# On the Rokhlin lemma for infinite measure-preserving bijections

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**Abstract**

We study the Rokhlin lemma in the context of infinite measure-preserving bijections, and completely classify such bijections up to $\lambda$-approximate conjugacy, where $\lambda$ is the infinite measure which is preserved. This sharpens the classical version of the Rokhlin lemma, which only provides such a classification up to $\mu$-approximate conjugacy where $\mu$ is a probability measure equivalent to $\lambda$.

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## 1 Introduction

In the probability measure-preserving (pmp) context, Rokhlin's lemma states that given any aperiodic pmp bijection $T$ of a standard probability space $(X, \mu)$, for any $N \geqslant 1$ and $\varepsilon > 0$ there is a Borel subset $A \subseteq X$ such that $A, T(A), \dots, T^{N-1}(A)$ are pairwise disjoint and

$$\mu(X \setminus (A \sqcup T(A) \sqcup \cdots \sqcup T^{N-1}(A))) < \varepsilon. \tag{1}$$

It has the following fundamental consequence, often also called the Rokhlin lemma: all aperiodic pmp bijections are $\mu$-approximately conjugate, namely for all $\varepsilon > 0$ and all aperiodic pmp bijections $T_1$ and $T_2$, we can find another pmp bijection $S$ such that

$$\mu(\{x \in X : ST_1S^{-1}(x) \neq T_2(x)\}) < \varepsilon. \tag{2}$$

In this note, our aim is to investigate versions of the Rokhlin lemma in the infinite measure-preserving setup, but where we keep working with the infinite measure rather than replacing it with a finite measure equivalent to it.

So let us fix a standard $\sigma$-finite infinite measured space $(X, \lambda)$ and an aperiodic measure-preserving bijection $T$ of $(X, \lambda)$. Let us first point out that the most basic version of Rokhlin's lemma (namely that given by Equation (1)) holds if one replaces $\lambda$ by $\mu$, for $\mu$ a probability measure equivalent to $\lambda$ (see Proposition 2.3 for details), which yields that all aperiodic infinite measure-preserving bijections are $\mu$-approximately conjugate (see Proposition 2.5). However, it is not true that all aperiodic infinite measure-preserving bijections are $\lambda$-approximately conjugate in the following sense.

**Definition 1.1.** Two measure-preserving bijections $T_1$ and $T_2$ of a standard infinite space $(X, \lambda)$ are **$\lambda$-approximately conjugate** if for all $\varepsilon > 0$ we can find another measure-preserving bijection $S$ such that

$$\lambda(\{x \in X : ST_1S^{-1}(x) \neq T_2(x)\}) < \varepsilon. \tag{3}$$

The main reason why aperiodic infinite measure-preserving bijections can fail to be $\lambda$-approximately conjugate is that this notion detects *dissipativity.* Recall that $T$ is dissipative when it is aperiodic and it admits a fundamental domain, namely a Borel subset $D$ such that $D$ intersects almost every $T$-orbit exactly once. Up to conjugacy by measure-preserving bijections, dissipative bijections are uniquely determined by the measure of their fundamental domains, and a model for a dissipative bijection whose fundamental domains have measure $d$ is given by $x \mapsto x + d$ on $X = \mathbb{R}$ endowed with the Lebesgue measure. Our first observation is that any two dissipative bijections are $\lambda$-approximately conjugate if and only if their fundamental domains have the same measure.

More generally, Hopf's decomposition associates to every infinite measure-preserving bijection $T$ a unique (up to a null set) maximal $T$-invariant set $\mathfrak{D}_T$ such that the restriction of $T$ to $\mathfrak{D}_T$ is dissipative, and then the restriction of $T$ to $\mathfrak{C}_T$ is conservative (see Definition 2.1 for the intrinsic definition of conservativity). We can now state our main result, which is a classification of aperiodic infinite measure-preserving bijections up to $\lambda$-approximate conjugacy.

**Theorem 1.2.** *Let $T_1, T_2 \in \mathrm{Aut}(X, \lambda)$ be aperiodic. Then $T_1$ and $T_2$ are $\lambda$-approximately conjugate if and only if their dissipative parts admit fundamental domains of equal measure. In particular if both $T_1$ and $T_2$ are conservative, then they are $\lambda$-approximately conjugate.*

The key ingredient for the conservative case is that given any two aperiodic conservative transformations $T_1$ and $T_2$, there are two small subsets with respect to which $T_1$ and $T_2$ have the exact same return time distribution (see Theorem 3.3). One can then match the Kakutani-Rokhlin towers associated to $T_1$ and $T_2$ to construct the desired conjugation. When the dissipative parts are not trivial, we are able to "absorb" the orbits of the conservative part into those of the dissipative part, see Lemma 3.4.

Going back to the conservative case and Theorem 3.3, let us highlight that one cannot just specify a return time distribution and hope that it shows up in any conservative aperiodic bijection. For instance, a return time distribution with only even return times will force the even levels of the Kakutani-Rokhlin towers to be $T^2$-invariant, so that if $T$ does have such a return time distribution, then $T^2$ is not ergodic. Nevertheless, it would be interesting to understand whether there is some return time distribution which shows up in any infinite measure-preserving conservative aperiodic bijection.

**Remark 1.3.** As pointed out to us by Emmanuel Roy, there exist infinite measure-preserving ergodic bijections whose return time distributions on finite measure sets always have infinite Shannon entropy (explicit examples can be found in [AP09, Prop. 2.1] and [JdlR12]). In particular, if there is a universal return time distribution (namely, a return time distribution that shows up in any conservative aperiodic transformation), then it must have infinite Shannon entropy.

Our paper is organized as follows: after presenting the setup and proving a version of Rokhlin's lemma in the preliminary Section 2, we prove Theorem 1.2 in Section 3. We finally comment on possible extensions of our result to infinite measure-preserving actions of amenable groups in Section 4.

## 2 Preliminaries

Let us fix once and for all a *standard $\sigma$-finite infinite measure space* $(X, \lambda)$, namely a standard Borel space $X$ endowed with a $\sigma$-finite infinite atomless measure $\lambda$. Such a measured space is isomorphic to the real line endowed with the Lebesgue measure. We are interested in the group $\mathrm{Aut}(X, \lambda)$ of all measure-preserving bijections $T : X \to X$. As usual, we neglect what happens on null sets.

**Definition 2.1.** A measure-preserving bijection $T \in \mathrm{Aut}(X, \lambda)$ is called

- **aperiodic** when all its orbits are infinite;
- **dissipative** when there is a Borel subset $D \subseteq X$ such that

$$X = \bigsqcup_{n \in \mathbb{Z}} T^n(D),$$

  this is then equivalent to $T$ being aperiodic and admitting a Borel fundamental domain, namely a subset $D$ intersecting each $T$-orbit exactly once;
- **conservative** if for all Borel $A \subseteq X$ of positive measure, every $x \in A$ eventually returns to $A$ via $T$, or in a formula:

$$A \subseteq \bigcup_{n \geqslant 1} T^{-n}(A).$$

Recall that the Hopf decomposition associates to any $T \in \mathrm{Aut}(X, \lambda)$ a unique $T$-invariant partition $X = \mathfrak{C}_T \sqcup \mathfrak{D}_T$ such that the restriction of $T$ to $\mathfrak{C}_T$ is conservative, while the restriction of $T$ to $\mathfrak{D}_T$ is dissipative. These respective restrictions are called the **conservative part** and the **dissipative part** of $T$. Observe that all fundamental domains of the dissipative part of $T$ must have the same measure since one can be obtained from the other by translating piecewise by powers of $T$.

Finally, recall that the **support** of $T \in \mathrm{Aut}(X, \lambda)$ is the Borel set of all $x \in X$ such that $T(x) \neq x$.

**Lemma 2.2.** *Let $(X, \lambda)$ be a standard infinite measure space, and $T \in \mathrm{Aut}(X, \lambda)$ be conservative and aperiodic on its support. For any $\varepsilon > 0$ there exists a Borel subset $B$ of $\mathrm{supp}\, T$ such that $B$ intersects all nontrivial $T$-orbits, and $\lambda(B) < \varepsilon$.*

*Proof.* Let $\varepsilon > 0$. Since $\lambda$ is atomless and $\sigma$-finite we can write $\mathrm{supp}\, T$ as $\mathrm{supp}\, T = \bigsqcup_{n \in \mathbb{N}} X_n$, with $\lambda(X_n)$ finite for all $n \in \mathbb{N}$. By conservativity, for $\lambda$-almost all $x \in X_n$ the $T$-orbit of $x$ intersects $X_n$ infinitely many times, in other words after throwing away a null set, the restriction $\mathcal{R}_n$ of the equivalence relation $\mathcal{R}_T = \{(x, T^k x) \colon x \in X, k \in \mathbb{Z}\}$ to $X_n$ is aperiodic. By the marker lemma [KM04, Lem. 6.7], for every $n$ there is a decreasing sequence $(B_{n,k})_k$ of Borel subsets of $X_n$ which intersects every $\mathcal{R}_n$-class and satisfies $\bigcap_k B_{n,k} = \emptyset$. In particular, since $\lambda(X_n)$ is finite, for some large enough $k_n$ we have $\lambda(B_{n,k_n}) < \varepsilon 2^{-n-1}$. It follows that $B \coloneqq \bigsqcup_n B_{n,k_n}$ is as desired. □

Given a conservative $T \in \mathrm{Aut}(X, \lambda)$ and $A \subseteq X$ Borel, we define the return time function $\tau_{A,T}$ by: for all $x \in A$,

$$\tau_{A,T}(x) = \min\{n \geqslant 1 \colon T^n(x) \in A\}.$$

We then have that the $T$-saturation of $A$ decomposes as a **Kakutani-Rokhlin partition** as follows:

$$\bigcup_{n\in\mathbb{Z}} T^n(A) = \bigsqcup_{n=1}^{\infty} \bigsqcup_{k=0}^{n-1} T^k(\tau_{A,T}^{-1}(n)) \tag{4}$$

For every $n \geqslant 1$, we say that $\bigsqcup_{k=0}^{n-1} T^k(\tau_{A,T}^{-1}(n))$ is a tower of height $n$, and that $T^k(\tau_{A,T}^{-1}(n))$ is the level $k$ of such a tower. Lemma 2.2 and the notion of Kakutani-Rokhlin partition are all we need in order to prove a natural version of Rokhlin's lemma in the infinite measure-preserving setup. The proof is a straightforward adaptation the probability measure case (as in [Gla03, Thm. 15.4]).

**Proposition 2.3.** *Given any aperiodic measure-preserving bijection $T$ of a standard infinite measure space $(X, \lambda)$, for any $N \geqslant 1$ and $\varepsilon > 0$ there is a Borel subset $A \subseteq X$ such that $A, T(A), \ldots, T^{N-1}(A)$ are pairwise disjoint and*

$$\lambda(X \setminus (A \sqcup T(A) \sqcup \cdots \sqcup T^{N-1}(A))) < \varepsilon. \tag{5}$$

*Proof.* On the dissipative part of $T$, if we let $D$ be a fundamental domain, we simply define the $\mathfrak{D}$-part of $A$ as $A_{\mathfrak{D}} = \bigsqcup_{k\in\mathbb{Z}} T^{Nk}(D)$. On the conservative part, take $\varepsilon' = \frac{\varepsilon}{N}$ and let $B$ intersect every $T_{\restriction \mathfrak{C}_T}$-orbit and have measure less than $\varepsilon'$, as provided by Lemma 2.2. We then have the Kakutani-Rokhlin partition

$$\mathfrak{C}_T = \bigsqcup_{n=1}^{\infty} \bigsqcup_{k=0}^{n-1} T^k(\tau_{B,T}^{-1}(n)).$$

It then suffices to take for $A_{\mathfrak{C}}$ the elements of the towers whose level is a multiple of $N$, and which still have $N-1$ levels above them, namely

$$A_{\mathfrak{C}} = \bigsqcup_{n=N}^{\infty} \bigsqcup_{q=0}^{\lfloor \frac{n}{N} \rfloor - 1} T^{qn}(\tau_{B,T}^{-1}(n))$$

and let $A = A_{\mathfrak{D}} \sqcup A_{\mathfrak{C}}$, noting that $X \setminus (A \sqcup T(A) \sqcup \cdots \sqcup T^{N-1}(A)))$ is contained in $\bigcup_{k=1}^{N} T^{-k}(B)$ and hence has measure at most $N \cdot \frac{\varepsilon}{N} = \varepsilon$. □

**Remark 2.4.** The above proposition is also true if one only require that $T$ is non-singular: in this case one has to modify the above procedure on the conservative part by finding $B$ intersecting all $T$-orbits such that $\mu(T^{-1}(B) \cup \cdots \cup T^{-N}(B)) < \varepsilon$ (see [Aar97, Thm. 1.5.9] for the ergodic case).

We finish this section by recalling how the above result yields $\mu$-approximate conjugacy of all aperiodic infinite measure-preserving bijections.

**Proposition 2.5.** *Given any two aperiodic measure-preserving bijections $T_1$, $T_2$ of a standard infinite measure space $(X, \lambda)$ and a probability measure $\mu$ equivalent to $\lambda$, for any $\varepsilon > 0$ there exists $S \in \mathrm{Aut}(X, \lambda)$ such that*

$$\mu(\{x \in X \colon ST_1S^{-1}(x) \neq T_2(x)\}) < \varepsilon.$$

*Proof.* By absolute continuity there is $\varepsilon' > 0$ such that for all $C \subseteq X$, $\lambda(C) < \varepsilon'$ implies $\mu(C) < \varepsilon/2$. Let us fix $N$ such that $\frac{1}{N} < \varepsilon/2$. Find $A_1$ as in Proposition 2.3 for $T_1$, up to shrinking $A_1$ a bit we have the following equality:

$$\lambda(X \setminus (A_1 \sqcup T_1(A_1) \sqcup \cdots \sqcup T_1^{N-1}(A_1))) = \varepsilon'.$$

Notice that since $T$ preserves $\lambda$, the above equality still holds if we replace $A_1$ by $T_1^k(A_1)$ for some $k \in \mathbb{Z}$. Moreover, since the first $N$ translates of $T_1^{N-1}(A_1)$ by $T_1$ are disjoint, one of them has $\mu$-measure $< \frac{1}{N}$. So up to replacing $A_1$ by some well-chosen $T_1^k(A_1)$ where $0 \leqslant k \leqslant N-1$, we may further assume $\mu(T_1^{N-1}(A_1)) \leqslant \frac{1}{N}$.

Similarly, we can find $A_2 \subseteq X$ such that $\mu(T_2^{N-1}(A_2)) \leqslant \frac{1}{N}$ and

$$\lambda(X \setminus (A_2 \sqcup T_2(A_2) \sqcup \cdots \sqcup T_2^{N-1}(A_2))) = \varepsilon'.$$

We then observe that both $A_1$ and $A_2$ must have infinite $\lambda$-measure, so as $(A_1, \lambda_{\restriction A_1})$ and $(A_2, \lambda_{\restriction A_2})$ are both standard $\sigma$-finite spaces, there is a measure-preserving bijection $\varphi : A_1 \to A_2$. We extend it to

$$\psi : A_1 \sqcup T_1(A_1) \sqcup \cdots \sqcup T_1^{N-1}(A_1) \to A_2 \sqcup T_2(A_2) \sqcup \cdots \sqcup T_2^{N-1}(A_2)$$

by letting for, for all $k \in \{0, ..., N-1\}$ and $x \in A_1$:

$$\psi(T_1^k(x)) = T_2^k(\varphi(x)),$$

and further extend $\psi$ to some $S \in \mathrm{Aut}(X, \lambda)$ arbitrarily (which is possible since the complement of $A_1 \sqcup T_1(A_1) \sqcup \cdots \sqcup T_1^{N-1}(A_1)$ has the same measure $\varepsilon'$ as the complement of $A_2 \sqcup T_2(A_2) \sqcup \cdots \sqcup T_2^{N-1}(A_2)$). Then we do have $S(T_1(x)) = T_2(S(x))$ except when $x$ belongs to the complement of $A_1 \sqcup T_1(A_1) \sqcup \cdots \sqcup T_1^{N-1}(A_1)$ (which has $\mu$-measure $< \varepsilon/2$) or to $T_1^{N-1}(A_1)$ (which has $\mu$-measure $\leqslant \frac{1}{N} < \varepsilon/2$). □

# 3 Proof of the main result

The proof of Theorem 1.2 is divided into three parts. In Section 3.1, we first show that any two aperiodic conservative infinite measure-preserving bijections share a common return time distribution. Section 3.2 is devoted to our dissipative absorption lemma, implying that any $T \in \mathrm{Aut}(X, \lambda)$ with nonempty dissipative part is $\lambda$-approximately conjugate to its dissipative part. These two results are finally put together in Section 3.3.

## 3.1 Finding the right return times

Here is the key lemma, which allows us to modify the return time distribution of any $B \subseteq X$ of finite measure intersecting all orbits.

**Lemma 3.1.** *Let $T \in \mathrm{Aut}(X, \lambda)$ be conservative and aperiodic. Let $B \subseteq X$ such that $\lambda(B) < +\infty$ and $B$ intersects all $T$-orbits. Fix $n \in \mathbb{N}^*$, and a sequence $(\lambda_k)_{k \geqslant n}$ such that for all $k \geqslant n$, $\lambda_k > \lambda(\tau_{B,T}^{-1}(k))$. Then there exists $C \subseteq X$ disjoint from $B$ such that*

(i) *For all $k < n$, we have $\tau_{B \sqcup C,T}^{-1}(k) = \tau_{B,T}^{-1}(k)$.*

(ii) *$\lambda(\tau_{B \sqcup C,T}^{-1}(n)) = \lambda_n$.*

(iii) *For all $k > n$, we still have $\lambda(\tau_{B \sqcup C,T}^{-1}(k)) < \lambda_k$.*

*Proof.* Let us begin by observing that since $B$ intersects almost all $T$-orbits, the Kakutani-Rokhlin decomposition of the infinite measure space $X$ as

$$X = \bigsqcup_{N \geqslant 1} \bigsqcup_{k=0}^{N-1} T^k(\tau_{B,T}^{-1}(N))$$

yields that

$$\sum_{N \geqslant 1} N\lambda(\tau_{B,T}^{-1}(N)) = \lambda(X) = +\infty.$$

Note that since $B$ has finite measure, all the terms in the above sum are finite. Let

$$\delta = \min \left\{ \lambda_k - \lambda(\tau_{B,T}^{-1}(k)) \;\middle|\; k \in \{n+1, \dots, 2n\} \right\} > 0.$$

Fix $K \in \mathbb{N}$ such that

$$K\delta > \lambda_n - \lambda(\tau_{B,T}^{-1}(n)). \tag{6}$$

Then every tower of height $k \geqslant Kn+n+1$ can be decomposed as $\lfloor \frac{k-n-1}{n} \rfloor \geqslant K$ subtowers of height $n$ stacked onto each other plus one final top subtower whose height belongs to $\{n+1, \dots, 2n\}$. The plan is now to add the top of each of these height $n$ subtowers to our set $B$ until we reach the desired return time distribution (the added elements correspond to the hatched set $C$ in Figure 1).

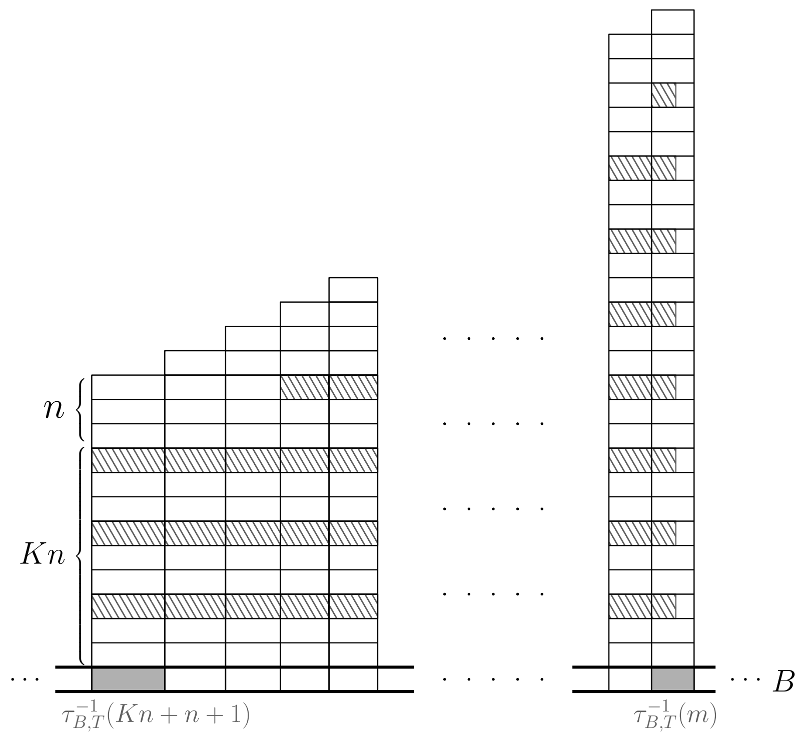


Figure 1: Illustration for $n = 3$, where the hatched part represents $C$ and $T$ translates every rectangle to the rectangle on top of it (except for the top rectangles, which are only known to be taken back into $B$).

Let us carry out this construction in details. Let $m$ be the first integer such that

$$\sum_{k=Kn+n+1}^{m} \left\lfloor \frac{k-n-1}{n} \right\rfloor \lambda(\tau_{B,T}^{-1}(k)) \geqslant \lambda_n - \lambda(\tau_{B,T}^{-1}(n)),$$

whose existence is guaranteed by the fact that $\sum_{N\geqslant 1} N\lambda(\tau_{B,T}^{-1}(N)) = +\infty$. For every $k \in \{Kn+n+1, \dots, m-1\}$ (which might be empty), we let

$$C_k = \bigsqcup_{l=1}^{\lfloor \frac{k-n-1}{n} \rfloor} T^{nl}(\tau_{B,T}^{-1}(k)),$$

which has measure $\lfloor \frac{k-n-1}{n} \rfloor \lambda(\tau_{B,T}^{-1}(k))$, and we finally pick a subset $D \subseteq \tau_{B,T}^{-1}(m)$ satisfying

$$\lambda(D) = \frac{1}{\lfloor \frac{m-n-1}{n} \rfloor} \left( \lambda_n - \lambda(\tau_{B,T}^{-1}(n)) - \sum_{k=Kn+n+1}^{m-1} \lambda(C_k) \right), \tag{7}$$

whose existence is guaranteed by the choice of $m$, and let

$$C_m \coloneqq \bigsqcup_{l=1}^{\lfloor \frac{m-n-1}{n} \rfloor} T^{nl}(D).$$

We then let $C = \bigsqcup_{k=Kn+n+1}^{m} C_k$, as in Figure 1. Since for every $k \geqslant Kn+n+1$ we have $\lfloor \frac{k-n-1}{n} \rfloor \geqslant K$, the definition of $C_k$ yields that for all $k \in \{Kn+n+1, \dots, m-1\}$.

$$K\lambda(\tau_{B,T}^{-1}(k)) \leqslant \lambda(C_k). \tag{8}$$

Similarly to Equation (8), for $k = m$ we have

$$K\lambda(D) \leqslant \lambda(C_m). \tag{9}$$

Let us now understand how the return times are affected when adding $C$ to the set $B$. Observe that $C$ is contained in the Kakutani-Rokhlin towers over $B$ whose heights belong to $\{Kn+n+1, \dots, m\}$, in particular the return times of points belonging to the bases of other towers are unaffected.

Fix some $k \in \{Kn+n+1, \dots, m-1\}$, let us compute the return times of the elements of $B \sqcup C$ in the tower of height $k$ over $B$:

(a) For each $l \in \{0, \dots, \lfloor \frac{k-n-1}{n} \rfloor - 1\}$, every $x \in T^{nl}(\tau_{B,T}^{-1}(k)) \subseteq C_k \sqcup B$, the $B \sqcup C$-return time of $x$ is equal to $n$, and

(b) if $x \in T^{n\lfloor \frac{k-n-1}{n} \rfloor}(\tau_{B,T}^{-1}(k)) \subseteq C_k$ then the $B \sqcup C$-return time of $x$ is equal to

$$k - n\left\lfloor \frac{k-n-1}{n} \right\rfloor \in \{n+1, \dots, 2n\}.$$

For $k = m$ we have similarly that

(a') for each $l \in \{0, \dots, \lfloor \frac{m-n-1}{n} \rfloor - 1\}$, and every $x \in T^{nl}(D) \subseteq C_m \sqcup B$, the $B \sqcup C$-return time of $x$ is equal to $n$, while

(b') if $x \in T^{n\lfloor \frac{m-n-1}{n} \rfloor}(D) \subseteq C_m$ its $B \sqcup C$-return time is equal to

$$m - n\left\lfloor \frac{m-n-1}{n} \right\rfloor \in \{n+1, \dots, 2n\}.$$

It follows from the above description of the new return times that for all $k < n$, we have $\tau^{-1}_{B\sqcup C,T}(k) = \tau^{-1}_{B,T}(k)$, which establishes Condition (i). Moreover, it follows from (a) and (a') that

$$\tau^{-1}_{B\sqcup C,T}(n) = \tau^{-1}_{B,T}(n) \sqcup \left( \bigsqcup_{k=Kn+n+1}^{m} T^{-n}(C_k) \right),$$

so that $\lambda(\tau^{-1}_{B\sqcup C,T}(n)) = \lambda(\tau^{-1}_{B,T}(n)) + \lambda(C)$. By Equation (7) and the definition of $m$, the last set $C_m$ used to change part of the height $m$ tower was chosen so as to obtain

$$\lambda(C) = \lambda_n - \lambda(\tau^{-1}_{B,T}(n)), \tag{10}$$

Condition (ii) thus holds. Finally for all $k \in \{n+1, \dots, 2n\}$, the new elements with return time equal to $k$ must come from (b) or (b'), hence we have

$$\tau^{-1}_{B\sqcup C,T}(k) \subseteq \tau^{-1}_{B,T}(k) \sqcup \left( \bigsqcup_{l=Kn+n+1}^{m-1} T^{n\left\lfloor \frac{l-n-1}{n} \right\rfloor}(\tau^{-1}_{B,T}(l)) \right) \sqcup T^{n\left\lfloor \frac{m-n-1}{n} \right\rfloor}(D).$$

We thus have

$$\begin{aligned}
\lambda(\tau^{-1}_{B\sqcup C,T}(k)) - \lambda(\tau^{-1}_{B,T}(k)) &\leqslant \left( \sum_{l=Kn+n+1}^{m-1} \lambda(\tau^{-1}_{B,T}(k)) \right) + \lambda(D) \\
&\leqslant \left( \sum_{l=Kn+n+1}^{m-1} \frac{\lambda(C_k)}{K} \right) + \frac{\lambda(C_m)}{K} \qquad \text{(by Equations (8) and (9)).}
\end{aligned}$$

Moreover, since $C = \bigsqcup_{k=Kn+n+1}^{m} C_k$, we have $\lambda(C) = \sum_{k=Kn+n+1}^{m} \lambda(C_k)$, and hence

$$\lambda(\tau^{-1}_{B\sqcup C,T}(k)) - \lambda(\tau^{-1}_{B,T}(k)) \leqslant \frac{\lambda(C)}{K} = \frac{\lambda_n - \lambda(\tau^{-1}_{B,T}(n))}{K} < \delta,$$

using first Equation (10) and then Inequality (6). It then follows from the definition of $\delta$ that Condition (iii) holds for $k \in \{n+1, \dots, 2n\}$. To conclude, note that for all $k \geqslant 2n+1$ we have $\tau^{-1}_{B\sqcup C,T}(k) \subseteq \tau^{-1}_{B,T}(k)$ so that Condition (iii) fully holds. □

**Lemma 3.2.** *Let $T \in \mathrm{Aut}(X, \lambda)$ be conservative and aperiodic. Let $A \subseteq X$ intersect almost all $T$-orbits. Then for every sequence $(\lambda_n)_{n\geqslant 1}$ such that $\lambda_n > \lambda(\tau^{-1}_{A,T}(n))$ for all $n \geqslant 1$, there is $B$ such that $A \subseteq B$ and for all $n \in \mathbb{N}^*$, $\lambda(\tau^{-1}_{B,T}(n)) = \lambda_n$.*

*Proof.* Lemma 3.1 allows us to inductively build a sequence of disjoint subsets $(C_n)_{n\geqslant 1}$ such that for all $n \geqslant 1$, the following hold:

(i) For all $k < n$, we have $\tau^{-1}_{A\sqcup C_1 \sqcup \cdots \sqcup C_n,T}(k) = \tau^{-1}_{A\sqcup C_1 \sqcup \cdots \sqcup C_{n-1},T}(k)$.

(ii) $\lambda(\tau^{-1}_{A\sqcup C_1 \sqcup \cdots \sqcup C_n,T}(n)) = \lambda_n$.

(iii) For all $k > n$, we have $\lambda(\tau^{-1}_{A\sqcup C_1 \sqcup \cdots \sqcup C_n,T}(k)) < \lambda_k$.

It is then a straightforward consequence of (i) and (ii) that the set $B \coloneqq A \sqcup \bigsqcup_{n\in\mathbb{N}} C_n$ is as desired. □

We can now state and prove the core result of our paper.

**Theorem 3.3.** *Let $T_1, T_2 \in \operatorname{Aut}(X, \lambda)$ be conservative aperiodic. For any $\varepsilon > 0$ there exists two subsets $B_1$ and $B_2$ of $X$ meeting almost all $T_1$ and $T_2$-orbits respectively, of measure $\lambda(B_1) = \lambda(B_2) < \varepsilon$, and such that for any $n \in \mathbb{N}^*$ we have: $\lambda(\tau_{B_1,T_1}^{-1}(n)) = \lambda(\tau_{B_2,T_2}^{-1}(n))$.*

*Proof.* By Lemma 2.2, we can fix two Borel sets $A_1$ and $A_2$ of measure $< \frac{\varepsilon}{2}$, meeting almost all $T_1$-orbits and $T_2$-orbits respectively. We then pick a sequence $(\lambda_n)_{n\geqslant 1}$ such that for all $n \geqslant 1$, $\lambda_n > \max\left\{\lambda(\tau_{A_1,T_1}^{-1}(n)), \lambda(\tau_{A_2,T_2}^{-1}(n))\right\}$ while $\sum_n \lambda_n < \varepsilon$. We conclude by applying Lemma 3.2 to $T_1$ and $A_1$, and then to $T_2$ and $A_2$, both times with this sequence $(\lambda_n)$. □

## 3.2 The dissipative absorption lemma

In order to prove Theorem 1.2, we require the following lemma, which allows the dissipative part to "absorb" the conservative part.

**Lemma 3.4.** *Let $\varepsilon > 0$ and $T \in \operatorname{Aut}(X, \lambda)$ whose conservative and dissipative parts both have positive measure, and let $D$ be a fundamental domain for the dissipative part of $T$. Then there exists a dissipative $\widetilde{T} \in \operatorname{Aut}(X, \lambda)$ such that $\lambda(\{x \in X : T(x) \neq \widetilde{T}(x)\}) < \varepsilon$ and $D$ is also a fundamental domain for $\widetilde{T}$.*

*Proof.* We obtain $\widetilde{T}$ by patching together the dissipative part and the conservative part of $T$ (see Figure 2 for an illustrative example).

Let $D$ be a fundamental domain of the dissipative part of $T$. By Lemma 2.2 we find $B \subseteq \mathfrak{C}_T$ such that $\lambda(B) < \frac{\varepsilon}{2}$, $\lambda(B) \leqslant \lambda(D)$, and $B$ intersects every orbit of the conservative part $\mathfrak{C}_T$ of $T$. We then fix $A \subseteq D$ such that $\lambda(A) = \lambda(B)$. Finally, let $\phi : A \to B$ and $\psi : T^{-1}(B) \to T(A)$ be measure-preserving bijections.

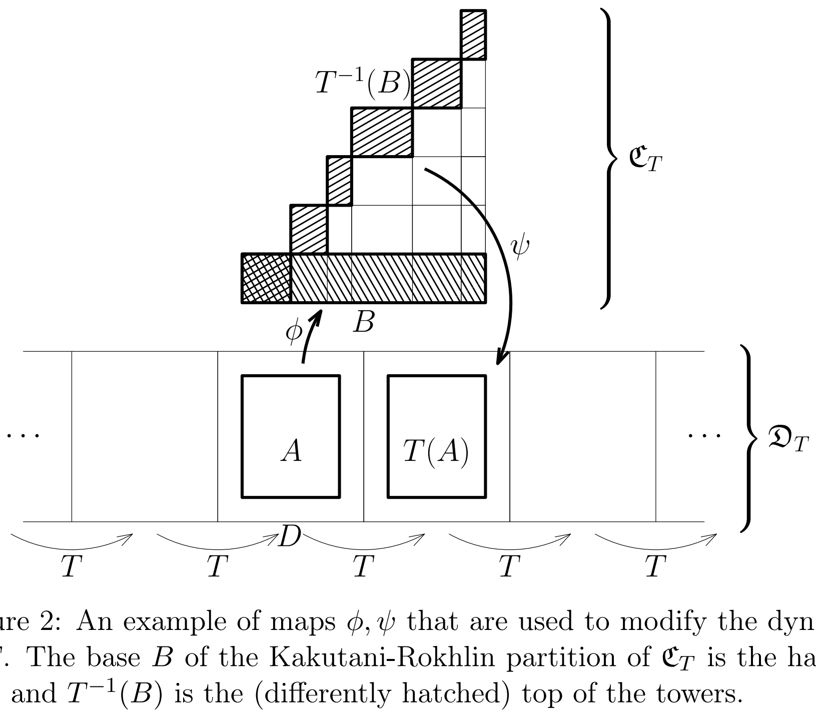


Figure 2: An example of maps $\phi, \psi$ that are used to modify the dynamics of $T$. The base $B$ of the Kakutani-Rokhlin partition of $\mathfrak{C}_T$ is the hatched part and $T^{-1}(B)$ is the (differently hatched) top of the towers.

We define $\widetilde{T}$ as follows:

$$\widetilde{T}(x)=\begin{cases} T(x) & \text{if } x\in X\setminus A\sqcup T^{-1}(B),\\ \phi(x) & \text{if } x\in A,\\ \psi(x) & \text{if } x\in T^{-1}(B).\end{cases}$$

Let us check that $\widetilde{T}$ is dissipative with fundamental domain $D$. Since $D$ is a fundamental domain of the dissipative part of $T$ and since $\widetilde{T}$ and $T$ coincide on $\bigsqcup_{n\leqslant 0} T^n(D)$, all the negative $\widetilde{T}$-translates of $D$ are disjoint from each other. Hence $D$ intersects every $\widetilde{T}$-orbit in at most one point, and $D$ intersects the $\widetilde{T}$-orbit of every element of $\bigsqcup_{n\leqslant 0} T^n(D)$. Denote by $Y$ the $\widetilde{T}$-saturation of $D$, we have to show $Y=X$, and for now we know that $Y\supseteq \bigsqcup_{n\leqslant 0} T^n(D)$. Since $\widetilde{T}$ and $T$ coincide on $\bigsqcup_{n\geqslant 0} T^n(D\setminus A)$, we further have $Y\supseteq \bigsqcup_{n\geqslant 1} T^n(D\setminus A)$. Now because $\widetilde{T}(A)=B$, we have $B\subseteq Y$. If $x\in\mathfrak{C}_T\setminus B$, let $k\geqslant 1$ be the first integer such that $T^{-k}(x)\in B$. Then since $T^{-1}$ and $\widetilde{T}^{-1}$ coincide on $\mathfrak{C}_T\setminus B$, we have $\widetilde{T}^{-k}(x)\in B$, which yields the inclusion $\mathfrak{C}_T\subseteq Y$. Finally, by construction $\bigsqcup_{n\geqslant 1}\widetilde{T}^n(T^{-1}B)=\bigsqcup_{n\geqslant 1} T^n(A)$ so $Y\supseteq\bigsqcup_{n\geqslant 1} T^n(A)$ so $Y=X$, which concludes the proof. □

## 3.3 Proof of Theorem 1.2

Before proving the main result, we isolate one last lemma which shows that the condition on the measure of the dissipative fundamental domains is necessary for $\lambda$-approximate conjugacy to hold.

**Lemma 3.5.** *Let $T_1\in\mathrm{Aut}(X,\lambda)$, let $d_1$ be the measure of a fundamental domain of its dissipative part, let $\varepsilon>0$, and $T_2\in\mathrm{Aut}(X,\lambda)$ such that $\lambda(\{x\in X\mid T_1(x)\neq T_2(x)\})\leqslant\varepsilon$. Then $T_2$ has a dissipative fundamental domain of measure $\geqslant d_1-\varepsilon$.*

*Proof.* Let $D_1$ be a fundamental domain for the dissipative part of $T_1$, then let $C_1$ denote the set of $x\in D_1$ whose $T_1$-orbit intersects the set $A:=\{x\in X\mid T_1(x)\neq T_2(x)\}$, which satisfies $\lambda(A)\leqslant\varepsilon$ by assumption. Since $T_1$ is measure-preserving and $C_1$ is contained both in the fundamental domain $D_1$ and in the $T_1$-saturation of $A$, we then have $\lambda(C_1)\leqslant\lambda(A)\leqslant\varepsilon$ and thus $\lambda(D_1\setminus C_1)\geqslant d_1-\varepsilon$. By construction $T_1$ and $T_2$ coincide on the $T_1$ orbit of any $x\in D_1\setminus C_1$ so $D_1\setminus C_1$ is contained in a fundamental domain for the dissipative part of $T_2$, which finishes the proof. □

*Proof of Theorem 1.2.* We begin by the direct implication: assume that $T_1$ and $T_2$ are $\lambda$-approximately conjugate. Let $d_1$ and $d_2$ denote the respective measures of fundamental domains for the dissipative parts of $T_1$ and $T_2$. Then by Lemma 3.5 we have $d_1\leqslant d_2$, but then by symmetry $d_2\leqslant d_1$ so that $d_1=d_2$.

Let us now show the converse. Denote by $d$ the common measure of a fundamental domain of $T_1$ and $T_2$.

Let us begin with the case $d=0$. We apply Theorem 3.3 and thus get $B_1$, $B_2$ meeting almost all $T_1$ and $T_2$ orbits respectively, with $\lambda(B_1)=\lambda(B_2)<\varepsilon$, and such that for all $n\in\mathbb{N}^*$ we have $\lambda(\tau^{-1}_{B_1,T_1}(n))=\lambda(\tau^{-1}_{B_2,T_2}(n))$. Observe that since $B_1$ meets all $T_1$-orbits, we have a Kakutani-Rokhlin partition of $X$ given by:

$$X=\bigsqcup_{n\geqslant 1}\bigsqcup_{k=0}^{n-1} T_1^k(\tau^{-1}_{B_1,T_1}(n)),$$

and similarly for $T_2$:

$$X = \bigsqcup_{n\geqslant 1} \bigsqcup_{k=0}^{n-1} T_2^k(\tau_{B_2,T_2}^{-1}(n)),$$

Let us fix a measure-preserving bijection $\varphi : B_1 \to B_2$ satisfying $\varphi(\tau_{B_1,T_1}^{-1}(n)) = \tau_{B_2,T_2}^{-1}(n)$ for any $n \in \mathbb{N}^*$. We extend it to a tower-preserving element $S$ of $\mathrm{Aut}(X, \lambda)$ by letting for, for all $n \in \mathbb{N}^*$, all $x \in \tau_{B_1,T_1}^{-1}(n)$ and all $k \in \{0, \dots, n-1\}$

$$S(T_1^k(x)) = T_2^k(\varphi(x)).$$

By construction, we always have $ST_1(x) = T_2S(x)$, except when $x$ is on the top of some $T_1$-tower, *i.e.* when $x \in \bigsqcup_{n\geqslant 1} T_1^{n-1}(\tau_{B_1,T_1}^{-1}(n))$. So

$$\begin{aligned}\lambda(\{x \in X \colon ST_1S^{-1}(x) \neq T_2(x)\}) &\leqslant \sum_{n=1}^{\infty} \lambda(T_1^{n-1}(\tau_{B_1,T_1}^{-1}(n))) \\ &= \sum_{n=1}^{\infty} \lambda(\tau_{B_1,T_1}^{-1}(n))) = \lambda(B_1) < \varepsilon,\end{aligned}$$

as desired.

We are now left with the case $d > 0$ (it can be infinite). If both $T_1$ and $T_2$ are dissipative, then they are actually conjugate, in particular they are $\lambda$-approximately conjugate. By transitivity of $\lambda$-approximate conjugacy, we may then assume without loss of generality that $T_1$ is dissipative but $T_2$ is not: its conservative part has positive measure.

Let us now fix $\varepsilon > 0$ and apply Lemma 3.4 so as to obtain $\widetilde{T}_2$ dissipative which shares a fundamental domain with $T_2$ and satisfies

$$\lambda(\{x \in X \colon T_2(x) \neq \widetilde{T}_2(x)\}) < \varepsilon.$$

Since $T_1$ and $\widetilde{T}_2$ are both dissipative with a fundamental domain of measure $d > 0$, they are conjugate, so there is $S \in \mathrm{Aut}(X, \lambda)$ such that $\lambda(\{x \in X \colon T_2(x) \neq ST_1S^{-1}(x)\}) < \varepsilon$. Since this can be done for all $\varepsilon > 0$, we conclude that $T_2$ and $T_1$ are $\lambda$-approximately conjugate as wanted, which concludes the proof. □

# 4 Beyond $\mathbb{Z}$-actions

In the probability context, given an infinite amenable group $\Gamma$, Ornstein and Weiss' quasi-tiling machinery yields that all free probability measure-preserving $\Gamma$-actions on $(X, \mu)$ are $\mu$-approximately conjugate [OW87]. More generally, it is a result of Elek that given any infinite countable group $\Gamma$, any two hyperfinite probability measure-preserving $\Gamma$-actions are $\mu$-approximately conjugate if and only if they share the same invariant random subgroup [Ele12, Thm. 9] (see also [Gir22, Sec. 2] for a recent more direct proof).

In the infinite measure-preserving world however, our work shows that even for $\Gamma = \mathbb{Z}$, the distribution of point stabilizers does not provide the whole information for $\lambda$-approximate conjugacy since dissipativity is not detected. Trying to extend our main result to other amenable groups, one way around this problem might be to require *elementwise conservativity* of the action, as recently introduced by Glasner and Lederle [GL25].

**Question.** Let $\Gamma$ be an infinite amenable group. Is it true that all elementwise conservative free $\Gamma$-actions on a standard infinite measure space $(X, \lambda)$ are $\lambda$-approximately conjugate, namely for any two such actions $\alpha_1$, $\alpha_2$, all $F \subseteq \Gamma$ finite and all $\varepsilon > 0$, is there $S \in \mathrm{Aut}(X, \lambda)$ such that for all $\gamma \in F$,

$$\lambda(\{x \in X : S\alpha_1(\gamma)S^{-1}x \neq \alpha_2(\gamma)(x)\}) < \varepsilon?$$

We don't know the answer to this question even for $\Gamma = \mathbb{Z}^2$ or $\Gamma$ virtually $\mathbb{Z}$.

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