so we conclude that $gF \cap U \neq \emptyset$ as wanted. In particular, the G-action on $\mathcal{F}(X)$ is Borel.

This action preserves $\mathbb{P}^X_{\kappa,\lambda}$ by uniqueness. Indeed consider the random variable F of law $\mathbb{P}^X_{\kappa,\lambda}$, for any finite measure open subset U of X and any g in G we have

$$
\begin{aligned}
\mathbb{P}^X_\kappa(|F \cap U| = k) &= e^{-\kappa\lambda(U)} \frac{(\kappa\lambda(U))^k}{k!} \\
&= e^{-\kappa\lambda(g^{-1} \cdot U)} \frac{(\kappa\lambda(g^{-1} \cdot U))^k}{k!} \\
&= \mathbb{P}^X_\kappa(|F \cap g^{-1} \cdot U| = k) \\
&= \mathbb{P}^X_\kappa(|g \cdot F \cap U| = k).
\end{aligned}
$$

Being a spatial measure-preserving action on a standard probability space, the G-action on $\mathcal{F}(X)$ it is trivial by Theorem 6.9. We will now see why this cannot be.

We first claim that we can find an open subset U of X with $0 < \lambda(U) < \infty$ and $g \in G$ which satisfies $U \cap g \cdot U = \emptyset$. Indeed, since G acts continuously non-trivially on (X, λ) the set of non fixed points $X_0 := \{x \in X \mid \exists g \in G, g \cdot x \neq x\}$ has positive measure. By continuity of the G-action on X, for every $x \in X_0$ we find $g_x \in G$ and an open subset $U_x \subseteq X$ such that $g_x U_x \cap U_x = \emptyset$. By Lindelöf's theorem, the cover $(U_x)_{x \in X_0}$ admits a countable subcover $(U_{x_i})_{i \in \mathbb{N}}$. Since $\lambda(X_0) > 0$, we conclude that some U_{x_i} satisfies $\lambda(U_{x_i}) > 0$, and thus taking $g = g_{x_i}$ and $U = U_{x_i}$, we have $\lambda(U) > 0$ and $gU \cap U = \emptyset$ as desired.

Now consider the set $\mathcal{E}_U$ of all $F \in \mathcal{F}(X)$ such that $|F \cap U| = 1$ and $|F \cap g^{-1} \cdot U| = 0$, which has positive $\mathbb{P}^X_{\kappa,\lambda}$ measure by Proposition 6.7. But by construction the positive measure set $\mathcal{E}_U$ is sent by g to a subset of $\mathcal{F}(X)$ which is disjoint from $\mathcal{E}_U$, yielding the desired contradiction since the action on $(\mathcal{F}(X), \mathbb{P}^X_{\kappa,\lambda})$ should be trivial. $\square$

Remark 6.11. Theorem 6.10 applies to $\mathrm{Aut}(X, \lambda)$ when λ is locally finite. Indeed, Giordano and Pestov have proved in [GP07, Thm. 4.2] that it is a Lévy group when endowed with its weak topology. However, Proposition 5.14 does not follow from the above result, as there are no topological requirements on the spatial model.

Remark 6.12. The hypothesis that λ has no atoms is not important: if it does, then G acts by permutation on these atoms, but the associated Bernoulli shift must be trivial by the Glasner-Weiss-Tsirelson theorem, so G acts trivially on the atomic part of λ.

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