Université Paris Cité
École Doctorale des Sciences Mathématiques de Paris Centre (ED 386)
Institut de Mathématiques de Jussieu–Paris Rive Gauche (UMR 7586)

Thèse de Doctorat

Spécialité : Mathématiques

On measure-preserving actions of Polish groups on spaces of infinite measure

Présentée par Fabien HOAREAU

Dirigée par François LE MAÎTRE et par Pierre FIMA

Présentée et soutenue publiquement le 2 Juillet 2026 à Paris

Devant un jury composé de :

Emmanuel Roy, MCF	Université Sorbonne Paris-Nord	Rapporteur
Anush Tserunyan, Assoc. Prof.	McGill University	Rapportrice
Claire Debord, PU	Université Paris Cité	Examinatrice
Todor Tsankov, PU	Université Claude Bernard	Examinateur
Pierre Fima, MCF	Université Paris Cité	Directeur
François Le Maître, PU	Université Bourgogne Europe	Directeur

Contents

Résumé

Titre : Sur les actions de groupes polonais préservant une mesure infinie

Mots-clés : Groupes polonais, mesures infinies, mesures localement finies, actions booléennes, actions spatiales, modèles spatiaux, groupes pleins

Résumé : Dans cette thèse nous nous intéressons aux actions préservant la mesure d'un groupe polonais G sur un espace Borélien standard X pourvu d'une mesure non-atomique σ-finie infinie λ. Dès que G est indénombrable, une distinction claire entre l'action spatiale et l'action booléenne associée, c'est-à-dire l'action sur les boréliens identifiés à mesure nulle près, se dessine. De plus, lorsque l'on s'intéresse à des mesures infinies un nouvel obstable apparaît : la mesure peut ne pas être localement finie par rapport à une topologie compatible sur X, ce qui signifie que la topologie et la mesure ne s'accordent pas entre elles.

Dans le premier chapitre nous exposons le contexte de l'étude et en rappelons certains résultats incontournables. Dans le deuxième chapitre, qui est basé sur un travail en commun avec François Le Maître, nous nous attaquons à la recherche de modèles spatiaux sur des espaces polonais équipés de mesures localement finies pour les G-actions préservant la mesure : le résultat principal affirme que l'on peut toujours trouver un tel modèle lorsque G est polonais localement compact. Nous donnons également une caractérisation des actions admettant des modèles spatiaux localement finis et étudions les actions d'autres classes de groupes polonais. Le troisième chapitre est dédié à l'étude des groupes pleins préservant une mesure infinie, et généralise les travaux de Carderi et Le Maître dans le contexte des mesures de probabilité. Plus spécifiquement, nous utilisons les modèles spatiaux du chapitre 2 pour munir tout groupe plein orbital provenant de l'action préservant la mesure d'un groupe polonais localement compact d'une topologie polonaise. Nous présentons ensuite quelques résultats découlant de l'existence d'une telle topologie, parmi lesquels un résultat d'unicité et de comparaison avec les topologies existantes et déjà bien connues, ainsi qu'une caractérisation des éléments génériques de ces groupes pleins.

Abstract

Title: On measure-preserving actions of Polish groups on spaces of infinite measure

Keywords: Polish groups, infinite measures, locally finite measures, Boolean actions, spatial actions, spatial models, full groups

Abstract: In this thesis we are interested in measure-preserving actions of a Polish group G on a standard Borel space X endowed with an atomless infinite σ-finite measure λ. As soon as G is uncountable, there is a clear distinction between the spatial action on the space and the associated boolean action on the Borel subsets, identified up to measure 0. Moreover, with infinite measures a new hurdle appears: the measure can fail to be locally finite with regard to a compatible topology on X, meaning that the topology and the measure do not behave well together.

In the first chapter we describe the context and recall some of the major results of the relevant fields of study. In the second chapter, which is based on a joint work with François Le Maître, we tackle the problem of finding spatial models on Polish spaces endowed with locally finite measures for infinite measure-preserving G-actions: the main result is that we can always find such a model when G is a locally compact Polish group. We also give a characterization of actions admitting locally finite spatial models, and discuss what happens for other classes of Polish groups and their actions. The third chapter is dedicated to the study of full groups preserving infinite measures, generalizing the work of Carderi and Le Maître in the probability measure-preserving context. More specifically, we use the spatial models of chapter 2 to endow any orbit full groups coming from an infinite measure-preserving action of a Polish locally compact group with a Polish topology. We then present some properties that follow from the existence of such a topology, including a uniqueness result and a comparison with the other well-known existing topologies, and a characterization of the generic elements in those full groups.

We're going to make this world a more open place. We'll confront the existing order and expand new horizons for future generations. And how do we do that? Even the most eloquent voice will fade if it ceases to speak. Even the firmest pillar will decay. Everyone dies. Everything is in flux. Even so, we know the way. The way to strengthen our convictions, visualize our thoughts, and turn seconds into eternity. We will carry this moment far away to some distant future or to the hands of another. We know how it can be done. Our method is impactful. We have to power to dissolve the existing order and social classes. We will make it happen. Let us now begin our publishing operation.

Schmitt in *On the movements of the Earth*

Remerciements

We are not simple creatures. You dream that with memories will come knowledge, and from knowledge understanding. But for every answer you find, a thousand new questions arise. All that we were has lead us to where we are, but tells us little of where we're going.

Apsalar in *Deadhouse Gates*, Steven Erikson

En tout premier lieu je tiens à sincèrement remercier mes directeurs de thèse, Pierre Fima et François Le Maître. Merci à Pierre, sans qui cette thèse n'aurait pas été possible, et pour son soutien depuis le cours de M2. François, merci de m'avoir fait comprendre l'essence de la théorie de la mesure[1]. Même si les maths c'est dur, merci d'avoir pris le temps de m'expliquer autant de choses, (presque[2]) toujours sans la moindre hésitation. Plus sérieusement, merci pour ta patience, ta confiance, et la générosité avec laquelle tu partages tes idées et tes opinions, mais également pour n'avoir jamais compromis sur la rigueur. Merci d'avoir cru en moi, et d'avoir contrecarré mon pessimisme avec un optimisme encore plus marqué.

I would like to sincerely thank the two referees of this thesis, Anush Tserunyan and Emmanuel Roy. I was genuinely very happy when you both accepted to review this thesis. Thank you for all the kind and encouraging words, and for your valuable feedback. Un grand merci également à Claire Debord et à Todor Tsankov, qui ont accepté de faire partie du jury de cette thèse. C'est à la fois un honneur et un plaisir.

Merci à tous les membre de l'équipe AO, au sein de laquelle, peu importe les circonstances, je me suis toujours senti à ma place. Merci en particulier à Georges et à Stéphane, qui ont réussi à imposer le sourire comme tenue de travail. Un grand merci également à Romain, qui a été un très bon tuteur, et dont la bonne humeur est si communicative.

Des remerciements très spéciaux sont adressés à Amandine, Colin, et Matthieu. Amandine, merci de montrer la voie à suivre de manière aussi exemplaire, à moi et à beaucoup d'autres. Merci pour ta gentillesse, c'est toujours un si grand plaisir de te croiser, à Sophie Germain ou en conf, j'espère que ça arrivera encore beaucoup de fois. Merci Colin pour ce super séjour à Dresde, et merci d'être la personne que tu es, quelqu'un avec qui je peux avoir les discussions qu'on a eues, qui comptent beaucoup pour moi, mais avec qui je peux aussi jouer à Hadès pendant des heures. Merci Matthieu de m'avoir invité à donner mon tout premier exposé en séminaire, et merci surtout pour ta disponibilité et ton attention. Je suis très heureux que les discussions dans le bureau 709 (et ses alentours) soient toujours aussi agréables, et stimulantes, malgré un certain changement. Je n'aurais pas pu espérer mieux.

Merci à tout le personnel administratif, dont l'aide a été si souvent précieuse. Je pense

[1] "Cet énoncé est vrai... Avec une grande probabilité." – F. Le Maître, 08/2023

[2] "C'est bon j'ai confiance, je te fais la preuve ! [...] Hmm... Ça a pas l'air facile en fait..." – F. Le Maître, 04/23

en particulier à Dominique Brena, Claire Lavollay, Amina Hariti, et Julia Roux.

Merci à tous les gens que j'ai rencontrés au fil des conférences, séminaires, groupes de travail ou autres occasions diverses, et qui contribuent à rendre ce milieu mathématique si vivant : Claire Anantharaman, Bruno Duchesnes, Amaury Freslon, Damien Gaboriau, Camille Horbez, Tomás Ibarlucía, David Kerr, Sébastien Martineau, Julien Melleray, Samuel Petite, Jean Renault, Yves Stalder, Andreas Thom, Alain Valette, ainsi que Pénélope Azuelos, Rémi Barritault, Corentin Le Bars, Sasha Bontemps, Anna Cascioli, Max Carter, Soham Chakraborty, Moudrik Chamoux, Antoine Derimay, Alon Dogon, Milan Donvil, Tom Ferragut, Felipe Flores, Martin Gilabert Vio, Yassine Guerch, Sasha Gurieva, Waltraud Lederle, Samuel Mellick, Noé de Rancourt, Spyridon Petrakos, Antoine Poulin, Eduardo Silva, Chinmay Tamhankar. Special thanks to Nachi Avraham Re'em, who does some very interesting maths, and shares them with kindness.

Merci à tous les doctorants et post-docs du bâtiment Sophie Germain, que j'ai cotoyés pendant de nombreuses (calmez-vous) années. Si ça a (presque) toujours été un plaisir de venir travailler ici, c'est en grande partie grâce à vous tous.

Merci tout d'abord à la génération qui m'a accueillie pendant mon stage de M2. Merci Alice, Charles, et Macarena, je me suis senti à l'aise si vite dans notre bureau. Merci à Tancrède, qui réapparaîtra dans ces remerciements, et merci à Omar, que je remercie dans ce paragraphe, car il reste pour moi le "sheriff goûter". Par peur d'oublier des gens (c'était il y a quand même un bout de temps) je ne vais pas essayer de citer plus de gens, mais je garde de très bons souvenirs de cette période.

Merci à tous les membres du bureau 745, passés et présents. Merci à Laura et Oussama qui m'y ont accueilli avec gentillesse (et avec des biscuits) il y a de cela quelques années déjà. Merci à Paul avec qui on a plus discuté de theorycrafting de Pokémon Gen 3 que de maths. Merci à Kémo, qui entame même des discussions maintenant, à mon plus grand plaisir. Thank you Kostya, we pick fights with each other all the time (well mostly you, to be honest), but we still manage to have a lot of fun. Merci Yihui, Marius et Paulo, d'avoir contribué à rendre l'ambiance de ce bureau aussi confortable, et pour avoir subi mes (quelques (?)) crashouts sans broncher. Vous êtes tops.

Merci à Élie dont la venue (semi-aléatoire) à Sophie Germain me fait énormément plaisir, à Tal et Gabriel qui ont fait tant d'efforts pour le bien-être des doctorants, à Alexandre, avec qui j'ai pu discuter de sujets très variés, toujours de manière très agréable, à Juan qui m'a toujours impressionné par la diversité et la richesse de ses connaissances, à Pierre, qui a un jour élu domicile dans mon bureau, et qui a failli m'oublier dans ses remerciements, malgré mes menaces. Merci à Merlin et Vincent, organisateurs du sympathique Tea seminar. Merci à Antonio, qui partage son expérience, et ses expréiences uniques, toujours avec bonne humeur. Merci également à Dorian, Jérémie, Maud, Maxime, Mingkun, Victor, et Skander.

Merci à Vincent, qui me fûme comme pas possible et grâce à qui j'ai eu un diplôme (!), à Mathieu, qui a parfois été mon compagnon d'infortune à rester au labo jusqu'à des heures pas possibles, à Léo, que je vais déranger bien trop souvent, et que j'espère écouter jouer à nouveau. Merci à Werner, pour son aide en (La)TeX bien sûr, mais plus généralement pour sa générosité et sa bonne humeur, merci à Paul, un peu étourdi parfois, mais toujours si attentionné, merci à Ivory, au sens de l'humour "tout ou rien", et qui mathématiquement raconte plein de trucs super intéressants, et merci à Alexis, que j'ai appris à connaître, et qui a appris à me connaître. Merci à Brian, qui partage tant de centres d'intérêt avec moi, même s'il est encore plus à la bourre sur One Piece, à Razvan, avec qui j'espère rediscuter d'Animes, à Enzo, –dit Tanguy–, que j'apprécie beaucoup même si c'est une cible facile, et à Céline, qui regardera un jour mes recommandations, j'espère. Merci à MC, avec qui ça a été un réel plaisir de discuter de tout, d'Arcane ou de choses déprimantes (et avec souvent des avis assez similaires), à Lior, qui a magnifiquement (?) décoré notre bureau, et pour les commandes de Laurettes, à Juan Ra qui a apporté tant de vie dans le labo et à Francesca, universellement reconnue pour sa gentillesse. Merci à An(y)atole, qui est très cool même si parfois un peu (!) pointilleux (et il n'y a aucun mal à ça !), à Armande, pour ses petits mots gentils, meême si on se croise malheureusement de moins en moins à SG, to Filipp who is so good at video games, and most likely at math as well, although I really can't tell, as I don't understand what he's doing. Many thanks also to Alessio, who was only here for a short, but oh-so pleasant time, et merci à Eliot, qui, en tant que stagiaire, à réussi à s'intégrer en un temps record. Merci également à Lucio, Alessandro, Michele, Hector, Ariane, Yousra, Dia, Quentin, et Francesca, ainsi qu'à tous ceux que j'ai oublié, et qui, je l'espère, sauront me pardonner. Merci également à Jérôme, doctorant honoraire (?) tous les jours, de midi à quatorze heures.

Merci à William, qui m'a si gentiment demandé de l'aider pour son four, et qui envoie des memes encore plus cursed que les miens somehow, et à Agathe, avec qui on a si souvent partagé nos déboires. Merci à Arthur, pour son amitié, et pour avoir accepté mon introversion sans trop me charrier, à Alberto, que j'espère apprendre à mieux connaître, et à Mario, qui crée un vide impossible à combler les jours où il n'est pas au labo.

Merci à Corentin –dit Corenlol– qui m'a tant aidé que je lui pardonnerai presque d'avoir soutenu avant son "grand(?) frère" de thèse. Merci pour tes nombreuses relectures, et pour avoir partagé tes maths avec autant d'enthousiasme. Pour la peine, je serai partant pour une deuxième tentative d'empois... un deuxième resto sichuanais, et ce malgré ton humour, qui, malgré toutes mes plaintes, a un (tout petit) peu déteint sur moi.

Merci à mes collègues de l'EN, en particulier à ma tutrice Cathy au collège Jean Rostand, aux collègues du lycée Alain, qui ont été si gentils avec moi, et bien sûr aux collègues du lycée Dumas, où j'ai passé les cinq dernières années. J'espère qu'ils se reconnaîtront et me pardonneront de ne pas tous les nommer. Des remerciements tout particuliers sont adressés à Tancrède, mon double-collègue qui m'a gentiment proposé de khôller, ainsi qu'à Claudine, qui s'attache à défendre tous ses collègues.

Il est temps de remercier mes amis de longue date, qui m'ont, de par leurs encouragements incessants, donné la force de tenir bon dans les moments difficiles. Si je n'ai pas baissé les bras, c'est en grande partie grâce à vous tous et ce que vous m'avez apporté.

Merci de tout mon coeur aux "Champions" : Perrine, Julie, Laurène, Etienne, Julien, Christian, Arnaud, Valentin, et Manu. À chaque fois qu'on repasse du temps ensemble, même après parfois des mois de silence radio de ma part, je réalise de nouveau la chance que j'ai de vous avoir. Merci de m'accepter comme je suis. Un merci encore plus spécial à Perrine et Julie, qui ont si souvent une pensée pour moi. Je ne sais pas ce que j'ai fait pour mériter ça, mais je ne vais certainement pas m'en plaindre.

Merci à Pablo, qui partage ses passions (et son temps) avec tant d'enthousiasme qu'il est impossible de ne pas se laisser emporter, à Iris, chez qui je suis toujours accueilli royalement, et avec qui j'espère échanger encore des recommandations de restaurants. Merci à Paul, avec qui je me sens si compris quand on discute de nos expériences communes, et à Aude, qui m'a traité si gentiment dès la première fois où je l'aie rencontrée. Merci Masanet, 90% du temps dans nos conversations on s'envoie des memes sans répondre aux messages précédents, et les 10% restants sont à peine explicables à vrai dire. Merci de me supporter depuis aussi longtemps.

Merci à Nhi et à Wing Szr pour les Fatous, où on sort des restos en roulant, et pour les desserts à la mangue. Je suis très content d'avoir repris contact, et à bientôt j'espère pour de nouvelles aventures (gastronomiques). Merci également à Hugo, avec qui on a repris contact ces dernières années, alors qu'on était tous les deux en région Parisienne... Pas surprenant venant de nous à vrai dire.

Si j'avais un chien, je le remercierais.

Marie, merci infiniment à toi. Pour ton amitié, pour les restos, d'organiser des choses

quand on ne s'est pas vus depuis longtemps, et de me laisser du temps quand j'en ai besoin. Quand on fera immanquablement (!) un film sur ta vie, j'espère que je pourrais jouer mon rôle. Merci à Guillaume, de m'avoir fait une place dans ta zone de confort. C'est toujours un plaisir de discuter de Malazan, de MH, de HK, et j'en passe. Merci à tous les deux de m'avoir si bien accueilli chez vous que je m'y sens comme chez moi.

Enfin merci du plus profond de mon coeur à ma famille. À mes yiyis, mes tâties, mes tontons, mes cousins et cousines, grands ou petits, à Cory, à Nicoletta, mais surtout, merci à mes parents, à mon frère et à ma soeur. Merci pour votre soutien indéfectible, que j'apprécie même si je suis parfois peu démonstratif. Merci d'avoir transformé mes "Si j'y arrive, [...]" en "Quand tu y arriveras, [...]".

But one thing I now know. I made it through all of childhood, and not once did I learn about surrendering.

Wreneck in *Fall of Light*, Steven Erikson

Résumé long en français

Ce résumé long en français est une traduction de certains éléments essentiels de l'introduction générale en anglais.

Contexte mathématique et historique

La théorie ergodique des actions de groupes est un domaine qui s'intègre dans le cadre de l'étude des systèmes dynamiques, et s'intéresse aux actions de groupes sur des espaces mesurables munis de mesures (quasi)-invariantes. Ce point de vue étend l'étude plus classique des actions de $\mathbb{Z}$ au cas des groupes non discrets, incluant ainsi les actions de $\mathbb{R}$, aussi connues sous le noms de flots préservant la mesure. Dans cette thèse nous nous intéressons aux actions de groupes polonais, généralement notés G, sur des espaces Boréliens standards X. Ces espaces Boréliens standards sont équipés d'une mesure sans atome, notée ν dans le cas général. Tous les espaces mesurés standards de même masse totale finie sont isomorphes, justifiant la terminologie d'*espace de probabilité standard* (dans ce cas nous dénoterons par μ la mesure), et la notion d'*espace σ-fini standard* est également bien définie (dans ce cas nous dénoterons la mesure par λ). Le groupe des bijections de X préservant la mesure ν est noté $\mathrm{Aut}(X, \nu)$ (deux telles bijections étant identifiées si elles coïncident sur un ensemble de mesure pleine). Ce groupe sera pour nous un objet central, puisque qu'au travers d'une G-action préservant la mesure, on peut identifier un élément de G à un élément de $\mathrm{Aut}(X, \nu)$.

Commençons par le cas des actions de groupes dénombrables sur un espace de probabilité standard (X, μ). L'étude des actions préservant une mesure de probabilité (pmp) est un domaine très actif, et l'un de ses objectifs est la classification des ces actions à isomorphisme près. Bien que des résultats importants aient été obtenus, à l'instar de la classification des décalages de Bernoulli par leur entropie, un résultat d'Ornstein ([Orn70]), Foreman, Rudolph et Weiss ([FRW11]) ont démontré que la relation de conjugaison sur l'ensemble des bijections ergodiques de (X, μ) n'est pas Borélienne. Il est alors assez naturel de considérer un affaiblissement de cette notion : la notion plus grossière d'équivalence orbitale, ou orbite-équivalence (OE). Deux actions α et β de deux groupes dénombrables Γ et Λ respectivement sur (X, μ) sont orbitalement équivalentes (ce que l'on dénotera également par OE) s'il existe un élément $S \in \mathrm{Aut}(X, \mu)$ tel que pour presque tout $x \in X$:

$$S(\alpha(\Gamma, x)) = \beta(\lambda, S(x)).$$

Autrement dit, l'OE oublie l'action, et ne garde en mémoire que la partition de l'espace en orbites. Dye a prouvé 1959 que l'OE est trop grossière et ne permet pas de différencier des actions ergodiques de $\mathbb{Z}$ ([Dye59]), et Ornstein et Weiss ont généralisé ce résultat à toutes les actions ergodiques de groupes dénombrables moyennables ([OW80]). Mais bien que l'OE ne parvienne pas à distinguer différents systèmes ergodiques, elle permet quand même de capturer certaines propriétés du groupe agissant. On appelle traditionnellement théorie mesurée des groupes le domaine d'étude s'intéressant aux liens entre le groupe agissant et la partition en orbites associée, et c'est un domaine présentant des liens riches avec de nombreux autres domaines (voir [Gab11], [Fur11] pour des panoramas du sujet, voir également [Kec10] pour un point de vue plus global).

L'un des principaux outils utilisés par Dye est le groupe plein de la relation d'équivalence $\mathcal{R}_\Gamma$ associée à une action d'un groupe dénombrable Γ sur un espace de probabilité standard, définie par $(x, y) \in \mathcal{R}_\Gamma$ si et seulement s'il existe $\gamma \in \Gamma$ tel que $\gamma \cdot x = y$. Le groupe plein $[\mathcal{R}_\Gamma]$ de $\mathcal{R}_\Gamma$ est défini comme suit :

$$[\mathcal{R}_\Gamma] \coloneqq \{T \in \mathrm{Aut}(X, \mu) \mid \text{pour } \mu\text{-presque tout } x \in X : (x, T(x)) \in \mathcal{R}_\Gamma\} .$$

Ce groupe est constitué des éléments de $\mathrm{Aut}(X, \mu)$ respectant les Γ-orbites, et encode l'OE : deux actions de deux groupes dénombrables Γ et Λ sont OE si et seulement si leurs groupes pleins sont conjugués dans $\mathrm{Aut}(X, \mu)$. En outre, un impressionnant théorème de Dye, le Théorème de Reconstruction (voir le Théorème A.1.1), affirme que deux actions ergodiques sont OE si et seulement si leurs groupes pleins associés sont isomorphes en tant que groupes topologiques (et en fait en tant que groupes abstraits). Un problème naturel est alors d'examiner les liens entre les proprétés (mesurées) de l'action et les propriétés (topologiques) du groupe plein. Pour ce faire, une première étape naturelle est de munir les groupes pleins d'une topologie polonaise. Dans ce cadre restreint, une telle topologie existe toujours : la topologie uniforme, engendrée par la distance bi-invariante complète

$$d_u(S, T) \coloneqq \mu(\{x \in X \mid S(x) \neq T(x)\}),$$

est une topologie faisant de $[\mathcal{R}_\Gamma]$ un groupe polonais, et cette approche s'est montrée très fructueuse (voir [GP07], [KT10], [LM14b], [EG16]).

L'étape suivante étant alors de se demander ce qu'il se passe en dehors du monde dénombrable, considérons un groupe polonais G quelconque. Carderi et Le Maître considèrent dans [CLM16] la relation d'équivalence associée à une G-action Borélienne sur un espace de probabilité standard, et définissent le groupe plein *orbital* associée de manière analogue : $[\mathcal{R}_G] \coloneqq \{T \in \mathrm{Aut}(X, \mu) \mid \forall x \in X : (x, T(x)) \in \mathcal{R}_G\}$. La topologie uniforme peut ne plus être séparable sur $[\mathcal{R}_G]$ dès lors que G est indénombrable. La question naturelle est la suivante : peut-on trouver une topologie de groupe polonaise sur $[\mathcal{R}_G]$? Le résultat principal de [CLM16] est le suivant : tout groupe plein orbital peut être équipé d'une nouvelle topologie, appelée topologie de la convergence orbitale en mesure, qui en fait un groupe polonais. Carderi et Le Maître prouvent de plus que cette topologie est intermédiaire entre la topologie uniforme et la topologie faible, qui est l'autre topologie bien connue sur $\mathrm{Aut}(X, \mu)$ et qui en fait un groupe polonais.

Et en mesure infinie ?

La topologie faible et la topologie uniforme sont déjà bien définies sur $\mathrm{Aut}(X, \lambda)$, et ainsi le premier objectif de cette thèse a été d'étendre la définition de la topologie de la convergence orbitale en mesure au cas des actions préservant une mesure σ-finie infinie (∞-mp). De nombreuses difficultés sont à prévoir : le théorème de récurrence de Poincaré ne s'applique plus, les exemples classiques de bijections ergodiques préservant la mesure n'existent pas dans ce monde (voir l'Appendice B pour des exemples de telles bijections), la notion de mélange y est considérablement plus compliquée, *etc*... Le groupe $\mathrm{Aut}(X, \lambda)$ lui-même est plus compliqué que $\mathrm{Aut}(X, \mu)$, car il contient notamment le sous-groupe distingué constitué des bijections ayant un support de mesure finie, alors que $\mathrm{Aut}(X, \mu)$ est un groupe simple.

La principale difficulté pour étendre le résultat de Carderi et Le Maître a cependant été quelque peu inattendue. La topologie de la convergence orbitale en mesure provient de la topologie de la convergence en mesure sur l'espace $\mathrm{L}^0(X,\mu,G)$ des applications mesurables de X à valeurs dans G. Plus précisément, la topologie de la convergence orbitale est définie grâce à la fonction

$$\begin{array}{cccc} \Phi & : & \mathrm{L}^0(X,\mu,G) & \longrightarrow & \mathrm{L}^0(X,\mu,X) \\ & & f & \longmapsto & (x \mapsto f(x)\cdot x) \end{array}$$

et en voyant $\mathrm{Aut}(X,\mu)$ comme topologiquement plongé dans $\mathrm{L}^0(X,\mu,X)$. C'est là que réside le problème : les topologies sur $\mathrm{Aut}(X,\mu)$ et sur $\mathrm{L}^0(X,\mu,X)$ sont différentes, et ce plongement est lié à la topologie de X. L'espace X n'a cependant pas de topologie intrinsèque, en tant que Borélien standard, et pour donner du sens à cette construction les auteurs utilisent un *modèle spatial continu* de l'action, c'est-à-dire une réalisation de l'action sur un espace topologique compatible, indistinguable de l'action initiale du point de vue de la mesure. La difficulté principale est alors la suivante : lorsque X est muni d'une mesure infinie, il est possible que la mesure ne s'accorde pas bien avec la topologie donnée par un modèle spatial continu. Plus précisément, on dit que la mesure est *localement finie* lorsque tout point admet un voisinage de mesure finie, et il n'y a aucune garantie que cette propriété soit vérifiée lorsque l'on passe à un modèle spatial continu.

À la recherche de modèles spatiaux localement finis

Le travail relatif à cet axe de recherche est exposé dans le Chapitre 2, et a mené à un article, écrit en collaboration avec François Le Maître.

Décrivons brièvement la notion de modèle spatial dans un grand niveau de généralité. On fixe dans toute cette section un groupe polonais G et un espace mesuré standard (X,ν). Une application d'action $\alpha : G\times X\to X$ définie partout et telle que pour tout $g\in G$ l'application $\alpha(g,\cdot)$ soit une bijection préservant ν est appelée une action spatiale. D'autre part, on peut définir une action booléenne comme un homomorphisme de groupes continu entre G et $\mathrm{Aut}(X,\nu)$. On peut facilement associer à toute action spatiale une action booléenne, mais le sens retour, généralement nommé *problème de réalisation spatiale*, est loin d'être aisé. Glasner, Tsirelson et Weiss, dans [GTW05], ont donné un second souffle à l'étude de ce problème en montrant que les groupes de Lévy, une certaine classe de groupes polonais, ne pouvaient jamais avoir d'actions spatiales non triviales, alors qu'ils peuvent admettre des actions booléennes non-triviales.

La question du lien entre l'action booléenne et l'action spatiale associée (qui n'existe pas toujours) peut en fait être décomposée en deux parties complémentaires :

– **Le problème de réalisation spatiale :** Étant donnée une G-action booléenne, peut-on trouver une G-action spatiale donnant une action booléenne qui est booléennement isomorphe à l'action de départ ?

– **Le problème d'isomorphisme :** Si deux actions spatiales ont des actions booléennes associées qui sont isomorphes, sont elles isomorphes en tant qu'actions spatiales ?

La réponse est claire dans le cas où G est dénombrable, et en fait un théorème fondamental de Mackey répond positivement aux deux questions dans le cas plus général où G est un groupe localement compact polonais (de manière équivalente, groupe localement compact à base dénombrable d'ouverts, que l'on appellera généralement groupe lcsc).

Revenons-en à notre problème original. On fixe une action spatiale ∞-mp α d'un groupe polonais G sur un espace σ-fini standard (X,λ). Notre objectif est de trouver un modèle spatial continu qui est isomorphe à l'action de départ, et tel que la mesure soit localement finie par rapport à la topologie. Plus précisément, les deux questions à se poser sont les suivantes : l'action booléenne associée à α provient-elle d'une action continue sur un espace polonais muni d'une mesure localement finie ? A-t-on une réponse positive au problème d'isomorphisme entre α et cette nouvelle action ? Si la réponse à la première question est positive, on dit que l'action considérée est une *réalisation localement finie* de l'action booléenne associée à α.

Concernant le problème de réalisation localement finie, nous caractérisons l'existence de tels modèles en termes de fonctions G-continues, une classe de fonctions introduite dans [GTW05]. Ces fonctions sont les briques permettant de construire les espaces de nos modèles continus *via* l'isomorphisme de Gelfand, et la mesure est définie grâce au théorème de représentation de Riesz. Pour obtenir un modèle satisfaisant, le problème revient donc à *trouver assez de fonctions G-continues.* Nous prouvons le théorème suivant.

Théorème (voir Théorème C)**.** Soit G un groupe polonais, et soit $\alpha : G \to \mathrm{Aut}(X, \lambda)$ une G-action booléenne préservant la mesure sur un espace σ-fini standard (X, λ). Les assertions suivantes sont équivalentes :

(1) l'algèbre $\mathcal{G}$ des fonctions G-continues vérifie

$$\overline{\mathcal{G} \cap \mathrm{L}^1(X, \lambda)}^{\|\cdot\|_1} = \mathrm{L}^1(X, \lambda);$$

(2) l'algèbre $\mathcal{G}$ des fonctions G-continues vérifie

$$\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda);$$

(3) X peut être muni d'une topologie polonaise localement compacte induisant sa structure Borélienne standard et telle que λ soit une mesure de Radon, et α se relève en une G-action spatiale continue préservant la mesure.

Ce théorème s'applique à tout groupe polonais, mais le problème d'isomorphisme n'est pas résolu lorsque G n'est pas lcsc, et nous exhibons en fait des exemples d'actions de groupes non lcsc qui sont booléennement isomorphes, mais pas spatialement isomorphes (voir la Section 2.4.3). Nous obtenons cependant une conclusion satisfaisante à notre quête pour les groupes lcsc : en utilisant la convolution comme source de G-continuité, on peut vérifier que toute G-action satisfait les conditions équivalentes du théorème précédent, puis appliquer le théorème de Mackey pour résoudre le problème d'isomorphisme.

Théorème (voir Théorème A)**.** Soit (X, λ) un espace σ-fini standard, et soit G un groupe localement compact polonais. Soit de plus $\alpha : G \times X \to X$ une G-action sur X préservant la mesure. Alors α est spatialement isomorphe à une G-action continue préservant la mesure sur un espace polonais localement compact Y muni d'une mesure de Radon η.

Il est en fait possible d'obtenir mieux que la conclusion du théorème précédent : il est possible de transporter l'espace X tout entier pour en faire un modèle localement fini, et pas seulement un sous-ensemble de mesure pleine.

Théorème (voir Théorème B)**.** Soient (X, λ) un espace σ-fini standard, G un groupe localement compact polonais, et $\alpha : G \times X \to X$ une G-action préservant la mesure. Alors il existe une G-action continue préservant la mesure sur un espace polonais localement compact Y et une injection Borélienne G-équivariante $\Phi : X \to Y$ telle que la mesure $\Phi_*\lambda$ soit une mesure de Radon sur Y.

Nous obtenons également un résultat général sur l'existence de modèles spatiaux localement finis pour une grande classe de groupes : les groupes d'isométries d'espaces localement compacts séparables. Cette classe contient notamment les groupes lcsc, et les groupes polonais non-archimédiens. Le théorème suivant est une généralisation en mesure infinie du théorème de Kwiatkowska et Solecki ([KS11, Thm. 1.1]).

Théorème (voir Théorème D)**.** Soit G un groupe d'isométries d'un espace localement compact séparable. Toute G-action booléenne préservant la mesure sur un espace σ-fini standard (X, λ) peut se relever en une action spatiale continue préservant la mesure sur X, muni d'une topologie polonaise localement compacte induisant sa structure Borélienne, et telle que λ soit une mesure de Radon.

Enfin, revenons sur un résultat déjà mentionné, le théorème de Glasner, Tsirelson et Weiss sur la non-existence d'actions spatiales non triviales de groupes de Lévy. En construisant un Processus Ponctuel de Poisson sur X grâce à l'espace Borélien d'Effros, ne dépendant pas intrinsèquement d'une quelconque métrique sur X, nous prouvons le théorème suivant. Il est à noter que ce résultat a été prouvé précédemment et indépendamment par Avraham-Re'em et Roy dans [AR25], avec une construction plus générale que la notre.

Théorème (voir Théorème E)**.** Toute action spatiale préservant la mesure d'un groupe de Lévy G sur un espace polonais X muni d'une mesure σ-finie localement finie sans atome λ est triviale, c'est-à-dire que l'ensemble $\{x \in X \mid \forall g \in G, g \cdot x = x\}$ est de mesure pleine.

Groupe pleins préservant une mesure infinie

Nous pouvons désormais utliser les résultats du Chapitre 2 pour définir la topologie de groupe polonaise sur les groupes pleins orbitaux préservant une mesure infinie. Ce travail a mené à une seconde prépublication, qui est présentée dans le Chapitre 3 de ce manuscrit.

À nouveau, fixons un espace σ-fini standard (X, λ), et si rien de plus spécifique n'est mentionné, G désignera un groupe polonais quelconque. La version la plus générale de notre théorème principal est la suivante.

Théorème (voir Théorème G)**.** Soit G un groupe polonais agissant de manière continue sur un espace polonais (X, τ_X) muni d'une mesure σ-finie sans atome λ, qui est localement finie sur τ_X. Le groupe plein orbital $[\mathcal{R}_G]$ muni de la topologie de la convergence orbitale en mesure est un groupe polonais.

En particulier, en utilisant un modèle spatial localement fini comme ceux du Chapitre 2, on peut se ramener à ces hypothèses à partir d'une action préservant la mesure, sans hypothèses topologiques. C'est notamment toujours possible lorsque le groupe agissant est un groupe localement compact polonais !

Comme mentionné précédemment, les groupes pleins pmp provenant d'actions de groupes dénombrables sont des invariants complets d'OE pour les actions ergodiques. Dans [CLM16] les auteurs prouvent que cette caractérisation est toujours valide dans le cas où le groupe agissant est un groupe lcsc. En se servant du Théorème de Reconstruction de Fremlin, (l'Appendice A est dédié à ce théorème, qui généralise le Théorème de Reconstruction de Dye), nous adaptons la preuve de Carderi et Le Maître et obtenons le résultat suivant.

Théorème (voir Théorème H)**.** Soient G et H deux groupes polonais localement compacts agissant de manière ergodique en préservant la mesure sur un espace σ-fini standard (X, λ). Soit également $\Psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ un isomorphisme de groupe. Alors, il existe une équivalence orbitale S entre $\mathcal{R}_G$ et $\mathcal{R}_H$, qui préserve la mesure à multiplication par un élément de $\mathbb{R}_+^*$ près, et telle que Ψ soit la conjugaison par S.

Étant donné que les groupes pleins capturent certaines propriétés orbitales des actions même dans le cas où les groupes agissant ne sont pas dénombrables, il est naturel d'en étudier les propriétés algébriques et topologiques. Commençons par nous intéresser plus en détail au comportement de la topologie polonaise (existant parfois) sur les groupes pleins, dans un plus grand niveau de généralité.

Rappelons la définition originelle des groupes pleins, introduite par Dye dans [Dye59] : les groupes pleins obtenus par *découpage et recollement*. Un élément T de $\mathrm{Aut}(X, \lambda)$ est obtenu en découpant et recollant la suite (T_n) d'éléments de $\mathrm{Aut}(X, \lambda)$ s'il existe une partition (A_n) de X telle que pour tout $n \in \mathbb{N}$, $T_{\restriction A_n} = T_{n \restriction A_n}$. Un sous-groupe $\mathbb{G}$ de $\mathrm{Aut}(X, \lambda)$ est un *groupe plein* s'il est stable par découpage et recollement de toute suite d'éléments de $\mathbb{G}$.

Bien qu'il ne soit *a priori* pas possible d'équiper tout groupe plein d'une topologie de groupe polonaise, il est quand même possible d'étudier les liens entre les différentes topologies possibles, à savoir la topologie faible et la topologie uniforme. En particulier, les topologies de groupes polonaises sont très rigides, un résultat classique datant de Banach affirme que tout homomorphisme de groupes entre groupes polonais est continu dès lors qu'il est Borélien. Ce résultat de *continuité automatique* et sa généralisation au sens de Rosendal ([Ros05]) sont des ingrédients essentiels du théorème suivant, généralisation de [CLM16, Thm. 4.7].

Théorème (voir Théorème I)**.** Soit $\mathbb{G}$ un groupe plein ergodique sur un espace σ-fini standard. Alors $\mathbb{G}$ possède au plus une topologie de groupe polonaise, et une telle topologie est nécessairement plus fine que la topologie faible, et plus grossière que la topologie uniforme.

Les deux résultats suivants concernent un groupe déjà évoqué, qui n'a pas d'analogue en mesure finie : le groupe $\mathrm{Aut}_f(X, \lambda)$ formé par les éléments de $\mathrm{Aut}(X, \lambda)$ dont le support est de mesure finie. Définissons également le groupe $\mathbb{G}_f$ par $\mathbb{G}_f \coloneqq \mathbb{G} \cap \mathrm{Aut}_f(X, \lambda)$.

Théorème (voir les Théorèmes J et K)**.** Soit $\mathbb{G}$ un groupe plein ergodique sur un espace σ-fini standard.

- Le groupe $\mathbb{G}_f$ peut être muni d'une topologie de groupe polonaise si et seulement si $\mathbb{G}$ est le groupe plein d'une relation d'équivalence dénombrable.
- Le groupe $\mathbb{G}_f$ est un groupe simple, et c'est l'unique sous-groupe distingué non trivial de $\mathbb{G}$.

Notons que dans le premier point du théorème précédent, la topologie polonaise sur $\mathbb{G}_f$ raffine la topologie uniforme. Notons également que les deux points du théorème s'appuient sur un théorème de Ryzhikov ([Ryz85]) : tout élément d'un groupe plein peut s'écrire comme produit de trois involutions, et en ce sens, toute l'information contenue dans un groupe plein ergodique est capturée par les involutions d'échange qu'il contient, ce qui explique leur rôle prééminent.

Le dernier résultat mentionné dans ce résumé porte sur des questions de généricité dans les groupes pleins orbitaux. Ayant accès au théorème de Baire dès lors que l'on dispose d'une topologie (de groupe) polonaise, il est naturel de se poser la question suivante : à quoi ressemble un élément générique dans un groupe plein orbital ? Nous obtenons le résultat suivant, généralisation de [CLM16, Thm. 4.4], qui en particulier nous dit que les éléments apériodiques sont génériques, sauf si la la relation d'équivalence associée est à classes dénombrables (voir Corollaire 3.7.19).

Théorème (voir Théorème L)**.** Soit $\mathbb{G} = [\mathcal{R}_G]$ un groupe plein orbital ergodique sur un espace σ-fini standard associé à une action Borélienne d'un groupe polonais (G, τ_G). Si la topologie de la convergence orbitale en mesure est polonaise sur $\mathbb{G}$, alors les assertions suivantes sont équivalentes :

(1) les éléments apériodiques de $\mathbb{G}$ forment un sous-ensemble dense de $\mathbb{G}$;

(2) les éléments ergodiques de $\mathbb{G}$ forment un sous-ensemble dense de $\mathbb{G}$;

(3) la $\mathbb{G}_f$-classe de conjugaison d'un élément apériodique de $\mathbb{G}$ est dense dans $\mathbb{G}$;

(4) pour toute action essentiellement libre préservant la mesure d'un groupe dénombrable $\Gamma \curvearrowright (X, \lambda)$, il existe un G_δ dense d'éléments de $\mathbb{G}$ induisant une action libre de $\Gamma * \mathbb{Z}$.

Ces conditions équivalentes impliquent de plus la condition suivante :

(5) pour tout τ_G-voisinage de e_G, l'ensemble $\bigcup_{g\in V} \{x \in X \mid g \cdot x \neq x\}$ est de mesure pleine.

Enfin, (5) implique les conditions ci-dessus dès que l'espace X est muni d'une topologie polonaise compatible, sur laquelle la mesure λ est localement finie, et pour laquelle l'action de G est continue.

Introduction

Context and a bit of history

The ergodic theory of group actions is a domain that falls under the umbrella of dynamical systems, and focuses on the study of actions of groups on measurable spaces, which admit (quasi)-invariant measures. This point of view extends the study of $\mathbb{Z}$-actions commonly found in classical (time-reversible) dynamical systems to a broader context that notably includes the non discrete case of $\mathbb{R}$-actions, also known as measure-preserving flows. In this thesis, we will be interested in actions of groups that are Polish, on standard Borel spaces. These standard Borel spaces are endowed with a measure, which we denote by ν in general. All standard atomless spaces with the same finite total mass are isomorphic, meaning that it makes sense to talk about standard probability spaces (in which case we denote the measure by μ), and it also makes sense to talk about standard σ-finite spaces of infinite measure (in which case the measure will usually be denoted by λ). Since a measure-preserving action on a standard measured space (X, ν) allows us to see an element of the acting group as a measure-preserving bijection, the main objects of interest will exactly be those bijections. We will denote by $\mathrm{Aut}(X, \nu)$ the group that they form, where two elements are identified if they coincide on a conull set.

Let us start with the case of countable group actions on a standard probability space (X, μ). The study of probability measure-preserving (pmp) actions, has been a very active area of research in recent years, and one of its objectives has been the classification of pmp actions of countable groups up to conjugacy. While some impressive results have been obtained, such as Ornstein's classification of $\mathbb{Z}$-Bernoulli shifts by their entropy ([Orn70]), Foreman, Rudolph and Weiss have proved in [FRW11] that the conjugacy relation on the set of ergodic transformations of (X, μ) is not Borel, explaining why it is so intractable in general. It therefore makes sense to consider a natural weakening of the notion of conjugacy: the coarser notion of orbit equivalence (OE). Two actions α and β of two countable groups Γ and Λ respectively on (X, μ) are *orbit equivalent* (also denoted by OE) if there exists a element $S \in \mathrm{Aut}(X, \mu)$ such that for μ-a.e. $x \in X$ we have

$$S(\alpha(\Gamma, x)) = \beta(\Lambda, S(x)).$$

In other words, OE forgets the action, and only remembers the partition of the space in orbits. As it turns out Dye proved that OE is too coarse to differentiate classical ergodic dynamical systems, i.e. actions of $\mathbb{Z}$ ([Dye59]), a result later generalized by Ornstein and Weiss to any free ergodic actions of countable amenable groups ([OW80]). Although OE is a weak notion in that regard, it is still capable of capturing some properties of the group. The rich area of mathematics that was born from investigating the ties between the group and the orbit partition is traditionally called measured group theory. It has vibrant connections to other fields, including geometry of groups, cohomology, and von Neumann algebras, and while we will not mention all the related developments, we refer to [Gab11] and [Fur11] for surveys on orbit equivalence and measured group theory, and to the monograph [Kec10] for a more global point of view.

One of the main tools introduced by Dye is the full group of the equivalence relation associated with an action, which we will first define in the restricted context of a pmp action of a countable group on a standard probability space. Given such an action of a countable group Γ on (X, μ), we first recall that $\mathcal{R}_\Gamma$ is defined by

$$\mathcal{R}_\Gamma \coloneqq \{(x, y) \in X \times X \mid \exists \gamma \in \Gamma \text{ such that } \gamma \cdot x = y\}.$$

It follows that two actions are OE if and only if their associated equivalence relations are isomorphic, up to a null set. In his pioneering work, Dye defined the orbit full group $[\mathcal{R}_\Gamma]$ of $\mathcal{R}_\Gamma$ as the subgroup of $\mathrm{Aut}(X, \mu)$ comprised of the elements that respect the Γ-orbits:

$$[\mathcal{R}_\Gamma] \coloneqq \{T \in \mathrm{Aut}(X, \mu) \mid \text{for } \mu\text{-a.e. } x \in X : (x, T(x)) \in \mathcal{R}_\Gamma\}.$$

This group encodes orbit equivalence in the following sense: two actions of two countable groups are OE if and only if their associated full groups are conjugated in $\mathrm{Aut}(X, \mu)$. In fact, a stunning theorem of Dye, the Reconstruction Theorem (see Theorem A.1.1), asserts that two ergodic actions are OE if and only if their full groups are isomorphic, as topological groups (and in fact as purely abstract groups). A natural problem is therefore to examine the links between the (measure-theoretic) properties of the action, and the topological (and algebraic) properties of the full group. To this end, and as the full groups are very large groups, the first natural step is to endow them with a Polish topology, giving us access to tools such as the Baire Category Theorem. In this restricted case, such a topology exists: the uniform topology, generated by the bi-invariant complete metric

$$d_u(S, T) \coloneqq \mu(\{x \in X \mid S(x) \neq T(x)\}),$$

is a Polish group topology on the full group associated to a countable group action. This approach has proved very fruitful (see *e.g.* [GP07], [KT10], [LM14b], [EG16]).

The next step is naturally to understand what happens when we step out of the countable world, so let us now consider a more general Polish group G. Carderi and Le Maître consider in [CLM16] the equivalence relation $\mathcal{R}_G$ associated with a Borel G-action on a standard probability space, and define the orbit full group in the same way as before: $[\mathcal{R}_G] \coloneqq \{T \in \mathrm{Aut}(X, \mu) \mid \forall x \in X : (x, T(x)) \in \mathcal{R}_G\}$ (the equivalence relation is defined similarly as in the countable case). The first remark is that the previously defined uniform topology can fail to be separable on $[\mathcal{R}_G]$ when G is uncountable. Hence the following natural question: can $[\mathcal{R}_G]$ be endowed with a Polish group topology? One can ask more and wonder what the link between this topology and the uniform topology is. This is a good moment to mention that there is another natural topology on $\mathrm{Aut}(X, \mu)$, the weak topology, which is a Polish group topology on the group $\mathrm{Aut}(X, \mu)$, and is pretty well understood (see Figure 1 for its definition in the pmp setup). The whole group $\mathrm{Aut}(X, \mu)$ itself is a full group, corresponding to a transitive action of an uncountable group, for instance the circle group $\mathbb{S}^1$ acting on itself by translation. This at the very least hints that it is possible for an orbit full group coming from the action of an uncountable group to be endowed with a Polish group topology. The main result of [CLM16] is that the authors endow *any* orbit full group coming from a Borel Polish group action with a Polish group topology, called the topology of orbital convergence in measure (it is only called the topology of convergence in measure in the original paper, we chose this different name in order not to confuse it with the natural topology on the space of measurable functions, more on this later). Moreover, they prove that this topology is intermediate between the weak and the uniform topologies, thus providing a unifying framework for studying orbit full groups (see Theorem 1.3.21).

Weak topology: $T_n \to T$ iff $\mu(T_n(A)\Delta T(A)) \to 0$ for any measurable $A \subseteq X$	Topology of orbital convergence in measure	Uniform topology: $T_n \to T$ iff $\mu(\{x \in X \mid T_n(x) \neq T(x)\}) \to 0$

Figure 1: The three Polish topologies on orbit full groups.

The main idea behind this definition is more easily understood for free actions: given a Borel free G-action on X, to any element $T \in [\mathcal{R}_G]$ it is possible to associate a function $f_T : X \to G$, uniquely defined by $T(x) = f_T(x)\cdot x$. This allows us to view $[\mathcal{R}_G]$ as embedded in the space $\mathrm{L}^0(X, \mu, G)$ of measurable functions $f : X \to G$, identified up to measure 0. The topology of convergence in measure on $\mathrm{L}^0(X, \mu, Z)$ (for any Polish space Z) is defined by declaring that two functions are close whenever $\int_X d(f(x), g(x))d\mu(x)$ is small, for any compatible metric d on Z. It is known that $\mathrm{L}^0(X, \mu, G)$ is a Polish group for the pointwise product when endowed with this topology, which allows us to define the desired Polish group topology on $[\mathcal{R}_G]$ simply as the restriction of this topology.

What about infinite measures?

The weak and uniform topologies are already well-defined on $\mathrm{Aut}(X, \lambda)$, so the first objective of this thesis has been to extend the previous result concerning the topology of orbital convergence in measure to the case of σ-finite infinite measure-preserving (∞-mp) actions. Some difficulties are to be expected: the world of ∞-mp dynamical systems is notably less agreeable than that of the previously discussed pmp dynamical systems. First and foremost the Poincaré recurrence theorem does not necessarily hold anymore: the phenomenon of dissipativity makes its appearance, dissipative ∞-mp bijections eventually leave any finite measure part of the space. This means that for any given ∞-mp bijection, the space has a conservative part which is revisited, and a dissipative part which is *pushed towards infinity*. A lot of the reassuring examples of pmp ergodic bijections, such as the irrational rotations or the Bernoulli shifts do not appear in this ∞-mp world, or not under their habitual form, and even well understood notions such as mixing get considerably more complicated in this setting (see [AS18] for a survey of mixing notions in this context). Indeed merely constructing ∞-mp ergodic bijections and studying their sometimes surprising properties demands considerable work (see *e.g.* [KP63], [AFS97], [EHH98], [BFMS01]). A short and non exhaustive list of 'classical' ∞-mp ergodic bijections is presented in Appendix B. The group $\mathrm{Aut}(X, \lambda)$ itself is more complicated, indeed, contrary to $\mathrm{Aut}(X, \mu)$, it is not a simple group as the bijections whose support have finite measure form a non-trivial proper normal subgroup.

The main difficulty in extending the result of Carderi and Le Maître however, has been quite an unexpected one. As mentioned before, the topology of orbital convergence in measure comes from the topology of convergence in measure on the space of measurable functions from X to G. More specifically, the authors define the function

$$\begin{array}{rccc} \Phi & : & \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ & & f & \longmapsto & (x \mapsto f(x)\cdot x) \end{array}$$

and use the embedding of $\mathrm{Aut}(X, \mu)$ in $\mathrm{L}^0(X, \mu, X)$. Here comes the problematic part: the embedding of $\mathrm{Aut}(X, \mu)$ in $\mathrm{L}^0(X, \mu, X)$ is linked to both the measure and some topological properties on X itself, witnessed here by the compatible distance appearing in the definition of the convergence in measure on $\mathrm{L}^0(X, \mu, X)$. Recall now that there is a priori no topological structure on X itself, as it is a standard Borel space. Therefore in order to use the topology of $\mathrm{L}^0(X, \mu, X)$, the authors use a *continuous spatial model* of the action, *i.e.* a new realization of the action on a compatible topological space, which is indistinguishable from the original action, from a measured point of view. The main difficulty can now be summed up as follows: if X is endowed with an infinite measure, it is possible for the measure to be irreconcilable with the topology provided by a continuous spatial model, which would compromise the topological embedding of $\mathrm{Aut}(X,\mu)$ in $\mathrm{L}^0(X,\mu,X)$. Let us be more specific: we say that a measure is *locally finite* with regard to a topology if every point admits a neighbourhood of finite measure. The main theorem concerning continuous spatial model is the Becker-Kechris theorem (see Theorem 2.1.1). Given a Borel action of a Polish group G on a standard Borel space X (endowed in practice with a Borel measure), it provides a compatible Polish topology τ_X on X which makes X a Polish G-space. However the topology is built in a way that is completely estranged from the measure, so they have no reason to cooperate nicely, *i.e.* to be such that the measure is locally finite with regard to τ_X.

The quest for locally finite spatial models

The work that concerns this axis of research is exposed in Chapter 2, and it led to a paper written in collaboration with François Le Maître.

Let us start by quickly describing the notion of spatial model in a great level of generality. We fix in this whole section a Polish group G and a standard measured space (X,ν). An everywhere defined Borel action map $\alpha : G\times X\to X$ such that for every $g\in G$ the map $\alpha(g,\cdot)$ defines a measure-preserving bijection of (X,ν) is a spatial action. On the other hand, there are several equivalent ways to define a boolean measure-preserving action, and one of them is as a continuous group homomorphism from the acting group G to $\mathrm{Aut}(X,\nu)$. This yields an action on the measure algebra, which is the boolean algebra of Borel subsets identified up to measure zero, justifying the terminology. It is easy to see that a spatial action yields an associated boolean action. The converse direction however, which is interesting only for uncountable groups, is known as the *realization problem* and is far from straightforward. Glasner, Tsirelson and Weiss in [GTW05] gave a new breath to the area of research interested in solving this problem: they showed that Lévy groups (which is a class of very large Polish groups) cannot admit non-trivial spatial actions, although they can admit non-trivial boolean actions. Note that the group $\mathrm{Aut}(X,\nu)$ itself is a Lévy group.

The question of the relation between a boolean action and the (possibly non-existing) associated spatial action can be broken down into two complementary problems:

– **The realization problem:** Given a boolean G-action, can one find a spatial G-action which gives rise to a boolean action which is booleanly isomorphic to the first one?

– **The isomorphism problem:** If two spatial actions have isomorphic associated boolean actions, are they isomorphic as spatial actions?

As mentioned, the answer to both of these problems is clear for actions of countable groups, but as it turns out, in a fundamental paper Mackey completely answered both questions in the positive (see Theorem 2.2.7) for the broader class of locally compact Polish groups (equivalently, locally compact second countable groups, which we will sometimes denote by lcsc groups).

Going back to our original problem, we fix a spatial ∞-mp action α of a Polish group G on a standard σ-finite standard space (X,λ). Our main goal is to find a continuous spatial model which is spatially isomorphic to the original one, and such that the measure is locally finite with regard to the topology. More specifically, the two natural questions are the following: does the boolean action associated with α come from a continuous action on a Polish space endowed with a locally finite measure? Do we have a positive answer to the isomorphism problem between α and the newly obtained action? If the answer to the first question is positive, we will say that this action is a *locally finite realization* of the boolean action associated with α.

For the locally finite realization problem, we characterize the existence of such models in terms of G*-continuous* functions, a class of functions introduced in [GTW05]. These functions will be the bricks that allow us to build the space of our continuous model through the Gelfand isomorphism, while the measure will be defined using the Riesz representation theorem. To obtain a satisfying model, the problem therefore reduces to *finding enough* G*-continuous functions*. We prove the following.

Theorem (see Theorem C)**.** Let G be a Polish group, and let $\alpha : G \to \mathrm{Aut}(X, \lambda)$ be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The following are equivalent:

(1) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^1(X, \lambda)}^{\|\cdot\|_1} = \mathrm{L}^1(X, \lambda);$$

(2) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda);$$

(3) one can endow X with a locally compact Polish topology inducing its standard Borel space structure so that λ is Radon and α lifts to a continuous measure-preserving G-action on X.

While this holds for any general Polish group G, the isomorphism problem is still not resolved when the group is not lcsc. In fact we exhibit examples of actions of non-lcsc groups that are booleanly isomorphic but not spatially isomorphic (see Section 2.4.3). For lcsc groups however, we are able to obtain a satisfying conclusion to our search. We use convolution as a source of G-continuous functions to verify that our action satisfies the equivalent conditions of the previous theorem, then apply Mackey's theorem to solve the isomorphism problem.

Theorem (see Theorem A)**.** Let (X, λ) be a standard σ-finite space, let G be a locally compact Polish group and let $\alpha : G \times X \to X$ be a measure-preserving G-action on X. Then α is spatially isomorphic to a continuous measure-preserving G-action on a locally compact Polish space Y endowed with a Radon measure η.

In fact we get something a bit stronger than the previous statement: it is possible to transport the whole space X into a continuous model, and not just a conull subset, as it is the case when we have a spatial isomorphism. The following statement is very reminiscent of the previously mentioned Becker-Kechris theorem. The strategy of proof is not too different from our first approach with boolean actions. The main difference is that we start working with everywhere defined functions from the very beginning, instead of working in the more familiar context of L^p spaces, where functions are identified when they coincide on conull sets.

Theorem (see Theorem B)**.** Let (X, λ) be a standard σ-finite space, let G be a locally compact Polish group and let $\alpha : G \times X \to X$ be a measure-preserving G-action on X. Then there is a continuous measure-preserving G-action on a locally compact Polish space Y and a Borel G-equivariant injection $\Phi : X \to Y$ such that the pushforward measure $\Phi_*\lambda$ is a Radon measure on Y.

We also obtain a more general result on the existence of locally finite spatial models for a larger family of groups: the class of isometry groups of locally compact separable metric spaces contains both the locally compact Polish groups and the non-archimedean Polish groups. In particular, this yields results even for groups for which we know that the isomorphism problem cannot easily be solved. The following is an infinite measure generalization of the spatial model theorem of Kwiatkowska and Solecki ([KS11, Thm. 1.1]). It uses a very nice characterization of this large class of group that is established in their paper (see Theorem 2.4.5), itself based on another nice characterization of these groups from Gao and Kechris ([GK03]): these groups are exactly the closed subgroups of groups of the form

$$\prod_{n \in \mathbb{N}} \mathfrak{S}_\infty \ltimes H_n^{\mathbb{N}},$$

where H_n is a lcsc group for any n, and $\mathfrak{S}_\infty = \mathrm{Sym}(\mathbb{N})$ acts by permutation of the coordinates of $H_n^{\mathbb{N}}$.

Theorem (see Theorem D)**.** Let G be the isometry group of a locally compact separable metric space. Then every boolean measure-preserving G-action on a standard σ-finite space (X, λ) can be lifted to a continuous measure-preserving action on X where X is endowed with a locally compact Polish topology inducing its standard Borel structure and λ is a Radon measure.

Next, we go back to an already mentioned result of Glasner, Tsirelson and Weiss, the non-existence of non-trivial spatial actions of Lévy groups. More can be said about non-existence of spatial models, in particular the notion of *whirliness* also introduced by Glasner, Tsirelson and Weiss, which is a strong version of ergodicity, guarantees that an action cannot admit a spatial model (see [GTW05, Prop. 3.3]), but for this introduction let us remain on the subject of Lévy groups. We prove the following by constructing a Poisson Point Process linking an infinite measure preserving G-action to a certain pmp one, and we then apply their result: the G-action on the space of closed subsets preserves the previously defined probability measure, so it has to be trivial. One of the main difficulties is that Poisson Point Processes are generally defined on metric spaces, and the construction depends quite heavily on the metric. To obtain results in the most general settings for our needs, we work directly with the Effros Borel space, and define the process *via* a projective limit. We also refer to the article of Avraham-Re'em and Roy ([AR25]) for a more general construction, where the authors obtained the following result first.

Theorem (see Theorem E)**.** Every continuous measure-preserving spatial action of a Lévy group G on a Polish space X endowed with a σ-finite atomless locally finite measure λ is trivial, *i.e.* the set of fixed points $\{x \in X \mid \forall g \in G, g \cdot x = x\}$ is conull.

We summarize some of the main results of this part in Figure 2: by 'Realization problem' here we specifically mean the locally finite realization problem, and the 'Isomorphism problem' assumes that we start with a spatial action.

Group class	Realization problem	Isomorphism problem
Countable	✓(trivially)	✓(trivially)
Isometry groups of locally compact separable spaces	✓	✓ for lcsc
		✗ for non-archimedean
		? for others
Lévy	✗	? (in full generality)
Others	?	?

Figure 2: Locally finite models of measure-preserving Polish group actions.

Full groups preserving an infinite measure

We can now use the results from Chapter 2 to define the Polish group topology on orbit full groups preserving an infinite measure. This work led to a second paper, which is presented in Chapter 3 of this thesis.

Once again we fix a standard σ-finite space (X, λ), and if nothing more specific is mentioned, G will denote a Polish group. As mentioned before, the Polish topology on orbit full groups is linked to the topology of convergence in measure on the spaces of measurable functions, and the topological embedding we needed holds in the case of a continuous action on a space with a compatible locally finite measure. Without any mention of a locally finite spatial *model*, the most general form of our main result is the following.

Theorem (see Theorem G)**.** Let G be a Polish group acting continuously on a Polish space (X, τ_X) equipped with an atomless σ-finite measure λ which is locally finite on τ_X. The orbit full group $[\mathcal{R}_G]$, equipped with the topology of orbital convergence in measure, is a Polish group.

In particular, by using locally finite spatial models from the previous section, it is possible in the measure-preserving case to remove the continuity hypothesis for locally compact Polish group actions (see Theorem F), allowing us to remain in the purely measured setting. It is reasonable to wonder if the local finiteness hypothesis that appears is only superficial, but as it turns out, it is a fundamental condition (see Section 3.5.2), shedding more light on a phenomenon exclusive to dynamics in infinite measure spaces.

As mentioned earlier, the full groups of pmp actions of countable groups are complete invariants of OE, for ergodic actions. In [CLM16] Carderi and Le Maître actually prove that this characterization still holds if the acting group is locally compact. In this context, the OE is defined as follows: two actions of two locally compact groups G and H are OE if there exists a map $S \in \mathrm{Aut}(X, \mu)$, and a conull subset $A \subseteq X$ such that for all $x \in A$

$$S(G \cdot x) \cap A = (H \cdot S(x)) \cap S(A).$$

The easy direction of the equivalence is still easy: if two actions are OE, then their associated orbit full groups are conjugated in $\mathrm{Aut}(X, \mu)$. For the converse direction, Dye's Reconstruction Theorem still holds, and the resulting conjugating map is shown to be an orbit equivalence between the actions. In our ∞-mp setup, while we do not have access to the original Reconstruction Theorem, we luckily have access to a more general one, proved by Fremlin, which holds even in a broader context that what we use it for. Fremlin's Reconstruction Theorem is exposed in Appendix A, along with a complete proof, and a discussion on the links between the two theorems. Armed with this theorem, we can prove the following natural analogue of Carderi and Le Maître's result in infinite measure.

Theorem (see Theorem H)**.** Let G and H be two Polish locally compact groups acting in a Borel measure-preserving ergodic manner on a standard σ-finite space (X, λ). Let $\Psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ be a group isomorphism. Then there exists an orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$, which preserves the measure λ up to multiplication by an element of $\mathbb{R}_+^*$, such that Ψ is the conjugation by S.

The fact that orbit full groups capture the orbital properties of the actions even for lcsc groups encourages a systematic study of both their algebraic and topological properties, a program in the vein of Dye's work. As such, we prove several results on the behaviour of infinite measure preserving full groups. We start by investigating the behaviour of the (sometimes existing) Polish topology in more details, and in a greater level of generality.

Let us recall the more general definition of full groups, introduced by Dye in [Dye59]: the full groups obtained by *cutting and pasting*. An element T of $\mathrm{Aut}(X, \lambda)$ is obtained by cutting and pasting a sequence (T_n) of elements of $\mathrm{Aut}(X, \lambda)$ if there exists a partition (A_n) of X such that for every $n \in \mathbb{N}$, we have $T_{\restriction A_n} = T_{n \restriction A_n}$. A visual representation of this operation can be found in Section 1.3.2. We then say that a subgroup $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ is a *full group* if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$ (of course these notions are not exclusive to the ∞-mp setup).

While it is *a priori* not possible to endow every full group with a Polish group topology, it is still possible to investigate the links between their possible topologies. In particular, the structure given by a Polish group topology is pretty rigid: a classical result of Banach asserts that any Borel homomorphism between Polish groups is automatically continuous. This *automatic continuity* result is one of the building blocks of the following theorem, which generalizes [CLM16, Thm. 4.7], itself building upon [KT10, Thm. 3.1] and [Kal85]. More specifically, the property known as the *Steinhaus Property*, witnessed by a *Steinhaus exponent*, guarantees automatic continuity in the sense of Rosendal ([Ros05]) for the full group endowed with the uniform topology.

Theorem (see Theorem I)**.** Let $\mathbb{G}$ be an ergodic full group on a standard σ-finite space. Then $\mathbb{G}$ carries at most one Polish group topology, and such a topology is necessarily coarser than the uniform topology, but refines the weak topology, seen as topologies inherited from those on $\mathrm{Aut}(X, \lambda)$.

In the process of establishing automatic continuity for the ergodic full groups endowed with the uniform topology, a new factorization of ∞-mp bijections is proved, allowing us to improve Fremlin's unpublished proof of the same result, by decreasing the Steinhaus exponent. As we remain brief on this part, we refer to Section 3.6.3 for more details.

The next two results are centered on a very interesting group that does not have any analogue in the pmp setup, the group $\mathrm{Aut}_f(X, \lambda)$ of bijections of supports of finite measure, where we recall that the support of a Borel bijection is the part of the space where it acts non-trivially. We also define for any $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ the group $\mathbb{G}_f \coloneqq \mathbb{G} \cap \mathrm{Aut}_f(X, \lambda)$.

Theorem (see Theorem J and Theorem K)**.** Let $\mathbb{G}$ be an ergodic full group on a standard σ-finite space.

- The group $\mathbb{G}_f$ can be endowed with a Polish group topology if and only if $\mathbb{G}$ is the full group of a countable equivalence relation.
- The group $\mathbb{G}_f$ is simple, and is the only non-trivial normal subgroup of $\mathbb{G}$.

In the first item of the previous theorem, the Polish group topology on $\mathbb{G}_f$ is called the uniform finite topology, and it refines the uniform topology. Let us briefly mention that, although the first item of the previous theorem uses a nice argument of *expansion of support* while the second item uses an algebraic trick, both rely on a theorem from Ryzhikov ([Ryz85]): any element of a full group can be written as the product of three involutions. In this sense, all the information from an ergodic full group is captured by the exchanging involutions it contains, explaining why they play such a salient role.

The last result we will mention in this introduction is about genericity in orbit full groups. Polish groups naturally raise the following question: what does a generic element look like? A very brief overview of category (in the sense of Baire) results for $\mathrm{Aut}(X,\nu)$ is presented in Section 1.2.6. For ergodic orbit full groups, we obtain the following analogue of [CLM16, Thm. 4.4], which in particular tells us that aperiodic elements are generic unless the associated equivalence relation has countable classes (see Corollary 3.7.19).

Theorem (see Theorem L)**.** Let $\mathbb{G} = [\mathcal{R}_G]$ be an ergodic orbit full group on a standard σ-finite space associated with a Borel action of a Polish group (G, τ_G). If the topology of orbital convergence in measure is Polish on $\mathbb{G}$, then the following are equivalent:

(1) the aperiodic elements of $\mathbb{G}$ form a dense subset of $\mathbb{G}$;

(2) the ergodic elements of $\mathbb{G}$ form a dense subset of $\mathbb{G}$;

(3) the $\mathbb{G}_f$-conjugacy class of any aperiodic element of $\mathbb{G}$ is dense in $\mathbb{G}$;

(4) for any essentially free measure-preserving action of a countable group $\Gamma \curvearrowright (X,\lambda)$, there is a dense G_δ set of elements of $\mathbb{G}$ inducing a free action of $\Gamma * \mathbb{Z}$.

These equivalent conditions moreover imply the following one:

(5) for any τ_G-neighbourhood V of e_G, the set $\bigcup_{g\in V}\{x \in X \mid g\cdot x \neq x\}$ is conull.

Finally, (5) implies the above conditions if the space X is endowed with a compatible Polish topology, with regard to which the measure is locally finite and the G-action is continuous.

We summarize some of the results of this part in Figure 3, which extends Figure 1, now in the ∞-mp setup.

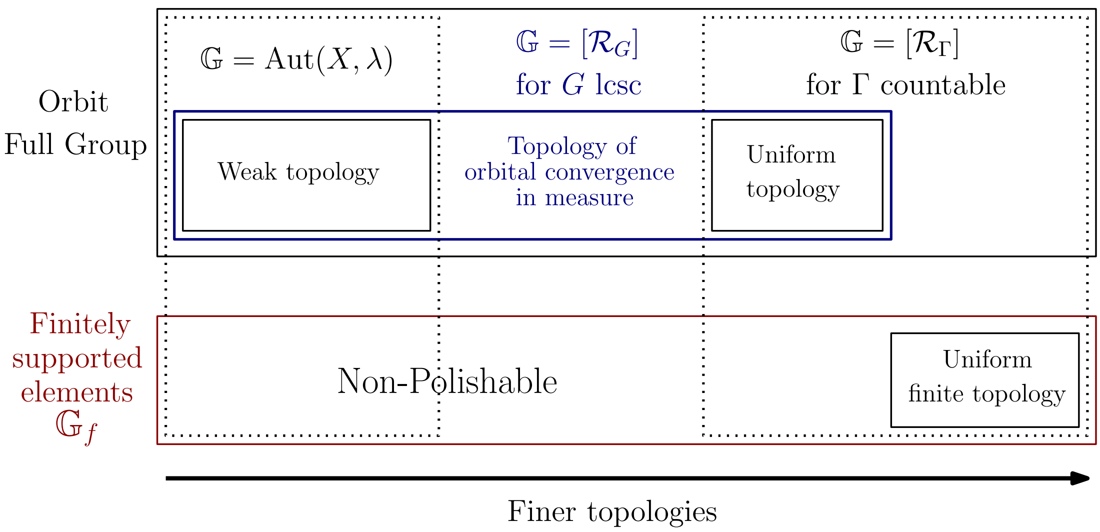

Figure 3: The possible Polish topologies on an ergodic full group $\mathbb{G}$, and on $\mathbb{G}_f$.

Outline of the thesis

This text is separated in three chapters and two appendices. The first chapter is a preliminary chapter, containing a short exposition of all the necessary concepts that will be used throughout, as well as the statements of some important contributions in the fields. It does not aim to be a complete overview of the state of the art, nor does it provide all the proofs of the mentioned statements. Its only goal is to gather the necessary ingredients at the same place, and only some proofs are given, when we believe that having them in mind can be beneficial for the rest of this text.

The second chapter is based on our first paper ([HLM24]), based on a joint work with our advisor François Le Maître. It explores the distinction between spatial and boolean actions on spaces with infinite measure, and provides some constructions of locally finite spatial models.

The third and final chapter is based on our second preprint ([Hoa25]), which as we mentioned, was our initial project. Compared to the preprint, we take advantage of the absence of page number constraints on this thesis so as to offer a more self-contained treatment. It partially answers the question of Polishness for ergodic full groups, and explores some of the topological properties that they demonstrate.

The first appendix is dedicated to the Fremlin Reconstruction Theorem, a generalization of Dye's Reconstruction Theorem. It was first written to be read independently, however to avoid repetitions (to a degree), some of the relevant definitions and propositions have been removed and referenced instead.

The second appendix is a rather informal and non-exhaustive list of ∞-mp ergodic bijections. Although the level of details in which they are exposed is uneven, more attention has been given to those that play a bigger role in our work. We believe such a list does not exist in the literature in this form, and hope it can become a useful resource for people discovering the world of dynamics in infinite measure.

Chapter 1

Preliminaries

Abstract

In this chapter we quickly present the framework of the subject matter, as well as some of the classical tools that will be necessary for our work. More specifically we give the very basics of standard Borel and measured spaces, as well as Polish groups. We then move on to the measure-preserving (and to a lesser extent measure class-preserving) bijections and the group they form. A special emphasis is put on some classical theorems of ergodic theory, notably the Poincaré recurrence theorem and the Rokhlin lemma. We then move on to some more specific preliminaries, about infinite measures, topologies on groups of measure-preserving bijections, and Baire category results. We conclude this preliminary chapter by describing the settings of measure-preserving actions of Polish groups –with of course an emphasis on infinite measure spaces–, and by briefly presenting measure-preserving full groups and their origins.

"Journey before destination. It cannot be a journey if it *doesn't have a beginning*."

Dalinar in *Oathbringer*, Brandon Sanderson

Contents

1.1 Measure-theoretic and topological context

This section aims to present the framework of standard measured spaces, along with some of its origins from descriptive set theory.

1.1.1 About Polish and standard Borel spaces

Definition 1.1.1. A **Polish space** is a separable and completely metrizable topological space. In particular, it is **Hausdorff** (or T_2), which means that any two distinct points admit respective neighbourhoods that are disjoint, and it is **second countable**, which means that the topology has a countable base.

Remark 1.1.2. There always exists a bounded (often by 1) metric which is compatible with the topology of a Polish space. Indeed, if d is a compatible metric, then $\min(1, d)$ is suitable, as it is equivalent to d. Moreover, $\min(1, d)$ has the same Cauchy sequences as d, in particular every Polish space admits a compatible complete bounded metric.

Remark 1.1.3. By virtue of being second countable, Polish spaces satisfy the Lindelöf Lemma: every open cover of a Polish space admits a countable subcover.

Note that in terms of cardinality, Polish spaces are either finite, countable, or of cardinality $2^{\aleph_0}$, (see [Kec95, Cor. 6.5]).

We also have the following fact about Polish spaces. Recall that a G_δ set is a countable intersection of open sets.

Proposition 1.1.4 ([Kec95, Thm. 3.11])**.** *Let (X, τ) be a Polish space. Then $Z \subseteq X$ is Polish for the induced topology if and only if Z is G_δ in X. In particular a closed subspace of a Polish space is Polish.*

Polish spaces are the topological spaces that give rise to a widely used measurable structure via their Borel sets. We recall that the Borel σ-algebra $\mathcal{B}(X)$ of a topological space is the smallest σ-algebra containing the open sets, *i.e.* the σ-algebra generated by the topology of the space. We say that the σ-algebra is countably generated when the topology is second countable, so in particular it is the case for Polish spaces. In fact, one can say more: Borel σ-algebras coming from Polish topologies are classified up to isomorphism by the cardinality of the underlying space (see Theorem 1.1.8 below).

Definition 1.1.5. A **standard Borel space** X is a measurable space with a σ-algebra $\mathcal{B}(X)$ of subsets that are Borel for some Polish topology on X.

The following gives a very useful property, allowing us to 'cut' standard Borel spaces while retaining standardness. In practice we will consider Borel subsets of standard Borel spaces without explicitly mentioning this result.

Proposition 1.1.6 ([Kec95, Cor. 13.4])**.** *Let $(X, \mathcal{B})$ be a standard Borel space, and $Y \in \mathcal{B}$ be a Borel subset. Then $(Y, \mathcal{B}_{\restriction Y} = \{A \subseteq Y \mid A \in \mathcal{B}\})$ is also a standard Borel space.*

Unless we specifically need it, the σ-algebra $\mathcal{B}(X)$ will usually be omitted in the notations. We have the following essential facts about standard Borel spaces.

Theorem 1.1.7 (Lusin-Suslin, see *e.g.* [Kec95, Thm. 15.2])**.** *Let X and Y be two standard Borel spaces, and let $f : X \to Y$ be an injective Borel map. Then for every Borel subset A of X, $f(A)$ is Borel.*

Note that with this definition there exist finite and countable 'standard Borel spaces'. We have the following, stating that all standard Borel spaces *of the same cardinality* are isomorphic, thus justifying the terminology.

Theorem 1.1.8 (The Isomorphism Theorem, [Kec95, Thm. 15.6])**.** *Any two standard Borel spaces X and Y are Borel isomorphic if and only if* $\mathrm{Card}(X) = \mathrm{Card}(Y)$.

Remark 1.1.9. We will almost exclusively be interested in uncountable standard Borel spaces, so in the rest of this text, a **standard Borel space** will refer to an uncountable one.

As an important example of a standard Borel space, we give the definition of the Effros Borel space, which will in particular be used in Section 2.6.

Definition 1.1.10. Let (X, τ) be a topological space. We denote by $F(X)$ the set of closed subsets of X. We endow $F(X)$ with the σ-algebra $\mathcal{B}_{\mathrm{Eff}}$ generated by the sets

$$\{F \in F(X) \mid F \cap U \neq \emptyset\},$$

for $U \in \tau$. The measurable space $(F(X), \mathcal{B}_{\mathrm{Eff}})$ is called the **Effros Borel space** of X.

Theorem 1.1.11 (see *e.g.* [Kec95, Thm. 12.6])**.** *The Effros Borel space $(F(X), \mathcal{B}_{\mathrm{Eff}})$ of a Polish space X is a standard Borel space.*

1.1.2 Standard measured spaces

In this text we will mostly be interested in atomless Borel measures (also called non-atomic, diffuse, continuous, etc...) on standard Borel spaces, or on Polish spaces if we need some topological considerations. We start with the theorem that will justify the terminologies.

Theorem 1.1.12 (The measure Isomorphism Theorem, [Kec95, Thm. 17.41])**.** *Let X be a standard Borel space, endowed with an atomless probability measure μ. Then (X, μ) is measure-isomorphic to $([0,1], \mathrm{Leb}_{\restriction[0,1]})$, i.e. there exists a Borel isomorphism φ between X and $[0,1]$ such that the pushforward measure $\varphi_*\mu$ is equal to $\mathrm{Leb}_{\restriction[0,1]}$.*

Definition 1.1.13. We call a pair (X, μ) as in the statement of Theorem 1.1.12 a **standard probability space**.

By virtue of Theorem 1.1.12, all standard probability spaces are measure-isomorphic, justifying the terminology.

We recall that a measure λ is σ-finite on a space X if we can write $X = \bigcup_{n\in\mathbb{N}} X_n$, where each X_n is measurable and satisfies $\lambda(X_n) < +\infty$. Equivalently, we can ask that the X_n are either disjoint, or increasing. We also say that (X, λ) is a σ-finite space. This definition, coupled with Theorem 1.1.12, yields the following.

Corollary 1.1.14. *Let X be a standard Borel space, endowed with an atomless σ-finite infinite measure λ. Then (X, λ) is measure-isomorphic to* $(\mathbb{R}, \mathrm{Leb})$.

Proof. First note that Theorem 1.1.8 and Theorem 1.1.12 yield that any standard Borel space endowed with an atomless finite measure μ is measure-isomorphic to $([0, \mu(X)[, \mathrm{Leb}_{\restriction})$, and $([0, \mu(X)[, \mathrm{Leb}_{\restriction})$ is measure-isomorphic to $([a, a + \mu(X)[, \mathrm{Leb}_{\restriction})$. We now consider a standard Borel space X endowed with an atomless σ-finite measure λ, such that $\lambda(X) = +\infty$. We write $X = \sqcup_{n\in\mathbb{N}} X_n$, with $\lambda(X_n) < +\infty$ for any n. We next write $\mathbb{R}_+ = \sqcup_{n\in\mathbb{N}} I_n$, where

$$I_n = \left[\sum_{m<n} \lambda(X_m), \sum_{m<n} \lambda(X_m) + \lambda(X_n)\right[,$$

and by the previous argument each X_n is measure-isomorphic to the corresponding I_n. As the partition is countable, this construction yields a measure-isomorphism between (X, λ) and $(\mathbb{R}_+, \mathrm{Leb})$. It is then clear that $(\mathbb{R}, \mathrm{Leb})$ and $(\mathbb{R}_+, \mathrm{Leb})$ are measure-isomorphic, so the proof is over. $\square$

Definition 1.1.15. We call a pair (X, λ) as in the statement of Corollary 1.1.14 a **standard σ-finite space**.

By virtue of Corollary 1.1.14, all standard σ-finite spaces are isomorphic, once again justifying the terminology. We now define the σ-algebras of sets that may not be Borel measurable, but which are close –up to measure 0– to Borel measurable sets.

Definition 1.1.16. Fix a Borel measure λ on a standard Borel space $(X, \mathcal{B})$.

1. We say that $A \subseteq X$ is **λ-null** if there exists $B \in \mathcal{B}$ with $\lambda(B) = 0$ and $A \subseteq B$.

2. We denote by Null_λ the σ-ideal (containing $\emptyset$, closed under countable unions and subsets) of λ-null sets for λ.

3. We write $\mu \sim \lambda$ to signify that two Borel measures μ and λ are **equivalent**, meaning that $\mathrm{Null}_\mu = \mathrm{Null}_\lambda$. We will denote by $[\mu]$ the equivalence class of a measure, comprised of all σ-finite measures equivalent to μ.

4. The σ-algebra generated by $\mathcal{B} \cup \mathrm{Null}_\lambda$, comprised of sets of the form $A \cup N$ with $A \in \mathcal{B}$ and $N \in \mathrm{Null}_\lambda$ is called the completed σ-algebra, and its elements are called **Lebesgue-measurable** for the measure λ (which is identified to its natural completion defined by $\widehat{\lambda}(A \cup N) := \lambda(A)$).

1.1.3 About Polish groups

Definition 1.1.17. A **topological group** is a group with a topology for which the applications $g \mapsto g^{-1}$ and $(g, h) \mapsto gh$ are continuous (for the topology on G and G^2 respectively). A **Polish group** is a topological group whose topology is Polish.

We have the following very basic closure properties for Polish groups.

Proposition 1.1.18 ([Gao09, Prop. 2.2.3])**.** *Any countable product of Polish groups is a Polish group for the product topology.*

Proposition 1.1.19 ([Gao09, Prop. 2.2.1])**.** *Let G be a Polish group, and endow $H \leqslant G$ with the subspace topology. Then the following are equivalent:*

1. *H is Polish;*
2. *H is G_δ in G;*
3. *H is closed in G.*

As previously defined, any Polish space admits a compatible complete metric, but there is *a priori* no reason for this metric to be **left (or right)-invariant** (*i.e.* $d(g, g') = d(hg, hg')$ for any $g, g', h \in G$). On the other hand a left-invariant metric always exists for any metrizable topological group from the Birkhoff-Kakutani theorem (see *e.g.* [Kec95, Thm. 9.1]), but in general it is not complete. Similarly, a right-invariant metric always exists, but again it is not complete in general.

We however have the following construction, which is similar in spirit to the notion of completion of a metric space, but in the context of topological groups, and uses Proposition 1.1.19.

Theorem 1.1.20 ([BK96, Thm. 1.1.2] and [BK96, Cor. 1.2.2])**.** *Let G be a topological group, with d a compatible left-invariant metric. Then the metric D defined on G by*

$$D(g, h) = d(g, h) + d(g^{-1}, h^{-1})$$

is a compatible metric on G. Moreover, the multiplication on G extends in a unique manner to the completion $\widehat{G}$ of (G, D), which is then a topological group. In particular, if G is Polish, then D is a compatible complete metric on G.

We now turn our attention to quotients, which require a bit more care. For a Polish group G, with $H \leqslant G$ a closed subgroup, we denote by G/H the set of left-cosets of H. If d is a right-invariant compatible metric on G, then d^*, defined by

$$\begin{aligned} d^*(g_1H, g_2H) &\coloneqq \inf\{d(k_1,k_2) \mid k_1 \in g_1H, k_2 \in g_2H\} \\ &= \inf\{d(g_1,k) \mid k \in g_2H\} \qquad \text{(by right-invariance)} \end{aligned}$$

is a metric on G/H. We then have the following.

Lemma 1.1.21 ([Gao09, Lem. 2.2.8]). *With the same notations as before, d^* is a compatible metric (with the quotient topology) on G/H.*

Using a result of Sierpinski (see *e.g.* [Gao09, Thm. 2.2.9]), we thus have the following.

Proposition 1.1.22 ([BK96, Prop. 1.2.3]). *Let G be a Polish group, and let $H \lhd G$ be a closed subgroup. Then G/H is a Polish space when endowed with the quotient, and if moreover H is a normal subgroup, then G/H is a Polish group.*

Let us also give a result about analytic morphisms between Polish groups, after recalling the definition of analyticity. It follows from the Pettis Lemma (see *e.g.* [Kec95, Thm. 9.9]). This result is linked to the study of automatic continuity, of which we will make use in Section 3.6.3.

Definition 1.1.23. Let X be a Polish space.

(1) A subset $A \subseteq X$ is **analytic** if there exists a Polish space Y and a continuous function $f : Y \to X$ with $f(Y) = A$.

(2) If Y is a Polish space, a map $f : X \to Y$ is **analytic** if the preimage of any open set is analytic.

In particular, Borel sets are analytic, but the converse does not hold (see *e.g.* [Sri98, Thm. 4.1.5]).

Theorem 1.1.24 ([BK96, Thm. 1.2.6]). *Let G and H be two Polish groups. Then any analytic homomorphism $\varphi : G \to H$ is continuous.*

It is of note that the previous theorem actually holds more generally for any Baire-measurable group homomorphism between a Baire topological group and a separable topological group. We refer to [Kec95, Thm. 9.10] for this statement.

Next we give a few examples of Polish groups, along with some classification results.

Example 1.1.25. We have the following list of classical examples of Polish groups.

1. Any Hausdorff locally compact second countable group (sometimes called lcsc group) is a Polish group. This in particular includes $(\mathbb{R}, +)$ and the Cantor group $((\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}, +)$. In the rest of this text we will refer to this class of groups as **locally compact Polish groups**.

2. The group $\mathfrak{S}_\infty$ of *all* bijections of an infinite countable set, say $\mathbb{N}$, is a Polish group for the topology of pointwise convergence, viewing $\mathbb{N}$ as a discrete set. We insist on the fact that we consider all bijections, not just those of finite support, as the group they form is also a group that is often of interest in the literature. It is a non-locally compact group, and a compatible left-invariant metric d is defined by

$$d(\sigma_1, \sigma_2) \coloneqq 2^{-n-1}, \text{ where } n \text{ is least such that } \sigma_1(n) \neq \sigma_2(n).$$

3. Any closed subgroup G of $\mathfrak{S}_\infty$ is indeed a Polish group by Proposition 1.1.19, and in fact this condition is equivalent to the following: G admits a countable neighbourhood basis at the identity consisting of open subgroups (see *e.g.* [BK96, Thm. 1.5.1]). Those groups are called the **non-archimedean Polish groups**.

4. For any locally compact separable metric space (X, d), the group $\mathrm{Iso}(X, d)$ of isometries of (X, d) is a Polish group for the topology of pointwise convergence. The class that these groups form is very general and contains both the class of lcsc groups and of non-archimedean Polish groups. We refer to Section 2.4.2 for more on this class of groups.

5. Let G be a Polish group, and (X, μ) be a standard probability space. The space $\mathrm{L}^0(X, \mu, G)$ of G-valued random variables, that is to say the Lebesgue-measurable functions $X \to G$ (where two such functions are identified if they coincide μ-a.e.), endowed with the pointwise multiplication, is a Polish group for the topology of convergence in measure. We refer to Section 3.3.2 for more on these groups, which play an important part in our work.

1.2 Around measure-preserving bijections

In this whole section we fix a standard Borel space X, and two atomless Borel measures μ and λ, such that (X, μ) is a standard probability space, while (X, λ) is a standard σ-finite space of infinite measure. We will use ν to denote a measure that can be either μ or λ, when we do not need to distinguish between those two cases.

1.2.1 Generalities on measure-preserving bijections

We say that a Borel bijection $T : (X, \nu) \to (X, \nu)$ (meaning here that both T and T^{-1} are measurable) is **ν-measure-preserving** if $\nu(A) = \nu(T(A))$ for every Borel subset $A \subseteq X$ (in particular this implies $\nu(A) = \nu(T^{-1}(A))$). We denote the group that these measure-preserving bijections form by $\mathrm{Aut}(X, \nu)$, when two such bijections are identified if they coincide on a conull set, more formally

$$\mathrm{Aut}(X, \nu) \coloneqq \{T : X \to X \mid T \text{ is a } \nu\text{-measure-preserving bijection}\}/\mathcal{R}_\nu,$$

where

$$(S, T) \in \mathcal{R}_\nu \iff \{x \in X \mid S(x) = T(x)\} \text{ is } \nu\text{-conull}.$$

In the case where $\nu = \mu$, elements of $\mathrm{Aut}(X, \mu)$ are preserving a probability measure, thus we will call them **pmp bijections**, where pmp stands for probability measure-preserving. In the case where $\nu = \lambda$, elements of $\mathrm{Aut}(X, \lambda)$ are preserving an infinite measure, thus we will call them **∞-mp bijections**.

It is important to know where the elements of the group $\mathrm{Aut}(X, \nu)$ act non trivially. The support of a bijection gives exactly this information.

Definition 1.2.1. We define the **support** of an element T of $\mathrm{Aut}(X, \nu)$ as follows:

$$\operatorname{supp} T \coloneqq \{x \in X \mid T(x) \neq x\},$$

where the equality is to be understood up to null sets.

Note that this definition also makes sense for Borel bijections, if we take genuine equalities instead of equalities up to null sets. We will use this point of view in Section A.2.2.

We recall some of the basic notions that we will be interested in, along with some basic examples.

Definition 1.2.2. We say that a measure-preserving bijection $T \in \mathrm{Aut}(X, \nu)$ is **periodic** if each orbit $\left\{T^k(x) \mid k \in \mathbb{Z}\right\}$ is finite, ν-a.e. It is **aperiodic** if $T^k(x) \neq x$ for all $k \neq 0$, ν-a.e. It is of note that an aperiodic measure-preserving bijection induces a free action of $\mathbb{Z}$, which is a point of view that we will adopt in the future.

When it will be useful, we will also consider for T the notion of being aperiodic on its support, meaning that $T_{\restriction \operatorname{supp} T}$ is aperiodic.

Definition 1.2.3. Let T be an element of $\mathrm{Aut}(X, \nu)$. We say that a Borel subset $A \subseteq X$ is **T-invariant** if $\nu(A \Delta T(A)) = 0$. In other words, up to a null set, a T-invariant set is a reunion of T-orbits. We say that T is **ergodic** if every T-invariant Borel subset is null or conull.

Ergodicity means that a bijection cannot be decomposed into two 'separate' ones in a non-trivial way. Yet another way to think of this is the following: any measurable property concerning the T-orbits holds either for almost all of them or almost nowhere. A lot of the work presented in this text will necessitate ergodicity.

Example 1.2.4. We quickly present maybe the three most well-known ergodic measure-preserving bijections of a standard probability space. For examples on a standard σ-finite space, see Appendix B.

1. Let (X, μ) be equal to $([0,1[, \mathrm{Leb}_{\restriction [0,1[})$, and θ be in $(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1[$. The **irrational rotation** of angle θ is the map
$$T_\theta : x \mapsto x + \theta \mod 1.$$
Equivalently, one can define it as a bijection of the unit circle, mapping x to $T_\theta(x) = xe^{2i\pi\theta}$. Fourier analysis can used to show that an irrational rotation is ergodic (see *e.g.* [Aar97, Prop. 1.2.5]), although it can be also be proved by more direct means (see *e.g.* [VO16, Exercise 4.2.6]).

2. We fix a sequence (q_n) of integers with $q_n \geqslant 2$ for every n. We set
$$X = \prod_{n \in \mathbb{N}} \{0, \ldots, q_n - 1\},$$
endowed with its natural product σ-algebra and with the product of the uniform probability measures on each finite set $\{0, \ldots, q_n - 1\}$. The **odometer** associated with X is the map $T : (x_n) \to (y_n)$ where (y_n) is obtained by addition of (x_n) with $(1, 0, 0, 0, \ldots)$, with carry over to the right whenever the addition at coordinate n goes over $q_n - 1$. Concretely, $(y_n) = (0, 0, \ldots, 0, x_{n_0} + 1, x_{n_0+1}, x_{n_0+2}, \ldots)$, with x_{n_0} being the first digit of (x_n) such that $x_n \neq q_n - 1$. For instance, if $q_n = 2$ for every n,
$$\begin{aligned} T((1,1,1,0,0,1,0,\ldots)) &= (1,1,1,0,0,1,0,\ldots) \\ &\quad + (1,0,0,0,0,0,0\ldots) \\ &= (0,0,0,1,0,1,0,\ldots). \end{aligned}$$
It remains to specify that T maps $(q_0 - 1, q_1 - 1, q_2 - 1, q_3 - 1, \ldots)$ to $(0, 0, 0, 0, \ldots)$ to get a self-bijection of X, and it is straightforward to check that it preserves the product measure. It can be shown that odometers are ergodic by using their action on cylinders (see *e.g.* [Cor25a, Ex. A.1.13]) or by using the fact that they induce the cofinite equivalence relation (see *e.g.* [LM14a, Ch. 2]).

3. We consider (Y, m) to be either a standard probability space or a finite or countable set endowed with a probability measure, and consider the product measured space $X =$

$Y^{\mathbb{Z}}$, endowed with the product measure $\mu = m^{\otimes\mathbb{Z}}$ defined on the σ-algebra generated by the cylinder sets of the form $[B_k, \dots, B_l] = \{(x_n)_{n\in\mathbb{Z}} \mid x_i \in B_i \text{ for } k \leqslant i \leqslant l\}$, where each B_i is in $\mathcal{B}(Y)$. The **Bernoulli shift** S is defined by

$$S((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}.$$

It can be shown that Bernoulli shifts are ergodic by showing that they are in fact *strong mixing*, *i.e.* that they satisfy $\mu(S^{-n}(A) \cap B) \to \mu(A)\mu(B)$ for any $A, B \in \mathcal{B}(X)$, which implies ergodicity (see *e.g.* [Cor25a, Sec. A.1.d]). One of the most commonly encountered version of the Bernoulli shift is the one that acts on the space $(\{0,1\}^{\mathbb{Z}}, (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes\mathbb{Z}})$.

1.2.2 Measure class-preserving bijections

Although the focus of this text is on σ-finite spaces, and thus on bijections preserving an infinite measure, we take a short detour through the world of measure class-preserving (also called non-singular, we will use both terms indiscriminately) bijections, as it is the right framework for many statements. First recall that λ being a σ-finite measure, there always exists a probability measure in $[\lambda]$, and as such, in this section we will assume that $\mu \sim \lambda$. For more on this very broad subject, we refer the great survey from Danilenko and Silva [DS22].

We say that a Borel bijection $T : (X, \mu) \to (X, \mu)$ is $[\mu]$-**measure class-preserving** if $\mu \sim T_*\mu$. More concretely, for any Borel subset $A \subseteq X$, we have $\mu(A) = 0$ if and only if $\mu(T(A)) = 0$, if and only if $\mu(T^{-1}(A)) = 0$. We denote the group that these measure-preserving bijections form by $\mathrm{Aut}(X, [\mu])$, when two such bijections are identified if they coincide on a conull set, more formally

$$\mathrm{Aut}(X, [\mu]) \coloneqq \{T : X \to X \mid T \text{ is a } [\mu]\text{-measure class-preserving bijection}\}/\mathcal{R}_\mu,$$

where once again

$$(S, T) \in \mathcal{R}_\mu \iff \{x \in X \mid S(x) = T(x)\} \text{ is } \mu\text{-conull}.$$

It is clear that $\mathrm{Aut}(X, [\mu]) = \mathrm{Aut}(X, [\lambda])$, and it is also clear that $\mathrm{Aut}(X, \mu) \subseteq \mathrm{Aut}(X, [\mu])$, and that $\mathrm{Aut}(X, \lambda) \subseteq \mathrm{Aut}(X, [\mu])$. The notions of support, periodicity and ergodicity from the previous section are all defined up to null sets, meaning that they are defined similarly for measure class-preserving bijections.

One of the easy consequences of ergodicity is the following.

Proposition 1.2.5 (see *e.g.* [Aar97, Prop. 1.0.9])**.** *Let T be in $\mathrm{Aut}(X, [\mu])$. Then T is ergodic if and only if any Borel function $f : X \to \mathbb{R}$ which satisfies $f = f \circ T^{-1}$ (μ-a.e.) is essentially constant.*

Proof. For every $q \in \mathbb{Q}$ consider the set $\{x \in X \mid f(x) < q\}$. It is T-invariant so it is null or conull, thus f is constant up to a null set. For the converse implication, one only has to consider the characteristic function $\mathbb{1}_A$ of a T-invariant Borel set A. $\square$

Next we quickly recall what the Radon-Nikodym derivative (or cocycle) is, then define the Krieger types.

Definition 1.2.6. Let T be in $\mathrm{Aut}(X, [\mu])$. The **Radon-Nikodym derivative** (or cocycle) of T with regard to μ is a real-valued measurable function defined by

$$c(T, \cdot) \coloneqq \frac{dT^{-1}{}_*\mu}{d\mu} = \frac{d\mu \circ T}{d\mu}.$$

This function is unique (modulo null sets) and satisfies $\mu(T(A)) = \int_A c(T,x)d\mu(x)$ for any Borel $A \subseteq X$. It is invertible, non-singular, belongs in $\mathrm{L}^1(X,\mu)$, and is equal to 1 μ-a.e. if and only if T is measure-preserving. Moreover, if T and S are in $\mathrm{Aut}(X,[\mu])$, the cocycle equation

$$c(ST,x) = \frac{d\mu \circ (ST)}{d\mu} = \frac{d\mu \circ (ST)}{d\mu \circ T}\,\frac{d\mu \circ T}{d\mu} = c(S,T(x))c(T,x)$$

is satisfied μ-a.e.

Definition 1.2.7. Consider an ergodic T in $\mathrm{Aut}(X,[\mu])$. We say that T is

- **type** II_1 if there exists a probability measure $\mu' \in [\mu]$ which is preserved by T;
- **type** II_∞ if there exists an infinite σ-finite measure $\lambda \in [\mu]$ which is preserved by T;
- **type** III otherwise.

By Proposition 1.2.5 (applied to the Radon-Nikodym derivative $\frac{d\mu'}{d\lambda}$, which is T-invariant since both measures are) we can see that any invariant measure is unique up to scaling, hence those cases are exclusive. It is of note also that the type III can be further refined into the types III_0, III_λ for $\lambda \in]0,1[$ and III_1, using the Radon-Nikodym derivative. We do not provide more details but refer to [DS22, 5.1] for more on this topic.

Example 1.2.8. The same examples as in the last section are measure class-preserving bijections, since they are measure-preserving, and it is not difficult to adapt the constructions to obtain non measure-preserving elements of $\mathrm{Aut}(X,[\mu])$.

1. The irrational rotation T_θ on $X = [0,1]$ is uniquely ergodic, meaning that it admits a unique invariant measure, which is the Lebesgue measure. Hence, when we endow X with any other (atomless) measure $\mu \in [\mathrm{Leb}_{\restriction [0,1]}]$, T_θ is an element of $\mathrm{Aut}(X,[\mu])$ which does not preserve the measure.

2. Similarly, we define non-singular odometers by replacing the measure on $X = \prod_{n\in\mathbb{N}} X_n$, where $X_n = \{0,\dots,q_n-1\}$, by a non-uniform one. Concretely, we let μ_n be a probability measure on X_n, such that $\mu_n(k) > 0$ for any $k \in X_n$. We also need to assume that $\prod_{n\in\mathbb{N}} \max\{\mu_n(k) \mid k \in X_n\} = 0$ for the measure to be atomless. Then if $\mu = \bigotimes_{n\in\mathbb{N}} \mu_n$, the previously defined odometer T is in $\mathrm{Aut}(X,[\mu])$, and is not measure-preserving as soon as there is a μ_n which is not the uniform product measure on X_n.

3. Finally, Bernoulli shifts can also be made into non-singular bijections by changing the measure into an arbitrary product measure. We endow the space $Y^{\mathbb{Z}}$, where Y is either a standard Borel space or a finite or countable set, with the measure $\mu = \bigotimes_{n\in\mathbb{Z}} \mu_n$, where each μ_n is a probability measure on Y, and such that $\prod_{n\in\mathbb{N}} \sup\{\mu_n(y) \mid y \in Y\} = 0$. Then the Bernoulli shift S is in $\mathrm{Aut}(X,[\mu])$, and is not measure-preserving as soon as the measures μ_n are not all equal.

It is of note also that while it is easy to check that non-singular irrational rotations and odometers are ergodic, the question of the ergodicity of non-singular Bernoulli shifts is a difficult one (see *e.g.* [BKV21]).

1.2.3 Around recurrence: Poincaré, Rokhlin

In this section we present two indispensable results of ergodic theory: the Poincaré recurrence theorem, along with the associated notions of first return time, induced bijection, Kakutani-Rokhlin partition, and then the Rokhlin lemma. As they are both very important results, and as we believe that their proofs are in the spirit of the work we present in this text, we provide complete proofs. We will remain in the probability context for this section.

Theorem 1.2.9 (Poincaré recurrence theorem, see *e.g.* [VO16, Thm. 1.2.1])**.** *Let T be in $\mathrm{Aut}(X,\mu)$. For any Borel subset $A \subseteq X$, if $\mu(A) > 0$, then for μ-a.e. $x \in A$, the set $\{n \in \mathbb{N}^* \mid T^n(x) \in A\}$ is infinite.*

Proof. We start by considering the set A_0 of points $x \in A$ that never return to A. We first notice that the pre-images $T^{-n}(A_0)$ are disjoint: for that we assume that $x \in T^{-n}(A_0) \cap T^{-m}(A_0)$ for $n > m \geqslant 1$. Then $T^m(x)$ is in A_0 and thus $T^{n-m}(T^m(x)) = T^n(x) \in A_0$, meaning that a positive power of T sends an element of A_0 back to A_0. By definition of A_0, each such intersection is null. We then have $\sum_{n\in\mathbb{N}^*} \mu(T^{-n}(A_0)) = \sum_{n\in\mathbb{N}^*} \mu(A_0)$ by virtue of T being pmp, and since μ is finite, all the terms of the sum have to be 0, so A_0 is null.

Now we let A_f be the set of points of A returning to A only finitely many times. We have $A_f \subseteq \bigcup_{n\in\mathbb{N}} T^{-n}(A_0)$, which again, is null. □

Remark 1.2.10. Although we will only consider bijections, it is of note that the previous proof does not rely on the fact that T is invertible.

Definition 1.2.11. Let $T \in \mathrm{Aut}(X,\mu)$, and $A \subseteq X$ be a Borel subset with $\mu(A) > 0$. The Poincaré recurrence theorem ensures us that for μ-a.e. $x \in A$ there exists a (finite) **smallest return time** in A denoted by $n_A(x)$, *i.e.*

$$n_A(x) \coloneqq \min\{n \in \mathbb{N}^* \mid T^n(x) \in A\}.$$

We can then define the **induced bijection** T_A by $T_A(x) = T^{n_A(x)}(x)$ for all $x \in A$ and $T_A(x) = x$ for all $x \notin A$. The family (A_n), where $A_n \coloneqq \{x \in A \mid n_A(x) = n\}$, is called the **Kakutani-Rokhlin partition** of A with regard to T. Note that the family of towers $\left\{T^i(A_n) \mid 0 \leqslant i < n\right\}$ is a partition of the saturation $T_{\mathrm{sat}}(A) = \bigcup_{k\in\mathbb{Z}} T^k(A) = \bigcup_{k\in\mathbb{N}} T^k(A)$ of A. In particular, for T ergodic, $T_{\mathrm{sat}}(A) = X$, so $X = \bigcup_{k\in\mathbb{N}} T^k(A)$.

Note that by Proposition 1.1.6, $(A, \mu_{\restriction A})$ is a standard measured space of total mass $\mu(A)$, and $T_A \in \mathrm{Aut}(A, \mu_{\restriction A})$.

Next is the acclaimed Rokhlin lemma, or Rokhlin tower theorem. Several versions of this result exist in the literature, we chose to state the aperiodic one, which will be the most useful for our needs. It can be useful for the second part of the proof to know the language of measure algebras, so we give the definition now.

Definition 1.2.12. The **measure algebra** of the standard probability space (X,μ) is denoted by $\mathrm{MAlg}(X,\mu)$ and defined as the set of Borel subsets of X, where two such subsets A and B are identified if $\mu(A\Delta B) = 0$ (here Δ denotes the symmetric difference).

Theorem 1.2.13 (Rokhlin lemma, see *e.g.* [KM10, Thm. 172])**.** *Let T be an aperiodic element of $\mathrm{Aut}(X,\mu)$. For any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists a Borel subset A of positive measure such that $A, T(A), T^2(A), \ldots, T^{N-1}(A)$ are disjoint, and such that*

$$\mu\left(X \setminus \bigsqcup_{k=0}^{N-1} T^k(A)\right) < \varepsilon.$$

Proof. (1). We start by proving the ergodic version of the result, following [Gla03, Thm. 15.4].

We fix a Borel subset C such that $0 < \mu(C) < \frac{\varepsilon}{N}$. By ergodicity, the positive saturation $\bigcup_{n\in\mathbb{N}} T^n(C)$ of C is conull. We start, thanks to the Poincaré recurrence theorem, by writing the return time decomposition $C = \bigsqcup_{k\in\mathbb{N}^*} C_k$, where $C_k = \{x \in C \mid n_C(x) = k\}$. We call a set of the form $T^i(C_k)$ for $0 \leqslant i < k$ a floor of the k-th tower. We then cut each tower $\bigsqcup\left\{T^i(C_k) \mid 0 \leqslant i < k\right\}$ into blocks of N floors, with a remaining number of floors not exceeding $N-1$. We then define our set A as the first floor of each block of N floors, in the towers of index k, for $k \geqslant N$. See Figure 1.1 for an illustration of the construction.

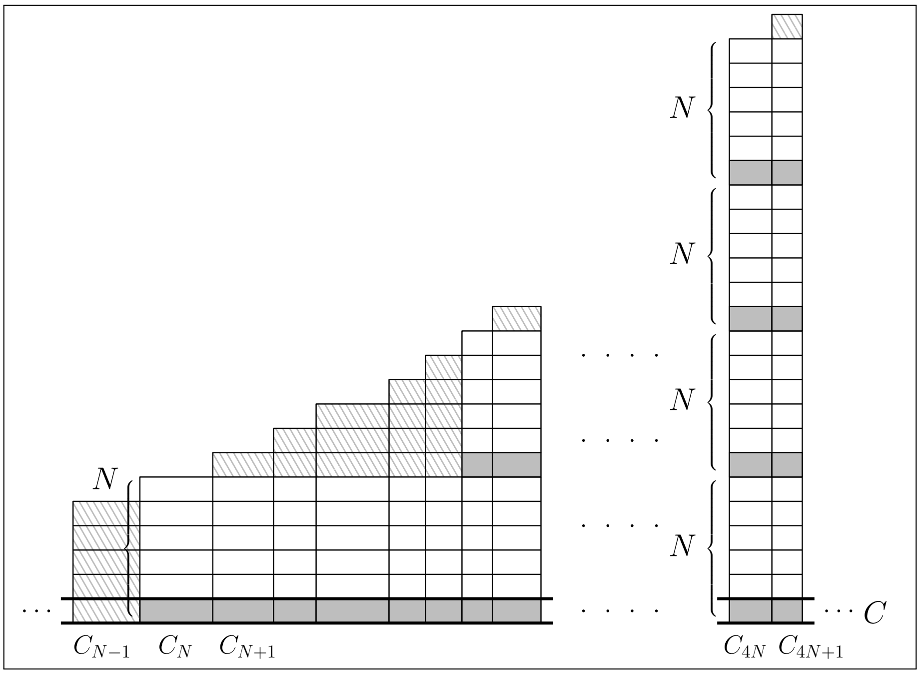

Figure 1.1: Visualisation of the proof of the Rokhlin Lemma (for $N = 6$). The greyed out part is the set A, the hatched part is the remainder.

By construction, the A-return time of elements of A by T is at least N, so the sets $A, T(A), \dots, T^{N-1}(A)$ are disjoint. Since the first non-modified towers have at most $N-1$ floors and the remainder floors in the modified towers number at most $N-1$, we can now easily check that

$$\mu\left(X \setminus (A \sqcup T(A) \sqcup \dots \sqcup T^{N-1}(A))\right) < N\mu(C) = \varepsilon.$$

(2). To extend this to the aperiodic context, we only need to find a set C of arbitrarily small measure whose T-saturation is conull, without the help of ergodicity. There are several ways to do that: it is possible to exhibit such a set via a measurable Zorn lemma, we can avoid doing so by using a result of existence of maximal elements in a measure algebra, using its completeness (see [LM14a, Prop. A.7], or Proposition 3.2.3 for a generalization). An even faster proof uses the following, which is immediate from the marker lemma, very interesting in its own right.

Lemma 1.2.14. *Let (X, μ) be a standard probability space, and $T \in \mathrm{Aut}(X, \mu)$ be aperiodic on its support. For any $\varepsilon > 0$ there exists a Borel subset C of $\operatorname{supp} T$ such that C intersects all non-trivial T-orbits, and $\mu(C) < \varepsilon$.*

Proof of the lemma. Let $\varepsilon > 0$. By aperiodicity of T, after throwing away a null set, the equivalence relation $\mathcal{R}_T = \{(x, T^k x) \mid x \in X, k \in \mathbb{Z}\}$ is aperiodic. By the marker lemma [KM04, Lem. 6.7], there is a decreasing sequence $(B_n)_n$ of Borel subsets of $\operatorname{supp} T$ which intersects every $\mathcal{R}_T$-class and satisfies $\bigcap_n B_n = \emptyset$. In particular for some large enough N we have $\lambda(B_N) < \varepsilon$, and choose this B_N to be our set C. $\diamondsuit$

The fact that the set C provided by the previous lemma intersects all T-orbits exactly means that its T-saturation is conull. $\square$

1.2.4 Around recurrence: dissipativity and conservativity

As stated before, the main interests of the present thesis lie in σ-finite spaces. Let us briefly discuss how recurrence behaves in this world.

The following definitions and results are classical, the statements are from [Aar97], which can be consulted for more details (see also [Kre85, § 1.3]). It is also notable that most of what follows hold in the broader context of non-singular transformations, although we will not go for this level of generality. We start with the quite technical notion of measurable union, before defining dissipativity and conservativity thanks to it. Note that the existence of the dissipative part could, just like in Theorem 1.2.13, be proved *via* [LM14a, Prop. A.7], but we chose to present the classical Exhaustion lemma, as we believe it is quite instructive.

Definition 1.2.15. A family $\mathcal{F}$ of Borel subsets of X is called **hereditary** if it is closed under subsets, *i.e.* for all $A \in \mathcal{F}$, and all Borel $A' \subseteq A$ we have $A' \in \mathcal{F}$. The **measurable union** $\bigcup_\lambda \mathcal{F}$ of a hereditary family $\mathcal{F}$ is a Borel subset that satisfies the following:

- $\bigcup_\lambda \mathcal{F}$ is a countable union of elements of $\mathcal{F}$;
- $\bigcup_\lambda \mathcal{F}$ **covers** $\mathcal{F}$, *i.e.* for any $A \in \mathcal{F}$, $A \subseteq \bigcup_\lambda \mathcal{F} \mod \lambda$;
- $\bigcup_\lambda \mathcal{F}$ is **saturated** by $\mathcal{F}$, *i.e.* for any Borel subset B of positive measure such that $B \subseteq \bigcup_\lambda \mathcal{F}$, there exists $A \in \mathcal{F}$ of positive measure such that $A \subseteq B \mod \lambda$.

Remark 1.2.16. The measurable union exactly corresponds to the union (*i.e.* the supremum) in the measure algebra, see Appendix A for more details.

Lemma 1.2.17 (Exhaustion lemma, [Aar97, 1.0.7])**.** *For any hereditary family $\mathcal{F}$ of Borel subsets of X, there exists a unique (up to a null set) measurable union $\bigcup_\lambda \mathcal{F}$.*

Proof. We first prove the uniqueness, which actually has nothing to do with exhaustion. If U_1 and U_2 are two measurable unions for the family $\mathcal{F}$, such that $\lambda(U_2 \setminus U_1) > 0$, then by saturation there exists $A \in \mathcal{F}$, with $A \subseteq U_2 \setminus U_1$ and $\lambda(A) > 0$. Since U_1 covers $\mathcal{F}$, $A \subseteq U_1$, which means that $\lambda(A) = 0$, a contradiction. The symmetric argument concludes.

Now we turn our attention to the exhaustion argument, used to prove existence. Since λ is σ-finite, there exists a probability measure in $[\lambda]$, which we will denote by μ. Define

$$\varepsilon_1 \coloneqq \sup \{\mu(A) \mid A \in \mathcal{F}\},$$

then find $A_1 \in \mathcal{F}$ such that $\mu(A_1) \geqslant \frac{\varepsilon_1}{2}$. By induction, defining

$$\varepsilon_n = \sup \left\{ \mu(A) \mid A \in \mathcal{F}, A \subseteq X \setminus \bigsqcup_{k<n} A_k \right\},$$

and finding $A_n \in \mathcal{F}$ such that $\mu(A_n) \geqslant \frac{\varepsilon_n}{2}$, we obtain sequences (ε_n) and (A_n). Since the A_n are disjoint, it is clear that $\sum_n \varepsilon_n \leqslant 2 \sum_n \mu(A_n) \leqslant 2$, so $\varepsilon_n \to 0$. It is clear that $\bigsqcup_n A_n$ is saturated by $\mathcal{F}$. It only remains to check that $\bigsqcup_n A_n$ covers $\mathcal{F}$. If not, there would exist $A \in \mathcal{F}$ of positive measure such that $A \cap A_n = \emptyset$ for any n, but $\mu(A)$ would have to be smaller than ε_n for any n, which is impossible by the previously proved convergence. Thus $\bigsqcup_n A_n$ is the measurable union $\bigcup_\lambda \mathcal{F}$ of the family $\mathcal{F}$. $\square$

With all the tools in hand let us now define dissipativity and conservativity, which we will notably use in Section 3.6.2.

Definition 1.2.18. Let T be a measure-preserving bijection of a standard σ-finite space (X, λ). A Borel subset $W \subseteq X$ is called a T**-wandering set** if the sets $T^n(W)$ are pairwise disjoint, for $n \in \mathbb{Z}$. We denote by $\mathcal{W}(T)$ the set of all T-wandering sets. Then

$$\mathfrak{D}(T) = \bigcup_\lambda \mathcal{W}(T)$$

is called the **dissipative part** of T. Do note ([Aar97, Prop. 1.1.2]) that by virtue of T being invertible, there exists a W in $\mathcal{W}(T)$ such that

$$\mathfrak{D}(T) = \bigsqcup_{n\in\mathbb{Z}} T^n(W),$$

meaning that W is a Borel fundamental domain (*i.e.* a set meeting each orbit in exactly one point) for the natural action of T on $\mathfrak{D}(T)$. This also implies that $\mathfrak{D}(T)$ is either null or of infinite measure. If T satisfies $\operatorname{supp} T \subseteq \mathfrak{D}(T)$ we say that T is **(totally) dissipative**. Do note also that $\mathfrak{D}(T)$ is indeed T-invariant.

Halmos' Recurrence Theorem ([Hal56], see also [Aar97, Thm. 1.1.1]) specifies the two possible behaviours for measure-preserving bijections on standard σ-finite spaces.

Theorem 1.2.19. *Let T be a measure-preserving bijection of a standard σ-finite space (X,λ). For any Borel subset A such that $\lambda(A)>0$ we have*

$$A\subseteq X\setminus\mathfrak{D}(T) \iff \forall\, B\subseteq A\colon \sum_{n\in\mathbb{N}} \mathbb{1}_B\circ T^n = +\infty\ \lambda\text{-a.e. on } B.$$

Motivated by this result, we give the following definition.

Definition 1.2.20. The measure-preserving bijection T is **(totally) conservative** if there exists no T-wandering sets of positive measure. Equivalently (by Theorem 1.2.19) T is conservative if it revisits any Borel set of positive measure at least once (equivalently, infinitely many times).

For any measure-preserving bijection T, the **conservative part** $\mathfrak{C}(T)$ is the measurable union with respect to λ of all T-invariants Borel sets that are revisited by T. In particular we have $\mathfrak{C}(T)=X\setminus\mathfrak{D}(T)$, and $\{\mathfrak{C}(T),\mathfrak{D}(T)\}$ is called the **Hopf decomposition** of T.

Furthermore for a conservative measure-preserving bijection T, we make the distinction between $\mathfrak{C}_f(T)$ and $\mathfrak{C}_\infty(T)$ the periodic and aperiodic conservative parts, comprised of the T-invariant Borel sets which consist of the finite and infinite T-orbits, respectively.

Remark 1.2.21. For any measure-preserving bijection T, we can then refine the Hopf decomposition: $X=\mathfrak{D}(T)\sqcup\mathfrak{C}_f(T)\sqcup\mathfrak{C}_\infty(T)$.

Remark 1.2.22. A measure-preserving bijection T with finite orbits is necessarily conservative.

We only prove the second item of the following facts, which again are from [Aar97, Cor. 1.1.4, Prop. 1.2.1], see also [Kre85, Thm. 3.4].

Proposition 1.2.23. *Let T be a measure-preserving bijection of a standard σ-finite space (X,λ).*

1. *$\forall\, n\in\mathbb{Z}\setminus\{0\}:\mathfrak{C}(T)=\mathfrak{C}(T^n)$ and $\mathfrak{D}(T)=\mathfrak{D}(T^n)$, up to a null set.*
2. *If T is ergodic, then it is totally conservative and aperiodic.*

Proof. *2.* It is clear that $T^{-1}\mathfrak{D}(T)=\mathfrak{D}(T)$, which means that T is either totally dissipative or totally conservative. Assume that it is totally dissipative, and consider a Borel fundamental domain W, satisfying $\lambda(W)>0$. The existence of a Borel subset $W'\subseteq W$ satisfying $\lambda(W')>0$ and $\lambda(W\setminus W')>0$, which is guaranteed by the fact that (X,λ) is standard (in particular atomless), contradicts the ergodicity of T. Hence T is totally conservative.

Next we notice that Halmos' Recurrence Theorem actually implies something stronger for conservative bijections. If we fix a Borel subset A such that $\lambda(A)>0$, then the set

$$\{x\in X \mid \exists \text{ infinitely many } n \text{ such that } T^n(x)\in A\}$$

is T-invariant (it is a property of the orbits), so it has to be conull. This means that for every Borel subset of positive measure A, for almost all $x \in X$, there exists infinitely many integers n such that $T^n(x)$ is in A. Cutting the space into countably many disjoint parts, all of which must be visited by T, yields aperiodicity. $\square$

Remark 1.2.24. Do note that the second point of the last statement holds because of the *atomless* assumption. Indeed, if atoms are allowed, consider $\mathbb{Z}$ endowed with the counting measure. The measure class-preserving bijection

$$\begin{array}{rccc} T & : \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & x+1 \end{array}$$

is ergodic, and clearly dissipative. It is the only such example, up to isomorphism (see [Aar97, Exercise. 1.2.1]).

Conservativity is exactly the right notion to define recurrence in the sense of Poincaré for bijections preserving an infinite measure. As such, Definition 1.2.11 naturally extends to this context. We give the precise statement in Definition 3.6.8.

1.2.5 Two topologies on the group of measure-preserving bijections

This text will mainly focus on the group $\mathrm{Aut}(X,\lambda)$ of ∞-mp bijections and some of its subgroups, but here we start by recalling some basic facts on the group $\mathrm{Aut}(X,\mu)$ of pmp bijections, which is more well-understood than its infinite measure-counterpart. Halmos' book can be consulted with great benefit for anything relevant to the material of this section (see [Hal56]). We can equip the previously defined $\mathrm{MAlg}(X,\mu)$ (see Definition 1.2.12) with the metric d_μ defined by $d_\mu(A,B) = \mu(A\Delta B)$. It is not too difficult to prove the following:

Proposition 1.2.25 ([LM14a, Prop. A.4], see also [LM22, Lem. 2.1])**.** *The metric space* $(\mathrm{MAlg}(X,\mu), d_\mu)$ *is separable and complete.*

Proof. Let us start with separability. As (X,μ) is standard it is isomorphic to $([0,1],\mathrm{Leb})$, and therefore finite unions of rational endpoints intervals are dense in $\mathrm{MAlg}(X,\mu)$.

Now for completeness, we let (A_n) be a d_μ-Cauchy sequence of Borel subsets (identified up to null sets). Passing to a subsequence if necessary, we may assume that $d_\mu(A_n, A_{n+1}) = \mu(A_n \Delta A_{n+1}) < 2^{-n}$ for any n. The Borel-Cantelli Lemma ensures us that

$$B \coloneqq \{x \in X \mid \exists N, \forall n \geqslant N : x \notin A_n \Delta A_{n+1}\}$$

is conull. Define then

$$A \coloneqq \{x \in X \mid \exists N, \forall n \geqslant N : x \in A_n\}.$$

If $x \in B \setminus A$, then there exists N big enough such that for any $n \geqslant N : x \notin A_n$ (indeed, if x is not in A, there exists infinitely many A_n that do not contain x, but if there also existed infinitely many A_n containing x, that would contradict the fact that x is in B). This is enough to conclude, as we then have $A\Delta A_N \subseteq \bigcup_{n\geqslant N} A_n \Delta A_{n+1}$, and since the measure of $\bigcup_{n\geqslant N} A_n\Delta A_{n+1}$ tends to zero, $d_\mu(A, A_N)$ also tends to zero. $\square$

Definition 1.2.26. The group $\mathrm{Aut}(X,\mu)$ can be endowed with the **weak topology**, which we will denote by τ_w, and is defined as follows. A sequence (T_n) of elements of $\mathrm{Aut}(X,\mu)$ converges weakly to $T \in \mathrm{Aut}(X,\mu)$ if $\mu(T_n(A)\Delta T(A)) \to 0$ as $n \to \infty$.

Equivalently, the weak topology on $\mathrm{Aut}(X,\mu)$ can be defined as the coarsest topology for which the maps $\mathrm{Aut}(X,\mu) \ni T \mapsto T(A) \in \mathrm{MAlg}(X,\mu)$ are continuous. The group $\mathrm{Aut}(X,\mu)$ can also be seen as the closed subgroup of the group of isometries of the separable and complete metric space $\mathrm{MAlg}(X,\mu)$ consisting of bijections T such that $T(\emptyset) = \emptyset$. By a classical result (see *e.g.* [GK03, Thm. 3.1]) and Proposition 1.1.19, we have the following.

Proposition 1.2.27. *The group* $\mathrm{Aut}(X,\mu)$ *is a Polish group when endowed with the weak topology.*

We also refer to the beginning of [Kec10] for more on $\mathrm{Aut}(X,\mu)$. For the analogous statements for $\mathrm{Aut}(X,\lambda)$, most of which adapt without much difficulty, we will wait until Section 3.3. Let us nonetheless note that Ionesco-Tulcea defined in [Ion65] a generalization of the weak topology (sometimes called the coarse topology, or even the strong topology, seeing as it comes from the strong operator topology) on the group $\mathrm{Aut}(X,[\mu])$ of non-singular bijections. Endowed with this topology, it is a Polish group. We refer to Section A.2 for more details.

Remark 1.2.28. As already stated, if $\mu\sim\lambda$, then both $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(X,\lambda)$ are subgroups of the group $\mathrm{Aut}(X,[\mu])$. For the natural Polish topology on $\mathrm{Aut}(X,[\mu])$, $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(X,\lambda)$ are *closed* subgroups. We refer to [DS22] for more details.

Before moving on to $\mathrm{Aut}(X,\lambda)$, let us define the second topology of interest of $\mathrm{Aut}(X,\mu)$.

Definition 1.2.29. We define the **uniform metric** d_u on $\mathrm{Aut}(X,\mu)$ by $d_u(S,T) = \mu(\{x\in X \mid S(x)\neq T(x)\})$. The topology it induces is called the **uniform topology**, and will usually be denoted by τ_u.

We will spend more time on this topology in Section 3.3.3, which is actually defined on the group $\mathrm{Aut}(X,[\mu])$, but let us quickly mention the following crucial facts.

Proposition 1.2.30 ([Hal56])**.** *The uniform metric on* $\mathrm{Aut}(X,\mu)$ *is complete, and the uniform topology refines the weak topology but is not separable.*

We quickly finish this section with the definitions of the analogous topologies on $\mathrm{Aut}(X,\lambda)$. The corresponding sections (Section 3.3.2 and Section 3.3.3) in chapter 3 can be consulted for more details.

Definition 1.2.31. The **weak topology** τ_w on $\mathrm{Aut}(X,\lambda)$ is defined in the following way: $T_n\to T$ if and only if for all $A\subseteq X$ Borel subset of finite measure, one has $\lambda(T_n(A)\Delta T(A)) = d_{X,\lambda}(T_n(A),T(A))\to 0$.

Definition 1.2.32. The **uniform topology** τ_u on $\mathrm{Aut}(X,\lambda)$ is the topology induced by the family of pseudometrics

$$d_{u,C} : S,T \mapsto \sup\left\{\lambda(C\cap(S(B)\Delta T(B))) \mid B\subseteq X \text{ Borel subset}\right\},$$

where C ranges among Borel subset of X of finite measure.

Proposition 1.2.33. *The uniform topology on* $\mathrm{Aut}(X,\lambda)$ *refines the weak topology, and the group* $\mathrm{Aut}(X,\lambda)$ *is a Polish group when endowed with the weak topology.*

1.2.6 Category results for $\mathrm{Aut}(X,\nu)$

In light of the Polishness of $(\mathrm{Aut}(X,\nu),\tau_w)$, a natural problem is to identify *generic* properties of measure-preserving bijections, that is to say properties that hold for a dense G_δ set of elements, which sometimes call generic elements. We present a brief and non exhaustive state of the art on the subject, and for a more thorough (although a bit dated now) survey we refer to [CP83].

We start with the results concerning $\mathrm{Aut}(X,\mu)$. The following is a direct consequence of the Rokhlin lemma (Theorem 1.2.13), proved by Halmos, and is obtained by cutting the remainder part into N parts of equal measure.

Theorem 1.2.34 (Uniform approximation theorem, see *e.g.* [Kec10, Thm. 2.2])**.** *The set of periodic elements of* $\mathrm{Aut}(X, \mu)$ *is uniformly dense (hence weakly dense) in* $\mathrm{Aut}(X, \mu)$.

In fact both the Rokhlin lemma and the uniform approximation theorem have been proved in the non-singular setting by Chacon and Friedman (see [CF65]).

Halmos has also proved the following about the class ERG of ergodic elements of $\mathrm{Aut}(X, \mu)$, the second point being a consequence of the uniform approximation theorem.

Theorem 1.2.35 ([Hal56, Uniform approximation, Category], see also [Kec10, Thm. 2.6])**.** *The following hold:*

(1) ERG *is a dense* G_δ *set in* $(\mathrm{Aut}(X, \mu), \tau_w)$*;*

(2) ERG *is nowhere dense in* $(\mathrm{Aut}(X, \mu), \tau_u)$*.*

Another consequence of the uniform approximation theorem is the following, also by Halmos, about density of conjugacy classes of aperiodic bijections. The main theorem of Section 3.7.3 is very much in the spirit of this result. We denote by APER the set of aperiodic elements of $\mathrm{Aut}(X, \mu)$.

Theorem 1.2.36 (see *e.g.* [Kec10, Thm. 2.4])**.** *We have the following:*

(1) APER *is dense* G_δ *in* $(\mathrm{Aut}(X, \mu), \tau_w)$*;*

(2) The conjugacy class $\{STS^{-1} \mid S \in \mathrm{Aut}(X, \mu)\}$ *of any* $T \in \mathrm{APER}$ *is* τ_u*-dense in* APER*;*

In particular, combining the two assertions yields that the conjugacy class of any aperiodic element is τ_w*-dense in* $\mathrm{Aut}(X, \mu)$*. Conversely, we have the following.*

(3) If the conjugacy class of a $T \in \mathrm{Aut}(X, \mu)$ *is weakly dense in* $\mathrm{Aut}(X, \mu)$*, then* $T \in \mathrm{APER}$.

Continuing with Halmos' (and Rokhlin's) work, he also proved (see [Hal56, Category]) that the set of strong mixing bijections (see Example 1.2.4) is meager in $(\mathrm{Aut}(X, \mu), \tau_w)$, while the set of weak mixing (which we will not define here) is dense G_δ in $(\mathrm{Aut}(X, \mu), \tau_w)$. It was only with Chacon's map (see Section B.3.1 for an infinite measure version) that an element of $\mathrm{Aut}(X, \mu)$ which is weak mixing but not strong mixing was explicitly presented.

We now move on from the probability world to the world of infinite measures. Before worrying about ergodicity, in light of the phenomena described in Section 1.2.4, the first order of business has been to study genericity of conservative bijections. This has been done by Krengel.

Theorem 1.2.37 ([Kre67, Thm. 8.1])**.** *The set of conservative elements of* $\mathrm{Aut}(X, \lambda)$ *is a dense* G_δ *set in both* $(\mathrm{Aut}(X, \lambda), \tau_w)$ *and* $(\mathrm{Aut}(X, \lambda), \tau_u)$*.*

The next meaningful results are from Sachdeva, who worked with non-necessarily invertible transformations, although we will only state the relevant results in our framework.

Theorem 1.2.38 ([Sac71, Thm. 2.2 and Thm. 3.1])**.** ERG *is dense* G_δ *in* $(\mathrm{Aut}(X, \lambda), \tau_w)$*, but is nowhere dense in* $(\mathrm{Aut}(X, \lambda), \tau_u)$*.*

Although it is not our primary concern in this text, category results for the group $\mathrm{Aut}(X, [\mu])$ also exist. In particular, Choksi and Kakutani proved the following.

Theorem 1.2.39 ([CK79, Thm. 3])**.** *The set* ERG *of ergodic elements of* $\mathrm{Aut}(X, [\mu])$ *is dense* G_δ *for the weak topology.*

Another result from their very rich paper is the following, about infinite-measure-preserving bijections, and of which we will actually make use in Section 3.7.3. It is of note also that the authors used this result to give another proof of the previously stated theorem of Sachdeva.

Proposition 1.2.40 ([CK79, Prop. 4])**.** *The set* ERG *of ergodic elements of* $\mathrm{Aut}(X,\lambda)$ *is* τ_u*-dense in* APER.

Let us conclude this section with a last result from the same paper, which we state in Section 3.7.3 in a slightly more general version.

Theorem 1.2.41 ([CK79, Thm. 7])**.** *Let T be a fixed aperiodic element in* $\mathrm{Aut}(X,\lambda)$.

(1) The set of $\mathrm{Aut}(X,\lambda)$*-conjugates of T is* τ_u*-dense in* APER*;*

(2) The set of $\mathrm{Aut}(X,\lambda)$*-conjugates of T is* τ_w*-dense in* $\mathrm{Aut}(X,\lambda)$.

1.3 Group actions and measure

As usual, (X,μ) will designate a standard probability space, (X,λ) a standard σ-finite space with $\lambda(X)=+\infty$, and a measure denoted by ν will be either μ or λ.

1.3.1 Actions of Polish groups on standard infinite measure spaces

Be it in the probability measure-preserving setup or the infinite measure-preserving one, so far we have mostly considered either classical dynamical systems, corresponding to $\mathbb{Z}$-actions, or general results on the whole group of measure-preserving bijections. However a lot of the objects we will be interested in this text actually come from ∞-mp group actions, sometimes even requiring ergodicity (which we will define in this context). We briefly recall basic facts about Polish group actions.

Definition 1.3.1. Let (G,τ_G) be a Polish group. A **G-action** on X is a map $\alpha : G\times X \to X$ (sometimes also denoted by $G \curvearrowright X$) such that $\alpha(e_G,x)=e_g\cdot x = x$, and

$$\alpha(h,\alpha(g,x))=\alpha(hg,x).$$

(We also sometimes use the notation $g\cdot x \coloneqq \alpha(g,x)$, when there is no risk of confusion.) We also say that X is a **G-space**. We will specifically be interested in the following three cases.

(1) If X is endowed with a metrizable topology τ_X, a **continuous G-action** is a G-action on X that is jointly continuous as a two variable function (equivalently separately continuous, see *e.g.* [Kec95, Thm. 9.14]). If τ_X is a Polish topology that is compatible with its standard Borel structure, we sometimes talk about a **Polish G-action**.

(2) A **Borel G-action** is a G-action on X that is Borel (as a two-variable function) with regard to the Borel σ-algebra generated by τ_G and to the Borel structure of X.

(3) A **ν-measure-preserving G-action** is a Borel G-action such that for every $g\in G$, the map $\alpha(g,\cdot)$ is an element of $\mathrm{Aut}(X,\nu)$.

We will also denote the **G-orbit** of x by $G\cdot x$, *i.e.* $G\cdot x \coloneqq \{g\cdot x \mid g\in G\}$.

With what we have already said, it is easy to see the following: from Theorem 1.1.24, Proposition 1.2.27 and Proposition 1.2.33, we get that any ν-measure-preserving G-action yields a continuous group homomorphism $\pi : G\to \mathrm{Aut}(X,\nu)$. As such, we will specifically be interested in subgroups of $\mathrm{Aut}(X,\nu)$. We can define ergodicity for such subgroups, hence we can define it for G-actions *via* ergodicity of the π-image of G in $\mathrm{Aut}(X,\nu)$.

Definition 1.3.2. We say that a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\nu)$ is **ergodic** if for every $A\subseteq X$ such that $\nu(T(A)\Delta A)=0$ for every T in $\mathbb{G}$, we have that A is either null or conull. For any Polish group G, we say that the G-action on (X,ν) is ergodic if the image of G in $\mathrm{Aut}(X,\nu)$ is ergodic.

We now point out a crucial fact: the homomorphism $\pi : G \to \mathrm{Aut}(X, \nu)$ allows us to see a G-action as a action on the measure algebra, acting on Borel subsets identified up to measure zero. Of course the genuine action on the points –called a **spatial action**– contains more information, and yields a **boolean action**. The possibility of going the other way, *i.e.* lifting a boolean action to a spatial action, is a very important problem, called the *realization problem*. Chapter 2 is partly dedicated to the study of the realization problem in the specific context of G-actions on standard infinite measure spaces.

We end this section by quickly going through some examples of ∞-mp group actions of uncountable groups.

Example 1.3.3. Let G be any non-compact lcsc group. Denote by m its Haar measure, which satisfies $m(G) = +\infty$. This makes (G, m) a standard σ-finite space of infinite measure. Any subgroup of G acts by translation on G, in a measure-preserving way. In particular, the G-action on (G, m) is always ergodic (see *e.g.* [Aar97, Prop. 1.6.8]).

With the 'trivial' examples out of the way, let us spend some time on the main class of examples: induced actions. We explicitly describe the case where we induce a $\mathbb{R}$-action from a $\mathbb{Z}$-action.

Example 1.3.4. One of the usual construction of a Borel flow (*i.e.* a Borel $\mathbb{R}$-action on a standard space) is the **suspension flow built under a function**. Starting from a ν-mp bijection $T : X \to X$, one can consider a roof (or ceiling) function $f : X \to \mathbb{R}_+^*$, which is Borel. We define the space Y to be the portion of the product space under the graph of f, *i.e.* $Y := \{(x, h) \in X \times \mathbb{R}_+ \mid 0 \leqslant h < f(x)\}$. We endow Y with the restriction of the product measure $\eta = \nu \otimes \mathrm{Leb}$. Notice that $\eta(X) = \int_X f d\nu$, so η is σ-finite infinite in two cases:

1. if $\nu = \lambda$, *i.e.* if the base (X, λ) has infinite measure,
2. if $\nu = \mu$ but f is not μ-integrable, *i.e.* the base has finite measure but the vertical component makes it infinite.

We define the flow $\alpha : \mathbb{R} \curvearrowright Y$ by setting

$$\alpha(t, (x, h)) := \begin{cases} (x, h+t) \qquad \text{if } h+t \in [0, f(x)[, \\ \left(T^n(x), h+t-\displaystyle\sum_{k=1}^{n} f(T^k(x))\right) \\ \qquad \text{if } h+t \in \left[\displaystyle\sum_{k=1}^{n-1} f(T^k(x)), \sum_{k=1}^{n} f(T^k(x))\right[, \\ \left(T^{-n}(x), h+t+\displaystyle\sum_{k=1}^{n} f(T^{-k}(x))\right) \\ \qquad \text{if } h+t \in \left[-\displaystyle\sum_{k=1}^{n} f(T^{-k}(x)), -\sum_{k=1}^{n-1} f(T^{-k}(x))\right[. \end{cases}$$

In other words, the α-action of a positive t moves the point (x, h) upwards along the x-fibre by translation by t, and if $h + t$ reaches the roof $(x, f(x))$, then it goes back to the base by applying T to the first coordinate, then keeps going upwards along the $T(x)$-fibre. Figure 1.2 is a classical illustration of the $\mathbb{R}$-action α on an example. This construction is a continuous analogue of the Kakutani skyscrapers defined in Section B.1.

The construction dates back to von Neumann, then Ambrose in [Amb41] and Ambrose and Kakutani in [AK42] have studied the problem of how simple can the roof function be. Notably, the Ambrose-Kakutani theorem asserts that any pmp ergodic flow is isomorphic to

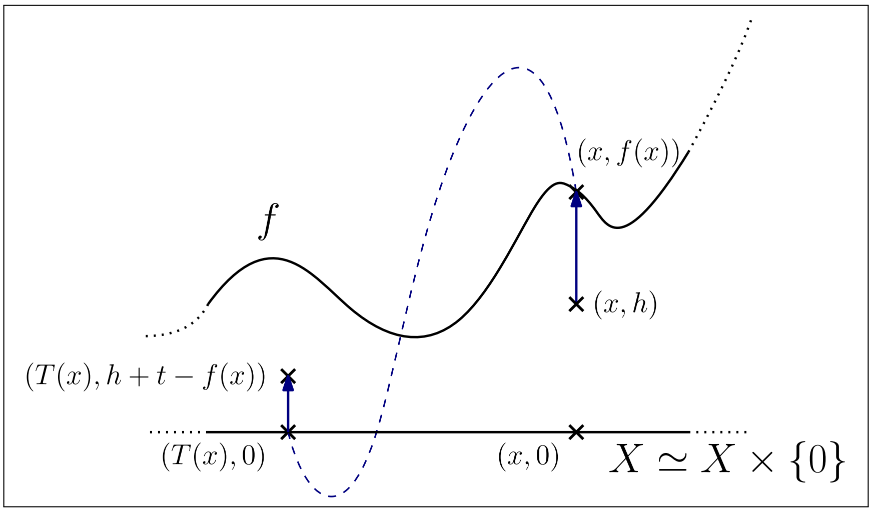

Figure 1.2: Example of a flow under a function.

a flow under a function, a result later strengthened by Rudolph in [Rud76], who expressed the roof function as a two-valued function.

This construction allows us to induce (∞-mp ergodic) $\mathbb{R}$-actions from (∞-mp ergodic) bijections, *i.e.* $\mathbb{Z}$-actions (see Appendix B for some examples), and multidimensional generalizations do exist, although we will not describe the constructions. We refer to [JŞ15] for an example of a ∞-mp *unit suspension*, extending a $\mathbb{Z}^2$-action to a $\mathbb{R}^2$-action.

A relative wealth of other examples of ∞-mp (ergodic) actions of lcsc groups exists in the literature, such as the rank-one flows from [IKSSW05], [DP11] or [DS07], or the Poissonian actions of non-(T) lcsc groups from [Dan22], [Dan23].

The same induction process applied to more general groups, providing in particular examples of ∞-mp G-actions of non locally compact Polish groups, is quickly described in Section 3.5.3.

1.3.2 Full groups and orbit equivalence

The notion of full groups was introduced by Dye in his pioneering work ([Dye59]) in order to study pmp bijections up to orbit equivalence. Orbit equivalence is a natural weakening of conjugacy, introduced partly because the problem of understanding pmp bijections up to conjugacy turned out to be very complicated (as it turns out, the isomorphism relation on $\mathrm{Aut}(X, \mu)$ is not Borel, see [FRW11]). We start with some background on orbit equivalence before defining full groups. The definition of orbit equivalence in infinite measure spaces will be given in Section 3.5.4.

Definition 1.3.5. Two actions from two groups G and H on the standard probability space (X, μ) are **orbit equivalent** if there exists $S \in \mathrm{Aut}(X, \mu)$ such that $S(G \cdot x) = H \cdot S(x)$ for μ-a.e. $x \in X$, and in that case we call S an orbit equivalence between the actions. In other words, two actions are orbit equivalent if they induce (up to a pmp automorphism) the same orbit partition of the space.

Dye proved the following for pmp group actions...

Theorem 1.3.6 ([Dye59])**.** *All the ergodic pmp actions of $\mathbb{Z}$ are orbit equivalent.*

...and Ornstein and Weiss gave the following generalization.

Theorem 1.3.7 ([OW80])**.** *Any two ergodic pmp actions of any two countable infinite amenable groups are orbit equivalent.*

Getting out of the countable and amenable case however, there is no reason for orbit equivalence to be the trivial relation. A good way to study an equivalence relation is to find an invariant. The full group of a group action (or more generally of an equivalence relation) plays this exact role in the case of countable group actions. We however start by giving the historical definition from Dye, *via* cutting and pasting.

Definition 1.3.8. Consider (T_n) a sequence of elements of $\mathrm{Aut}(X,\nu)$. An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have

$$T_{\restriction A_n} = T_{n\restriction A_n}.$$

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$. Figure 1.3 illustrates a pmp bijection obtained by cutting and pasting five pmp bijections along a partition of five non-trivial Borel subsets of $X = [0,1]$.

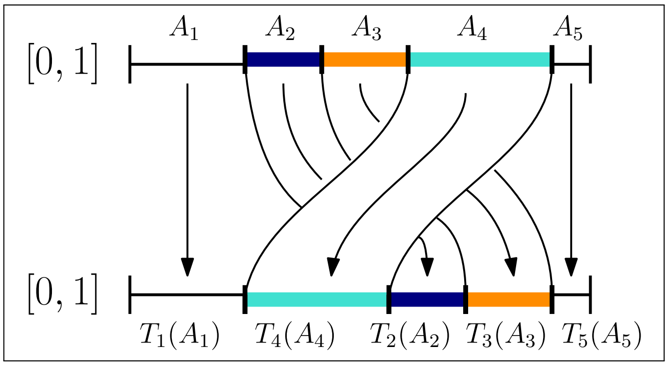

Figure 1.3: Construction of $T \in \mathrm{Aut}([0,1],\mathrm{Leb}_{\restriction[0,1]})$ by cutting and pasting $T_1,\dots,T_5$ along the partition $A_1,\dots,A_5$.

We give the rigorous definitions in Section 3.4, but it is immediate to generalize the previous definition to the case of a standard σ-finite space. Figure 1.4 illustrates the construction of a ∞-mp bijection by cutting and pasting.

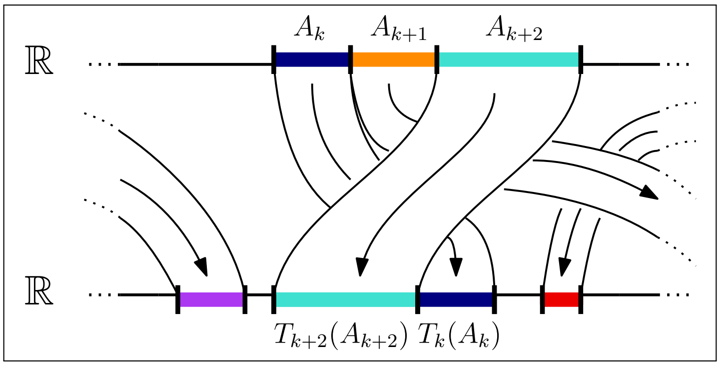

Figure 1.4: Construction of $T \in \mathrm{Aut}(\mathbb{R},\mathrm{Leb})$ by cutting and pasting (T_k) along the partition (A_k), only some arbitrary members of the partitions of $\mathrm{Dom}(T) = \mathbb{R}$ and $\mathrm{rng}(T) = \mathbb{R}$ are represented.

Remark 1.3.9. Although it is not the chosen framework for this text, full groups can be defined in the non-singular context as well. The first to do so was Krieger, in [Kri69], who worked with countably generated full groups. In Figure 1.5 we illustrate the construction of a non-singular bijection obtained by cutting and pasting five non-singular bijections along a partition of five non-trivial Borel subsets.

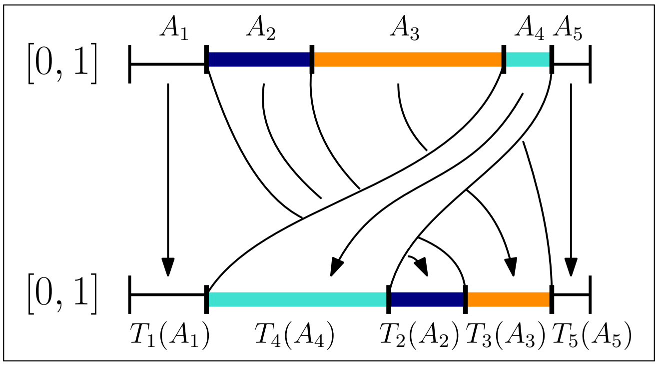

Figure 1.5: Construction of $T \in \mathrm{Aut}([0,1],[\mathrm{Leb}_{\restriction[0,1]}])$ by cutting and pasting $T_1,\dots,T_5$ along the partition $A_1,\dots,A_5$.

Of course, the main difference with the construction in Figure 1.3 is that the measure of the images of the A_k by the T_k does not have to be equal to the measure of the A_k, illustrated here by the length of the subintervals.

The most commonly found example of a full group is the full group of a (countable) equivalence relation. If a countable group Γ is acting on (X,μ), the equivalence relation

$$(x,y) \in \mathcal{R}_\Gamma \Leftrightarrow (\exists \gamma \in \Gamma, \gamma \cdot x = y)$$

allows us to define the full groups of the Γ-action. One can consult [Kec10, Sec. 3] for more on full groups of countable equivalence relations.

Definition 1.3.10. Let $\mathcal{R}$ be a Borel equivalence relation on X. The **full group of the countable equivalence relation** $\mathcal{R}$ is the subgroup $[\mathcal{R}]$ of $\mathrm{Aut}(X,\mu)$ defined by

$$[\mathcal{R}] \coloneqq \{T \in \mathrm{Aut}(X,\mu) \mid \text{for } \mu\text{-a.e. } x \in X : (x,T(x)) \in \mathcal{R}\}.$$

For the case of an equivalence relation $\mathcal{R}_\Gamma$ coming from the Borel action of a group Γ, we thus have the following rephrasing: $T \in [\mathcal{R}_\Gamma] \Leftrightarrow (T(x) \in \Gamma \cdot x$ for almost all $x \in X)$. It is also of note that by the theorem of Feldman and Moore (see *e.g.* [KM04, Thm. 1.3]), any countable equivalence relation can be realized as coming from the action of a countable group.

Once again, we view these groups as topological groups. As it turns out, the weak topology for which $\mathrm{Aut}(X,\mu)$ is a Polish topology is not the best topology to endow those full groups with: it is instead the previously defined uniform topology. The first thing to point out is the following.

Proposition 1.3.11 ([Dye59, Lem. 5.4])**.** *Any full group $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ is complete for the uniform metric d_u.*

In the case of full groups coming from a countable equivalence relation we can actually say more.

Proposition 1.3.12 ([CLM16, Prop. 3.8])**.** *A full group $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ is a Polish group when endowed with the restriction of τ_u if and only if it is the full group of a countable equivalence relation.*

As mentioned before, the full groups of countable equivalence relations are invariants of orbit equivalence. Now equipped with topology, let us give a bit more details. Assume that two actions

$$\alpha : \Gamma \curvearrowright (X,\mu) \qquad \text{and} \qquad \beta : \Lambda \curvearrowright (X,\mu)$$

of two countable groups are orbit equivalent, with $S \in \mathrm{Aut}(X,\mu)$ the associated orbit equivalence. The pmp bijection S conjugates $[\mathcal{R}_\Gamma]$ and $[\mathcal{R}_\Lambda]$, which are thus isomorphic as topological groups by Theorem 1.1.24 coupled with Proposition 1.3.12. Morally, being in a full groups means that one has to respect the orbits of the action, so up to the change of point of view which is represented by the conjugation by S, α and β have the same orbits, hence the same constraints, hence the same full groups, along with their topological structure.

The converse direction also holds: from an isomorphism between the full groups, one can deduce that the actions are orbit equivalent and 'reconstruct' the orbit equivalence. The statement is usually referred to as Dye's Reconstruction Theorem. We chose the statement from [CLM16, Thm. 3.27], which is less general than its original version, but has the merit of being a bit more digest with our vocabulary and framework.

Theorem 1.3.13 (Reconstruction Theorem, see [Dye63, Thm. 2])**.** *Let $\mathbb{G}_1$ and $\mathbb{G}_1$ be two ergodic full groups on a standard probability space (X,μ). Any group isomorphism $\Psi : \mathbb{G}_1 \to \mathbb{G}_1$ (and by this we mean an 'abstract' group isomorphism, not an isomorphism of topological groups) yields a $S \in \mathrm{Aut}(X,\mu)$ such that $\Psi(T) = STS^{-1}$ for any $T \in \mathbb{G}_1$.*

Another way to phrase all of this is to say that the full groups of countable group actions, seen as Polish groups for the uniform topology, are complete invariants of orbit equivalence. Let us quickly mention that while one of the main objectives of our work in Chapter 3 is to generalize results of this section, valid in the pmp setting to the ∞-mp setting, Dye's Reconstruction Theorem already exists in a more general form. It is this form that we use in Section 3.5.4, and we discuss the proof in Appendix A.

Based on what was just said, our next immediate goal is to link properties of the action with properties of the associated full group. For instance, ergodicity is visible from a topological point of view thanks to the following proposition. Note that it is stated in [Kec10] for full groups of countable equivalence relations, but the proof actually holds for more general full groups.

Proposition 1.3.14 ([Kec10, Prop. 3.1])**.** *A full group $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ is ergodic if and only if it is τ_w-dense in $\mathrm{Aut}(X,\mu)$.*

And as a consequence (using Theorem 1.1.24), one can observe the following.

Corollary 1.3.15 ([CLM16, Cor. 3.13])**.** *The group $\mathrm{Aut}(X,\mu)$ is the only ergodic full group that is Polish for the weak topology.*

More subtle properties of the group or of the action can also be reflected in the full group endowed with the uniform topology. For example, Giordano and Pestov proved the following.

Theorem 1.3.16 ([GP07, Thm. 5.7])**.** *Let Γ be a countable group acting in a ergodic manner on (X,μ). Then the following are equivalent:*

1. *Γ is amenable;*
2. *$[\mathcal{R}_\Gamma]$ is amenable;*
3. *$[\mathcal{R}_\Gamma]$ is a Lévy group.*

Again, in light of the Polishness of $([\mathcal{R}_\Gamma], \tau_u)$, questions of genericity are quite natural. In other words, what sort of category results from Section 1.2.6 can we hope to 'localize' from $(\mathrm{Aut}(X,\mu), \tau_w)$ to $([\mathcal{R}_\Gamma], \tau_u)$? As it turns out, quite a lot can be said, with some results requiring an ergodic action.

Theorem 1.3.17 (Uniform approximation theorem, see *e.g.* [Kec10, Thm. 3.3])**.** *Let Γ be a countable group, acting in a pmp manner on (X,μ). The set of periodic elements of $[\mathcal{R}_\Gamma]$ is uniformly dense in $[\mathcal{R}_\Gamma]$.*

Theorem 1.3.18 ([Kec10, Thm. 3.4])**.** *Let Γ be a countable group, acting in a pmp and ergodic manner on (X,μ). Then the $[\mathcal{R}_\Gamma]$-conjugacy class of any aperiodic element of $[\mathcal{R}_\Gamma]$, that is $\left\{STS^{-1} \mid S \in [\mathcal{R}_\Gamma]\right\}$, is τ_u-dense in $\mathrm{APER} \cap [\mathcal{R}_\Gamma]$.*

Theorem 1.3.19 ([Kec10, Thm. 3.6])**.** *Let Γ be a countable group, acting in a pmp and ergodic manner on (X,μ). Then the set $\mathrm{ERG} \cap [\mathcal{R}_\Gamma]$ is a τ_u-dense G_δ set in $\mathrm{APER} \cap [\mathcal{R}_\Gamma]$.*

We end this topic with the following theorem, which is very close in spirit to [CLM16, Thm. 4.4], and to our generalization in infinite measure, Theorem 3.7.16.

Theorem 1.3.20 ([Kec10, Thm. 3.9])**.** *Let Γ be a countable group, acting in a pmp and ergodic manner on (X,μ). The set of pairs (S,T) such that S and T generate a free group is a dense G_δ set in $([\mathcal{R}_\Gamma], \tau_u)^2$.*

So far for the study of orbit full groups we have remained in the world of either transitive actions (the full group is $\mathrm{Aut}(X,\mu)$ as a whole) or countable group actions, in which the theory is well-developed. But what about actions of more general Polish locally compact groups, or even of any Polish groups? The lack of a suitable natural Polish topology on the corresponding full groups has made their study very complicated, and as such, the theory developed in [CLM16] represents a major progress. Their main result is the following, unifying both $(\mathrm{Aut}(X,\mu), \tau_w)$ and $([\mathcal{R}_\Gamma], \tau_u)$ into a general framework, providing a topology for which *any* orbit full group coming from the action of a Polish group is a Polish group itself. Orbit full groups for general Polish groups actions are defined similarly to Definition 1.3.10.

Theorem 1.3.21 ([CLM16, Thm. 3.17])**.** *Let G be a Polish group acting in a Borel manner on (X,μ). Then there exists a topology τ_m^o on the associated orbit full group $[\mathcal{R}_G]$ which makes it a Polish group. Moreover, this topology is coarser than the uniform topology, but refines the weak topology.*

Generalizing this result has been the main objective of this PhD, and the corresponding result can be found in Theorem 3.5.7. Before concluding this section, let us quickly go back to the orbit equivalence. The authors also proved in [CLM16] that the fact that orbit full groups are complete invariants of orbit equivalence extends to orbit full groups coming from ergodic measure-preserving actions of Polish locally compact groups.

Theorem 1.3.22 ([CLM16, Thm. 3.26])**.** *Let G and H be two Polish locally compact groups acting in an ergodic and measure-preserving manner on (X,μ). Let $\Psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ be a group homomorphism. Then there exists an orbit equivalence $S \in \mathrm{Aut}(X,\mu)$ between $\mathcal{R}_G$ and $\mathcal{R}_H$, such that for all $T \in [\mathcal{R}_G]$ we have $\Psi(T) = STS^{-1}$.*

We also generalize this result to the II_∞ case in Theorem 3.5.31.

Chapter 2

Spatial models for boolean actions in the infinite measure-preserving setup

This chapter is based on the article [HLM24], which is a joint work with François Le Maître.

Abstract

We show that up to a null set, every infinite measure-preserving action of a locally compact Polish group can be turned into a continuous measure-preserving action on a locally compact Polish space where the underlying measure is Radon. We also investigate the distinction between spatial and boolean actions in the infinite measure-preserving setup. In particular, we extend Kwiatkowska and Solecki's Point Realization Theorem to the infinite measure setup. We finally obtain a streamlined proof of a recent result of Avraham-Re'em and Roy: Lévy groups cannot admit non-trivial continuous measure-preserving actions on Polish spaces when the measure is locally finite.

I'm a huge fan of space. Outer, personal, and as acted upon pointwise in a continuous manner by an uncountable Polish group.

Unknown (?)

Contents

2.1 Introduction

The notion of flow is a convenient framework for understanding global solutions of differential equations. In full generality, a **flow** on a set X is an action $\alpha : \mathbb{R} \times X \to X$ of the real line $\mathbb{R}$ on X. The connection with differential equations (on say $\mathbb{R}^n$ for simplicity) is simply the following: a *global solution* of the differential equation associated to the vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ is a flow $\alpha : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$\frac{d}{dt}\left[\alpha(t,x)\right]_{\restriction t=0} = V(x).$$

In particular, for all $x \in \mathbb{R}^n$, the map $t \mapsto \alpha(t,x)$ is everywhere differentiable. Provided a global solution exists and is unique, the regularity of the corresponding flow usually reflects that of the vector field. In particular, if V is continuous, we cannot expect the flow to be something better than continuous in general.

Many flows arising as solutions of a differential equation preserve a natural σ-finite measure which comes from the geometry of the ambient manifold, thus allowing one to use ergodic-theoretic methods. A notable example is given by Hopf's study of the geodesic flow on hyperbolic surfaces [Hop71]. From the ergodic point of view, flows are identified up to **conjugacy**, and make sense even when the flow is only assumed to be a measurable map.

To be more precise, recall that a **standard Borel space** is an uncountable measurable space X endowed with a σ-algebra $\mathcal{B}(X)$ of Borel subsets coming from some Polish topology on X. All such spaces are isomorphic, see [Kec95, Thm. 15.6], justifying the terminology. A **Borel flow** on X is then an action $\alpha : \mathbb{R} \times X \to X$ which is a Borel map $\mathbb{R} \times X \to X$, meaning that it is measurable if we endow $\mathbb{R} \times X$ with the usual product σ-algebra of the Borel σ-algebra of $\mathbb{R}$ and that of X. Given a Borel σ-finite measure λ on X, a **measure-preserving flow** on (X, λ) is simply a Borel flow such that for all $t \in \mathbb{R}$, the (automatically Borel) bijection $\alpha(t, \cdot)$ satisfies $\lambda(A) = \lambda(\alpha(t, A))$ for all $A \subseteq X$ Borel. Finally, two flows α, β on respective σ-finite measured standard Borel spaces (X, λ) and (Y, η) are called **spatially isomorphic** when there is a measure-preserving Borel bijection $\Phi : X_0 \to Y_0$ between two invariant conull Borel subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that for all $x \in X_0$ and all $t \in \mathbb{R}$, we have

$$\Phi(\alpha(t,x)) = \beta(t, \Phi(x)).$$

Given that certain measure-preserving flows come from differential equations, it is natural to wonder whether conversely every measure-preserving flow is spatially isomorphic to a flow coming from a differential equation. Our main result shows that as far as the *topology* is concerned, every measure-preserving flow is spatially isomorphic to a flow where the topology and the measure play nicely with each other, just as they do when they come from a differential equation. The key point is that the flow is continuous while the measure is *locally finite* (which implies that it is Radon since the ambient space is locally compact Polish, see Section 2.2.4). We state our result below in its most general form, replacing $\mathbb{R}$ by an arbitrary locally compact second-countable group, see Section 2.2.1 for relevant definitions.

Theorem A (see Cor. 2.4.4). *Let (X, λ) be a standard σ-finite space, let G be a locally compact Polish group and let $\alpha : G \times X \to X$ be a measure-preserving G-action on X. Then α is spatially isomorphic to a continuous measure-preserving G-action on a locally compact Polish space Y endowed with a Radon measure η.*

This result is new only when the measure λ is infinite. In order to explain why, we first recall the Becker-Kechris theorem, which holds in the even wider setup of Polish groups actions and does not require one to throw out null sets. Note that for locally compact Polish groups, item (1) below was first proven by Varadarajan, see [Var63, Thm. 3.2].

Theorem 2.1.1 ([BK96, Thm. 2.6.6 and Thm. 5.2.1])**.** *Let G be a Polish group and let $\alpha : G \times X \to X$ be a Borel G-action on a standard Borel space X.*

(1) There is a compact Polish space K endowed with a continuous G-action $\beta : G \times K \to K$, and a Borel injection $\Phi : X \to K$ which is G-equivariant: for all $x \in X$ and all $g \in G$,

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Moreover, β and K can be chosen so as not to depend on α.

(2) There exists a Polish topology on X inducing its Borel structure such that α is continuous.

From the first item, we deduce that if λ is any Borel finite measure on X and $\alpha : G \curvearrowright (X, \lambda)$ is any measure-preserving action of a Polish group G, then the map $\Phi : X \to K$ is a conjugacy of α with the continuous measure-preserving action $\beta : G \curvearrowright (K, \Phi_*\lambda)$, where K is endowed with the pushforward measure $\Phi_*\lambda$. Since finite Borel measures on compact Polish spaces are automatically Radon, *when λ is finite, Theorem A holds for any measure-preserving action of any Polish group G.* So our Theorem A says something new only for infinite measures (in the above argument the pushforward measure $\Phi_*\lambda$ fails to be locally finite, being an infinite measure over a compact space).

We have two ways of proving Theorem A. The first uses the associated *boolean action* (more on that later). The second one goes through the following measured version of the first item of the Becker-Kechris theorem.

Theorem B (see Thm. 2.5.4). *Let (X, λ) be a standard σ-finite space, let G be a locally compact Polish group and let $\alpha : G \times X \to X$ be a measure-preserving G-action on X. Then there is a continuous measure-preserving G-action on a locally compact Polish space Y and a Borel G-equivariant injection $\Phi : X \to Y$ such that the pushforward measure $\Phi_*\lambda$ is a Radon measure on Y.*

The above result is a strengthening of Theorem A: instead of just a full measure subset of X, the whole set X is taken into a continuous action with Radon measure. The main idea of its proof is to adapt Glasner, Tsirelson and Weiss' notion of G-continuous function (see [GTW05, Def. 2.1]) to the Borel setup, and to show that there is a countable set of integrable G-continuous bounded Borel functions that separates points. In order to prove the latter result, we use convolution, which is also crucial in our proof of Theorem A via boolean actions.

It is natural to ask whether Theorem A holds for (some) non locally compact Polish groups as well. However, there is a natural obstruction: the spatial G-action α on (X, λ) could have heavy local orbits in the following sense: for every $x \in X$ and every neighborhood of the identity $V \subseteq G$, the measurable set $\alpha(V, x)$ always has infinite measure (see Section 2.5.4 for an example when $G = \mathfrak{S}_\infty$ is the permutation group of the integers). Other than that, we do not see any obstruction and the following question is wide open.

Question 1. Let G be a non locally compact Polish group. Let $\alpha : G \curvearrowright (X, \lambda)$ be a measure-preserving action on an infinite σ-finite standard measured space (X, λ).

- When is α spatially isomorphic to a measure-preserving continuous G-action β on (Y, η) with Y locally compact Polish and η Radon?
- More generally, when is α spatially isomorphic to a measure-preserving continuous G-action β on (Y, η) with Y Polish and η locally finite?

Motivated by item (2) from Theorem 2.1.1, we ask another related question for general Polish groups.

Question 2. Let α be a measure-preserving action of a Polish group G on an infinite σ-finite standard measured space (X, λ). When can one endow X itself with a Polish topology inducing its standard Borel structure so that the action α becomes continuous and the measure λ becomes locally finite?

In the first version of this paper, we mentioned that we did not know the answer to this question even in the case where the acting group G is locally compact Polish. However, Nachi Avraham Re'em kindly pointed out to us that Theorem B can be combined with a stronger version of the Becker-Kechris theorem so as to obtain a positive answer when G is locally compact. We discuss this in Section 2.5.3, where we also explain why, even when G is discrete, one cannot in general endow X with a *locally compact* Polish topology so that the G-action becomes continuous (see Proposition 2.5.6).

We now move to the topic of boolean measure-preserving actions. These can be defined in many equivalent ways (see Section 2.2), but for this introduction the easiest is probably to describe them as continuous group homomorphisms $G \to \mathrm{Aut}(X, \lambda)$, where $\mathrm{Aut}(X, \lambda)$ is the group of all measure-preserving bijections of (X, λ) *identified up to measure zero*. A fundamental question is: which boolean actions lift to genuine spatial actions?

In order to answer it, Glasner, Tsirelson and Weiss developed in the setup of a probability measure-preserving boolean action α the following notion which we already briefly mentioned: a function $f \in \mathrm{L}^\infty(X, \lambda)$ is **G-continuous** if $\|f - f \circ \alpha(g, \cdot)\|_\infty \to 0$ as $g \to e_G$. Here is a natural extension of the conclusion of [GW05, Thm. 1.7] to the case where λ is possibly infinite (but still σ-finite).

Theorem C (see Thm. 2.3.5)**.** *Let G be a Polish group, and let $\alpha : G \to \mathrm{Aut}(X, \lambda)$ be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The following are equivalent:*

(1) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^1(X, \lambda)}^{\|\cdot\|_1} = \mathrm{L}^1(X, \lambda);$$

(2) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda);$$

(3) one can endow X with a locally compact Polish topology inducing its standard Borel space structure so that λ is Radon and α lifts to a continuous measure-preserving G-action on X.

Remark 2.1.2. We don't know if condition (3) can be changed so as to only require that X is Polish and λ is locally finite.

We use the above theorem to obtain natural infinite measure-preserving version of results of Kwiatkowska-Solecki, namely [KS11, Thm. 1.1]. Recall that given a metric space (X, d), its isometry group is naturally endowed with the topology of pointwise convergence, which makes it a Polish group as soon as (X, d) is separable and complete. While every Polish group arises as the isometry group of a separable complete metric space, the class of isometry groups of locally compact separable metric spaces is smaller, but it contains all locally compact Polish groups, and all Polish non-archimedean groups (see [KS11] for details). We can now state our generalization of [KS11, Thm. 1.1], which itself extends [GW05, Thm. 1.4(a) and Thm. 2.3].

Theorem D (see Thm. 2.4.7)**.** *Let G be the isometry group of a locally compact separable metric space. Then every boolean measure-preserving G-action on a standard σ-finite space (X, λ) can be lifted to a continuous measure-preserving action on X where X is endowed with a locally compact Polish topology inducing its standard Borel structure and λ is a Radon measure.*

When G is locally compact, this strengthens a result of Mackey which provides a purely Borel realization (see Theorem 2.2.7 and [Mac62] for the more general non-singular version). Furthermore, if the measure is in addition finite, our theorem follows by combining Mackey's result with [Var63, Thm. 3.2], but our proof strategy is very different. Indeed, we first treat the locally compact case by relying on convolution, which is very natural in the framework of G-continuous functions (see Theorem 2.4.3). As we briefly mentioned earlier, Theorem D can be used to give another proof of Theorem A using a result of Mackey (see Corollary 2.4.4).

In the general case, our proof is similar to Kwiatkowska and Solecki's, except that we begin with a simpler fixed point argument in L^2 relying on closed convex hulls, and then we work directly at the level of subalgebras of $\mathrm{L}^\infty(X, \lambda)$ (see the proof of Proposition 2.4.6).

We also adapt the Glasner-Tsirelson-Weiss results on whirly actions (see [GTW05, Sec. 3]) to the infinite measure-presering setup, showing in Section 2.7 that the tautological boolean $\mathrm{Aut}(X, \lambda)$-action cannot be lifted to a Borel spatial action.

Finally, we obtain the following result on Lévy groups, analogous to the Glasner-Tsirelson-Weiss result that says that all their continuous probability measure-preserving actions on compact metric spaces are trivial [GTW05, Thm. 1.1].

Theorem E (see Thm. 2.6.9)**.** *Every continuous measure-preserving spatial action of a Lévy group G on a Polish space X endowed with a σ-finite atomless locally finite measure λ is trivial, i.e. the set of fixed points $\{x \in X \mid \forall g \in G, g \cdot x = x\}$ is conull.*

While finishing the first version of our paper, we were informed by Emmanuel Roy that Nachi Avraham-Re'em and him had obtained the exact same result as above and put it on arXiv quite recently (see [AR25, Thm. 5]). Our proof actually follows the same two steps as theirs. First, we construct a Poisson point process without using counting measures, by directly working in the Effros standard Borel space (see Theorem 2.6.3, which is a minimalistic version of Theorem 3 in their paper). And then we conclude the proof using the Glasner-Tsirelson-Weiss theorem: the natural action of the group on the space of closed subsets preserves the Poisson point process probability measure, so it has to be trivial, which implies the triviality of the action we started with.

Since our construction of the Poisson point process appears to be more direct than Avraham-Re'em and Roy's, we decided to keep it in our paper, but the reader should definitely consult their work: they obtain a more precise description of the Poisson point process along with very nice applications of the above result to diffeomorphism groups, using the Maharam extension. They also have a complete description of the boolean probability measure-preserving actions one can obtain through this Poisson point process construction when starting from an infinite measure-preserving boolean action.

Finally, we don't know whether the conclusion of Theorem E holds for general infinite measure-preserving Borel actions, but recall that in general there are measure-preserving actions which cannot be spatially isomorphic to actions satisfying the hypothesis of the above theorem (see Section 2.5.4). Some related and intriguing open questions may be found in the last section of [AR25].

2.2 Preliminaries of Chapter 2

2.2.1 Spatial and boolean actions on standard σ-finite spaces

Recall that a **standard Borel space** is an uncountable measurable space X endowed with a σ-algebra $\mathcal{B}(X)$ of Borel subsets coming from some Polish topology on X. All such spaces are isomorphic, see [Kec95, Thm. 15.6], justifying the terminology.

Throughout the paper, (X, λ) will denote a standard Borel space X endowed with a non-trivial atomless σ-finite measure λ. Since rescaling the measure does not change the dynamics that we are interested in, there are only two fundamentally different cases:

- $\lambda(X) < +\infty$, in which case (X, λ) is isomorphic to $([0, \lambda(X)), \mathrm{Leb}_{\restriction})$ where Leb is the Lebesgue measure (this is a direct consequence of [Kec95, Thm. 17.41]);
- $\lambda(X) = +\infty$, in which case (X, λ) is isomorphic to $(\mathbb{R}, \mathrm{Leb})$ (to see this, observe that by σ-finiteness, X can be cut into a countable partition (X_n) such that $\lambda(X_n) < +\infty$ for all n, then cut $\mathbb{R}$ into a countable partition consisting of half-open intervals I_n of length $\mathrm{Leb}(X_n)$ and apply the previous case).

Given a standard σ-finite space (X, λ), a **measure-preserving bijection** of (X, λ) is a Borel bijection $T : X \to X$ such that $T_*\lambda = \lambda$, that is, for all $A \subseteq X$ Borel, $\lambda(A) = \lambda(T^{-1}(A))$.

Definition 2.2.1. Let G be a Polish group, let (X, λ) be a standard σ-finite space. A **spatial** measure-preserving G-action on (X, λ) is a Borel action map $\alpha : G \times X \to X$ such that for all $g \in G$, the map $\alpha(g, \cdot) : X \to X$ defines a measure-preserving bijection of (X, λ).

Our main goal in this paper is to contrast this notion to that of a boolean action, which to our knowledge first appeared in a paper of Mackey [Mac62]. We will give in the next section several characterizations of this notion, one of which justifies the terminology and the connection with Mackey's definition. In order to motivate the definition we are giving here, notice that by definition the fact that α is an action map means that for all $g, h \in G$ and for *all* $x \in X$ and, we have

$$\alpha(gh, x) = \alpha(g, \alpha(h, x)).$$

Definition 2.2.2. Let G be a Polish group, let (X, λ) be a standard σ-finite space. A **boolean** measure-preserving G-action on (X, λ) is a Borel map $\alpha : G \times X \to X$ such that:

(i) for all $g, h \in G$, we have that for λ-*almost* all $x \in X$,

$$\alpha(gh, x) = \alpha(g, \alpha(h, x));$$

(ii) for all $g \in G$, the map $\alpha(g, \cdot) : X \to X$ is measure-preserving bijection of (X, λ). (in particular, using $g = h = e_G$ in condition (i), we deduce that $\alpha(e_G, x) = x$ for λ-almost all $x \in X$.)

As is customary, we often write $\alpha(g)$ for the map $\alpha(g, \cdot) : X \to X$ when α is either a boolean or a spatial measure-preserving action.

Remark 2.2.3. By definition, Condition (i) means that for all $g, h \in G$ the following subset is conull:

$$X_{g,h} = \{x \in X : \alpha(gh, x) = \alpha(g, \alpha(h, x))\},$$

but $X_{g,h}$ does depend on g and h, in particular there does not need to be a conull subset $X_0 \subseteq X$ contained in all the $X_{g,h}$.

By definition, every measure-preserving spatial action is a measure-preserving boolean action. Spatial and boolean actions admit different natural notions of isomorphism.

Definition 2.2.4. Given measure-preserving spatial actions α on (X, λ) and β on (Y, η) of a Polish group G, a **spatial isomorphism** between them is a measure-preserving injection $\Phi : X_0 \to Y$, where X_0 is a conull $\alpha(G)$-invariant Borel subset of X, such that for all $g \in G$, and all x in X_0, we have

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

When there exists such a spatial isomorphism, we say that α and β are **spatially isomorphic**.

Remark 2.2.5. Since every Borel injection between standard Borel spaces is a Borel isomorphism onto its image [Kec95, Thm. 4.12], in the above definition the set $Y_0 := \Phi(X_0)$ is a Borel subset of Y which is conull since $\Phi_*\lambda_{\restriction X_0} = \eta$, and we conclude that being spatially isomorphic is a symmetric relation. It is not hard to check that it is also transitive, and hence an equivalence relation.

Definition 2.2.6. Given measure-preserving boolean actions α on (X, λ) and β on (Y, η) of a Polish group G, a **boolean isomorphism** between them is a measure preserving Borel injection $\Phi : X_0 \to Y$, where X_0 is a full measure Borel subset of X, such that for any g in G, there is a conull Borel subset $X_g \subseteq X_0$ such that for all $x \in X_g$,

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

When two boolean actions as above admit a boolean isomorphism between them, they are called **booleanly isomorphic**, and it is straightforward to check that this defines an equivalence relation on boolean actions. Let us point out that the above definition can be reinforced as follows: we can require X_0 to be equal to X and Φ to be bijective (see Proposition 2.2.8).

Observe that any spatial isomorphism between spatial actions is also a boolean isomorphism. Two natural problems arise from this for a fixed Polish group G

- the **realization problem**: given a boolean G-action, does it admit a *spatial realization* (or a *spatial model*), i.e. is it booleanly isomorphic to a spatial G-action?
- the **boolean to spatial isomorphism problem**: given a boolean isomorphism between spatial G-actions, is this isomorphism spatial?

When the answers to both these questions are positive, note that the spatial realization of a boolean action is unique up to spatial isomorphism. Using the fact that any countable intersection of conull sets is conull, it is not hard to see that when G is countable discrete, all its boolean actions are booleanly isomorphic to spatial actions, and all boolean isomorphisms of G-actions are spatial. More generally, a fundamental theorem of Mackey (see [Mac62]) states that this is still true when the acting group G is locally compact Polish. In our restricted context of measure-preserving actions, the statement is the following.

Theorem 2.2.7 ([Mac62, Thm. 1 and Thm. 2])**.** *Let G be a locally compact Polish group.*

(1) Let $\alpha : G \times X \to X$ be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). There exists a standard σ-finite space (Y, η) and a spatial G-action β on (Y, η) such that α and β are booleanly isomorphic.

(2) Let α and β be two spatial measure-preserving G-actions on standard σ-finite spaces (X, λ) and (Y, η) respectively. Any boolean isomorphism Φ between α and β coincides with a spatial isomorphism on a G-invariant conull Borel subset.

However, for non locally compact Polish groups, there might exist spatial actions which are booleanly isomorphic but not spatially. This is the case for $G = \mathfrak{S}_\infty$, and we will discuss this more thoroughly in Section 2.4.3.

We now go back to the definition of boolean isomorphism between actions, which can be reinforced via the following well-known proposition.

Proposition 2.2.8. *Two measure-preserving boolean actions α on (X, λ) and β on (Y, η) of a Polish group G are booleanly isomorphic if and only if there is a measure preserving bijection $\Phi : X \to Y$ such that for any g in G, there is a conull Borel subset $X_g \subseteq X$ such that for all $x \in X_g$,*

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Proof. By definition, assuming that α and β are booleanly isomorphic, we have a measure-preserving injection $\Phi : X_0 \to Y$ where X_0 is a full measure Borel subset of X satisfying our equivariance condition. Observe that it suffices to show that there is a measure-preserving bijection $\tilde{\Phi} : X \to Y$ which coincides with Φ on a conull Borel subset of X_0.

To this end, let $A \subseteq X_0$ be a Borel uncountable set of measure zero (such a set exists since λ is atomless, e.g. take for A the triadic Cantor subset of $(X, \lambda) = (\mathbb{R}, \mathrm{Leb})$).

Then $\tilde{A} \coloneqq (X \setminus X_0) \sqcup A$ and $\tilde{B} \coloneqq (Y \setminus \Phi(X_0)) \sqcup \Phi(A)$ are uncountable Borel subsets of standard Borel spaces, so by [Kec95, Cor. 13.4 and Thm. 15.6] there exists a Borel bijection $f : \tilde{A} \to \tilde{B}$, and we simply let $\tilde{\Phi} = f \sqcup \Phi_{\restriction X_0 \setminus A}$. □

The next remark was suggested by the first referee and requires the previous proposition. It can be summed up as follows: the realization problem actually asks whether we can turn the given boolean action into a spatial action without changing the underlying space.

Remark 2.2.9. Let α be a measure-preserving boolean action of a Polish group G on (X, λ) for which the realization problem has a positive answer. Then α is booleanly isomorphic to a spatial G-action β on (Y, η) through a measure-preserving bijection $\Phi : X \to Y$ provided by the previous proposition: for any $g \in G$, there is a conull Borel subset $X_g \subseteq X$ such that for all $x \in X_g$: $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$. If we define $\tilde{\alpha}$ by $\tilde{\alpha}(g, x) = \Phi^{-1}\beta(g, \Phi(x))$, then $\tilde{\alpha}$ is a spatial action on (X, λ) such that for any $g \in G$ and all $x \in X_g$ we have $\tilde{\alpha}(g, x) = \alpha(g, x)$. In particular $\tilde{\alpha}$ and α are booleanly isomorphic (through the identity map). This observation allows us to see the problem of spatial realization as finding an appropriate modification of α on the same space.

We finally observe that the definition of spatial isomorphism could naturally be weakened by requiring the $\alpha(G)$-invariant set X_0 to be Lebesgue measurable. This notion still implies boolean isomorphism of the corresponding boolean actions, so for locally compact Polish groups this does not change the definition of being spatially isomorphic thanks to Mackey's Theorem 2.2.7. However, we don't know what the situation is for general Polish groups.

2.2.2 Characterization of boolean actions

Following the introduction of [GTW05], we now provide other ways of understanding boolean actions, one of which will be used in our proof of Theorem A. We first need to describe the Polish group topology of the group of measure-preserving bijections.

Definition 2.2.10. Let (X, λ) be a standard σ-finite space. We denote by $\mathrm{Aut}(X, \lambda)$ the group of measure-preserving bijections, where two such bijections are identified if they coincide on a conull Borel subset of X.

The group $\mathrm{Aut}(X, \lambda)$ is endowed with **weak topology**, defined as follows: $T_n \to T$ if and only if for all $A \subseteq Y$ Borel subset of finite measure, one has $\lambda(T_n(A) \Delta T(A)) \to 0$.

In order to see that $\mathrm{Aut}(X,\lambda)$ is a Polish group, it is useful to recall the following definition.

Definition 2.2.11. The **measure algebra** $\mathrm{MAlg}_f(X,\lambda)$ is comprised of the Borel subsets of (X,λ) of finite measure, where two such subsets are identified if the measure of their symmetric difference is equal to zero. It is equipped with the metric d_λ, defined as follows :

$$d_\lambda(A,B) \coloneqq \lambda(A\Delta B).$$

Proposition 2.2.12 ([LM22, Prop. 2.2])**.** $\mathrm{Aut}(X,\lambda)$ *is equal to the group of isometries of* $(\mathrm{MAlg}_f(X,\lambda), d_\lambda)$ *which fix* $\emptyset$*. As such, it is a Polish group when endowed with the weak topology.*

The following proposition tells us that boolean G-actions on (X,λ) and continuous group homomorphisms $G \to \mathrm{Aut}(X,\lambda)$ are basically the same thing. This result is proved in the introduction of [GTW05] in the finite measure case, and it extends almost verbatim to our context. However, their *near actions* have a small difference with our boolean actions: we require $\alpha(g,\cdot)$ to be a measure-preserving bijection while they only require that it is a measure-preserving map, and need an additional axiom ensuring that in the end that it is a bijection *up to a null set*. Having a more restrictive definition which requires $\alpha(g,\cdot)$ to be a bijection of the whole space, we will need to work a little harder in order to prove the corresponding lifting property (item (III) below).

Proposition 2.2.13. *Let G be a Polish group, and (X,λ) a standard σ-finite space. The following hold:*

(I) Every boolean measure-preserving G-action α on (X,λ) yields a continuous group homomorphism $\pi_\alpha : G \to \mathrm{Aut}(X,\lambda)$.

(II) Given two boolean measure-preserving G-actions α, β on (X,λ), we have $\pi_\alpha = \pi_\beta$ if and only if the identity map is a boolean isomorphism between α and β.

(III) (lifting property) For every continuous group homomorphism $\pi : G \to \mathrm{Aut}(X,\lambda)$, there is a boolean measure-preserving G-action α on (X,λ) such that $\pi = \pi_\alpha$. Furthermore, such a lift α is unique up to boolean isomorphism.

Proof. *(I)*: Given a boolean measure-preserving action α of G on (X,λ), we can consider the natural mapping

$$\begin{array}{rcl} \pi_\alpha : G & \longrightarrow & \mathrm{Aut}(X,\lambda) \\ g & \longmapsto & \alpha(\,g\,,\cdot\,) \end{array}$$

which is well-defined by Condition (ii) from the definition of boolean action, and is a group homomorphism by Condition (i). Recall that any Borel group homomorphism between Polish groups is automatically continuous by a result which dates back to Banach (see e.g. [Ros09, Thm. 2.2]). We are thus left with showing that π_α is a Borel map. By definition of the weak topology on $\mathrm{Aut}(X,\lambda)$ it is enough to show that for every Borel finite measure subset A of X and every $\varepsilon > 0$, the set

$$\mathcal{B} \coloneqq \{g \in G : \lambda(\pi_\alpha(g)(A)\Delta A) < \varepsilon\}$$

is Borel. By the definition of π_α, we can rewrite $\mathcal{B}$ as $\mathcal{B} = \{g \in G : \lambda(\alpha(g,A)\Delta A) < \varepsilon\}$. Since the boolean action α is a Borel map, the subset $\Gamma \coloneqq \{(g,x) \in G \times X : \ x \in \alpha(g,A)\}$ is Borel and hence $\Gamma_A \coloneqq \Gamma\Delta(G \times A)$ is also Borel. By the Fubini-Tonelli theorem, this implies that the map

$$M : g \longmapsto \lambda\left(\{x \in X : \ x \in A\Delta\alpha(g,A)\}\right)$$

which associates to $g \in G$ the measure of the horizontal section of Γ_A above g is also Borel. So we can conclude that $\mathcal{B}$ is Borel by noting that $\mathcal{B} = M^{-1}([0, \varepsilon[)$, thus finishing the proof that π_α is Borel and hence continuous as desired.

(II): This is a straightforward consequence of the definition of $\mathrm{Aut}(X, \lambda)$, which identifies measure-preserving bijections which coincide on a conull set.

(III): We first observe that it suffices to show that the identity group homomorphism $\mathrm{id}_{\mathrm{Aut}(X,\lambda)} : \mathrm{Aut}(X, \lambda) \to \mathrm{Aut}(X, \lambda)$ can be lifted to a boolean measure preserving $\mathrm{Aut}(X, \lambda)$-action β on (X, λ), namely there is a Borel map $\beta : \mathrm{Aut}(X, \lambda) \times X \to X$ such that for all $T \in \mathrm{Aut}(X, \lambda)$, the map $\beta(T, \cdot)$ is a Borel measure-preserving bijection of (X, λ) whose equivalence class up to a null set is equal to T (in other words, $\mathrm{id}_{\mathrm{Aut}(X,\lambda)} = \pi_\beta$).

Indeed, granted that such a boolean action β exists, and given an arbitrary continuous group homomorphism $\pi : G \to \mathrm{Aut}(X, \lambda)$, we simply let $\alpha(g, x) = \beta(\pi(g), x)$. By construction we have $\pi = \pi_\alpha$, and the uniqueness of α is then a direct consequence of (II).

So let us prove *(III)* in the case $G = \mathrm{Aut}(X, \lambda)$ and $\pi = \mathrm{id}_{\mathrm{Aut}(X,\lambda)}$. As explained in Section 2.2.1 we may as well assume that X is an interval of $\mathbb{R}$ and λ is the Lebesgue measure restricted to that interval. Denote by $\mathrm{L}^0(X, \lambda, X)$ the set of measurable functions from X to itself, up to equality λ-a.e. Its Borel structure is defined by requiring the functions $f \mapsto \lambda(A \cap f^{-1}(B))$ to be measurable, for any Borel subsets A and B of X with A of finite measure. This makes $\mathrm{L}^0(X, \lambda, X)$ a standard Borel space, and it is straightforward to check that $\mathrm{Aut}(X, \lambda)$ is a Borel subset of $\mathrm{L}^0(X, \lambda, X)$.

Following Glasner-Tsirelson-Weiss, we can now define a Borel way to lift any element in $\mathrm{L}^0(X, \lambda, X)$ to a genuine function $X \to X$. Define $V : \mathrm{L}^0(X, \lambda, X) \times X \longrightarrow X$ by

$$V(f, x) = \limsup_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(z) d\lambda(z).$$

Applying the functional version of Lebesgue's density theorem (see [Fre03, Thm. 223A.]), for any f in $\mathrm{L}^0(X, \lambda, X)$, the function $x \mapsto V(f, x)$ is in the same λ-a.e. equality equivalence class as f. Restricting V to $\mathrm{Aut}(X, \lambda) \subseteq \mathrm{L}^0(X, \lambda, X)$, we obtain a Borel map $\beta_0 : \mathrm{Aut}(X, \lambda) \times X \to X$ verifying all the required conditions, except that for all $T \in \mathrm{Aut}(X, \lambda)$, the map $\beta_0(T) = V(T, \cdot) : X \to X$ is a measure-preserving bijection *up to a null set*.

In order to correct this, let us first define $\beta_1(T)$ as the partial Borel map whose graph is the intersection of the graph of $\beta_0(T, \cdot)$ with the inverse of the graph of $\beta_0(T^{-1}, \cdot)$. In other words,

$$\beta_1(T, x) = \beta_0(T, x) \text{ if } \beta_0(T^{-1}, \beta_0(T, x)) = x,$$

otherwise $\beta_1(T, x)$ is undefined.

Then $\beta_1(T)$ is a measure-preserving bijection between full measure subsets of X, and we denote by D_n^T the conull subset where $\beta_1(T)^n$ is defined, for $n \in \mathbb{Z}$. We finally define $\beta(T)$ by $\beta(T, x) = \beta_1(T, x)$ on $\bigcap_n D_n^T$, and by $\beta(T, x) = x$ elsewhere. By construction $\beta(T)$ is now a bijection of X itself, and it preserves the measure since it coincides with $\beta_1(T)$ on the conull Borel set $\bigcap_n D_n^T$.

The proof that β is still a Borel map boils down to the fact that for every $n \in \mathbb{Z}$, the set of $(T, x) \in \mathrm{Aut}(X, \lambda) \times X$ such that $x \in X \setminus D_n^T$ is Borel, which we leave to the reader. Since for every $T \in \mathrm{Aut}(X, \lambda)$, the bijection $\beta(T)$ coincides with $\beta_0(T)$ on a full measure set, the map β now satisfies all the axioms of a boolean action and $\mathrm{id}_{\mathrm{Aut}(X,\lambda)} = \pi_\beta$ as desired. □

By the above proposition, boolean measure-preserving G-actions are essentially the same thing as continuous group homomorphisms $G \to \mathrm{Aut}(X, \lambda)$. We can now justify the terminology properly: the space $\mathrm{MAlg}_f(X, \lambda)$ of finite measure Borel subsets up to measure zero can be equipped with the usual set theoretic operations Δ and $\bigcap$ so as to become a (non-unital) boolean ring. This boolean ring, when endowed additionally with the metric d_λ, has its automorphism group equal to $\mathrm{Aut}(X,\lambda)$ as a consequence of Proposition 2.2.12.

So continuous group homomorphisms $G \to \mathrm{Aut}(X,\lambda)$ are exactly continuous G-actions by automorphism on the metric boolean ring $(\mathrm{MAlg}_f(X,\lambda),\Delta,\cap,d_\lambda)$, and since these homomorphisms are the same thing as boolean actions by the above proposition, the terminology is now fully justified.

As we will see in the next section, we can also view boolean actions as trace preserving actions on the von Neumann algebra $\mathrm{L}^\infty(X,\lambda)$ satisfying an additional continuity condition.

2.2.3 Actions on function spaces

Given a standard σ-finite space (X,λ), we denote by $\mathrm{L}^0(X,\lambda)$ the space of measurable functions $X \to \mathbb{C}$, two such functions being identified if they coincide on a conull set. The following subspaces of functions are of importance to us:

- the space $\mathrm{L}^\infty(X,\lambda)$ of complex-valued bounded measurable functions on X, endowed with the essential supremum norm $\|\cdot\|_\infty$;
- the space $\mathrm{L}^1(X,\lambda)$ of complex-valued integrable functions on X, endowed with the L^1 norm given by $\|f\|_1 = \int_X |f|\,d\lambda$;
- the space $\mathrm{L}^2(X,\lambda)$of complex-valued square integrable functions on X, endowed with the Hilbert space structure given by the scalar product $\langle f,g\rangle = \int_X \overline{f} g d\lambda$.

We have a natural left action of $\mathrm{Aut}(X,\lambda)$ on $\mathrm{L}^0(X,\lambda)$ given by

$$T\cdot f(x) = f(T^{-1}(x)).$$

All the subspaces $\mathrm{L}^\infty(X,\lambda)$, $\mathrm{L}^2(X,\lambda)$ and $\mathrm{L}^1(X,\lambda)$ are invariant under this action. Moreover, they are acted upon by isometries (for the last two, this uses the fact that elements of $\mathrm{Aut}(X,\lambda)$ preserves the measure, while for $\mathrm{L}^\infty(X,\lambda)$ we only need that the measure class is preserved).

However, the $\mathrm{Aut}(X,\lambda)$-action on $\mathrm{L}^\infty(X,\lambda)$ is not continuous (for the norm $\|\cdot\|_\infty$), while the $\mathrm{Aut}(X,\lambda)$-action on $\mathrm{L}^2(X,\lambda)$ and $\mathrm{L}^1(X,\lambda)$ are both continuous. Indeed, for any Borel subset A of (X,λ) of finite measure, A is sent by an element T of $\mathrm{Aut}(X,\lambda)$ to a subset $T(A)$ which is close to A in $(\mathrm{MAlg}_f(X,\lambda),d_\lambda)$ whenever T is close to id_X. This suffices to prove the continuity, as both actions are by isometries and linear combinations of finitely many characteristic functions of finite measure subsets are dense in $\mathrm{L}^2(X,\lambda)$ and $\mathrm{L}^1(X,\lambda)$ for their respective norms.

We denote by $\mathcal{B}(\mathrm{L}^2(X,\lambda))$ the space of bounded operators on $\mathrm{L}^2(X,\lambda)$, that is to say the bounded linear maps from $\mathrm{L}^2(X,\lambda)$ to itself, and by M_f the operator of pointwise multiplication by a element f of $\mathrm{L}^\infty(X,\lambda)$. We recall that the map

$$\begin{array}{cccc} M : & \mathrm{L}^\infty(X,\lambda) & \longrightarrow & \mathcal{B}(\mathrm{L}^2(X,\lambda)) \\ & f & \longmapsto & M_f \end{array}$$

is an isometric $*$-isomorphism between $\mathrm{L}^\infty(X,\lambda)$ (endowed with the algebra structure provided by pointwise multiplication) and a (von Neumann) subalgebra of $\mathcal{B}(\mathrm{L}^2(X,\lambda))$, where $*$ is the involution taking $f\in\mathrm{L}^\infty$ to its complex conjugate $x\mapsto \overline{f(x)}$. The map M allows us to endow $\mathrm{L}^\infty(X,\lambda)$ with the **strong operator topology**: a net f_n converges to f if and only for all $\xi\in\mathrm{L}^2(X,\lambda)$, $M_{f_n}\xi \to M_f\xi$ in L^2 norm.

Remark 2.2.14. Suppose λ is infinite and μ is a finite measure equivalent to λ. Let $f = \frac{d\lambda}{d\mu}$ be the Radon-Nikodym derivative, then the map $g \mapsto \sqrt{f}g$ induces a surjective isometry $\mathrm{L}^2(X,\lambda) \to \mathrm{L}^2(X,\mu)$ which commutes with the action by multiplication of

$\mathrm{L}^\infty(X,\mu)=\mathrm{L}^\infty(X,\lambda)$. We conclude that strong convergence in $\mathrm{L}^\infty(X,\mu)$ is an intrinsic notion which does not depend on the choice of a (possibly infinite) σ-finite measure in the class of μ.

We finally identify $\mathrm{Aut}(X,\lambda)$ to the following group.

Proposition 2.2.15. *Given any standard σ-finite space (X,λ), the group $\mathrm{Aut}(X,\lambda)$ naturally identifies to the group of $*$-automorphisms of $\mathrm{L}^\infty(X,\lambda)$ which preserve the integral of elements of $\mathrm{L}^\infty(X,\lambda)\cap\mathrm{L}^1(X,\lambda)$.*

Proof. We have already observed at the beginning of this section that precomposition by the inverse allows to view every measure-preserving bijection T of (X,λ) as an integral-preserving automorphism α_T of $\mathrm{L}^\infty(X,\lambda)$, namely $\alpha_T(f)(x)=f(T^{-1}(x))$. Observe that for every $A\subseteq X$ Borel, $\alpha_T(\mathbb{1}_A)=\mathbb{1}_{T(A)}$.

Conversely, every integral-preserving $*$-automorphims α of $\mathrm{L}^\infty(X,\lambda)$ must take projections to projections. Since the projections which are integrable are naturally identified to characteristic functions of elements of $\mathrm{MAlg}_f(X,\lambda)$, every integral-preserving automorphism α of $\mathrm{L}^\infty(X,\lambda)$ defines a measure-preserving transformation T. Moreover, standard results on C^*-algebras yield that α is an isometry for $\|\cdot\|_\infty$ (see e.g. [Con00, Cor. 1.8]). Using the $\|\cdot\|_\infty$-density of the linear span of projections, we conclude that α is equal to α_T as wanted. $\square$

Having the above proposition and Proposition 2.2.13 in mind, we can view boolean actions on (X,λ) as integral-preserving actions by $*$-automorphisms [1] on $\mathrm{L}^\infty(X,\lambda)$ which are **continuous** in the following sense (derived from Proposition 2.2.12): whenever $g_n\to g$ and $A\in\mathrm{MAlg}_f(X,\lambda)$, we have $\int_X|g_n\cdot\mathbb{1}_A-g\cdot\mathbb{1}_A|\,d\lambda\to0$, or equivalently whenever $g_n\to e_G$ and $A\in\mathrm{MAlg}_f(X,\lambda)$, we have $\int_X|g_n\cdot\mathbb{1}_A-\mathbb{1}_A|\,d\lambda\to0$. Boolean isomorphisms then become the following.

Proposition 2.2.16. *Let α and β be two boolean G-actions on (X,λ) and (Y,η) respectively, both viewed as continuous actions by integral-preserving automorphisms of their respective L^∞ spaces. Then α and β are booleanly isomorphic if and only if there is an integral-preserving $*$-isomorphism $\rho:\mathrm{L}^\infty(X,\lambda)\to\mathrm{L}^\infty(Y,\eta)$ such that for all $f\in\mathrm{L}^\infty(X,\lambda)$, we have $\rho(\alpha(g,f))=\beta(g,\rho(f))$.*

Proof. We may as well assume that $X=Y$ and $\lambda=\eta$, so that by the above proposition ρ can be lifted to a measure-preserving bijection $\Phi:X\to X$. Using the uniqueness of lifts of elements of $\mathrm{Aut}(X,\lambda)$ up to measure zero, it is then straightforward to check that Φ satisfies the required conditions to be a boolean isomorphism between the actions. $\square$

2.2.4 Locally finite measures

In this section, we introduce the main property of a Borel measure on a Polish space that we will be interested in, and connect it with the notion of a Radon measure.

Definition 2.2.17. Let λ be a Borel measure on a Polish space X. We say that λ is **locally finite** if every $x\in X$ admits an open neighborhood U such that $\lambda(U)<+\infty$.

Of course this property is only interesting for infinite measures. Let us first observe that it implies σ-finiteness.

Lemma 2.2.18. *Every locally finite measure λ on a Polish space X is σ-finite.*

[1] This is not the right point of view if one wants to make sense of continuous actions on von Neumann algebras in general. In a more general setup, one conveniently uses the fact that any automorphism yields an isometry of the predual which completely determines it, thus getting a natural Polish topology on the automorphism group of any von Neumann algebra with separable predual.

Proof. By assumption, X is covered by finite measure open sets. Lindelöf's lemma grants us a countable subcover which witnesses the fact that λ is σ-finite. $\square$

Remark 2.2.19. A σ-finite measure on a standard Borel space needs not be locally finite. For instance, the counting measure on $\mathbb{Q}$ extends to a non locally finite σ-finite measure on $\mathbb{R}$. More interesting examples will be given later on.

We then recall a definition of Radon measures, following [Coh13, Sec. 7.2] where these are called regular measures.

Definition 2.2.20. A measure λ on the Borel σ-algebra of a Hausdorff topological space X is called **Radon** when it verifies the following:

1. $\lambda(K) < +\infty$ for any compact subset $K \subseteq X$,
2. for each open subset U of X, we have $\lambda(U) = \sup\{\lambda(K) \mid K \subseteq U, \text{ and } K \text{ is compact}\}$ (inner regularity on open sets).
3. for each Borel subset A of X, we have $\lambda(A) = \inf\{\lambda(U) \mid A \subseteq U, \text{ and } U \text{ is open}\}$ (outer regularity on Borel sets).

Proposition 2.2.21. *Every locally finite measure λ on a Polish space X is Radon.*

Proof. A straightforward compactness argument shows that λ is finite on compact sets.

For outer regularity, we adapt Cohn's proof of [Coh13, Prop. 7.2.3]. By local finiteness, X can be covered by finite measure open subsets. Lindelöf's lemma yields a countable cover $(U_n)_{n\in\mathbb{N}}$ of X such that $\lambda(U_n) < +\infty$ for any n. We define the finite Borel measures

$$\mu_n := \lambda(\,\cdot\, \cap U_n),$$

which are outer regular on X by [Coh13, Lem. 7.2.4] (indeed X satisfies the condition that any open set is F_σ by virtue of being metrizable). The rest of the proof is the same: we fix $\varepsilon > 0$ and a Borel set $A \subseteq X$. By outer regularity of μ_n we can find an open set V_n containing A such that

$$\mu_n(V_n) < \mu_n(A) + \frac{\varepsilon}{2^{n+1}},$$

therefore we get that

$$\lambda((V_n \cap U_n) \setminus A) < \frac{\varepsilon}{2}$$

for any $n \in \mathbb{N}$. As $V := \bigcup_n (V_n \cap U_n)$ is open and satisfies $A \subseteq V$ and $\lambda(V \setminus A) < \varepsilon$, it witnesses the outer regularity of λ, concluding the proof of the outer regularity.

Inner regularity on finite measure open sets directly follows from [Coh13, Prop. 8.1.12]. Finally, given an open set V of infinite measure, observe that $\lambda(V \cap (\bigcup_{n\leqslant N} U_n))$ becomes arbitrarily large as N grows, hence by inner regularity on finite measure open sets, the set V contains arbitrarily large measure compact subsets as desired. $\square$

Recall that a locally compact space is Polish if and only if it is second-countable (see [Kec95, Thm. 5.3]). In such spaces, Radon measures behave particularly nicely.

Proposition 2.2.22. *Let X be a locally compact Polish space, let λ be a Borel measure on X. Then λ is Radon if and only if it is locally finite.*

Proof. Suppose λ is Radon, then since it is finite on compact sets and X is locally compact, we conclude that λ is locally finite. The converse is a direct consequence of the previous proposition. $\square$

Remark 2.2.23. As noted in [Coh13, Prop. 7.2.6], σ-finiteness implies inner regularity on all Borel sets, not only open sets. Putting together the previous proposition and Lemma 2.2.18, we see that locally finite Borel measures on locally compact Polish spaces are always inner regular on all Borel sets.

We finally quote the Riesz-Markov-Kakutani representation theorem, which will allow us to build Radon measures on new spaces.

Theorem 2.2.24 (Riesz-Markov-Kakutani, see e.g. [Rud87, Thm. 2.14])**.** *Let Y be a Polish locally compact space, and let Ψ be a positive linear functional on the space of complex-valued compactly supported continuous functions $C_c(Y)$. Then there exists a unique Borel measure η on Y such that*

$$\Psi(f) = \int_Y f d\eta$$

holds for each f in $C_c(Y)$. Moreover, the measure η is Radon.

Remark 2.2.25. In the proof of Theorem 2.5.1, it is crucial that the uniqueness of η does not require it to be Radon. As observed in the beginning of the proof of [Rud87, Thm. 2.14] any Borel measure m such that $\Psi(f) = \int_Y f dm$ holds for each f in $C_c(Y)$ must be finite on compact sets, hence locally finite, which implies that it is Radon by Proposition 2.2.22, so the above statement does follow from [Rud87, Thm. 2.14] (see also [Rud87, Thm. 2.18]).

2.2.5 Gelfand spaces in the non unital case

In order to build spatial actions, we will as in [GTW05] crucially use separable C^* algebras, but in our setup these will be non-unital. We thus start by recalling the Gelfand-Naimark theorem in the case of a non-unital commutative C^*-algebra.

Definition 2.2.26. The **spectrum** of a commutative C^*-algebra $\mathcal{A}$ is the space of **characters** on $\mathcal{A}$, that is to say the space of non-zero homomorphisms $\mathcal{A} \longrightarrow \mathbb{C}$. We denote by $\mathrm{Sp}(\mathcal{A})$ the spectrum of $\mathcal{A}$. It is locally compact and Hausdorff when equipped with the topology of pointwise convergence, and if $\mathcal{A}$ is unital, it is a compact space.

Proposition 2.2.27. *If a commutative C^*-algebra $\mathcal{A}$ is separable, then $\mathrm{Sp}(\mathcal{A})$ is a locally compact Polish space.*

Proof. By [Mur90, Rem. 4.4.1], $\mathrm{Sp}(\mathcal{A})$ is second-countable. Since every locally compact second-countable Hausdorff space is Polish [Kec95, Thm. 5.3], the conclusion follows. □

Theorem 2.2.28 (Gelfand-Naimark, see e.g. [Mur90, Thm. 2.1.10])**.** *Let $\mathcal{A}$ be a non-zero commutative C^*-algebra. The Gelfand representation*

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & C_0(\mathrm{Sp}(\mathcal{A})) \\ f & \longmapsto & \widehat{f} \end{array}$$

where $\widehat{f}(z) = z(f)$, between $\mathcal{A}$ and the space of continuous functions that vanish at infinity on $\mathrm{Sp}(\mathcal{A})$ is an isometric $*$-isomorphism.

We now give the proof of the following fact, which is probably well-known, but for which we have found no reference in the literature. Combined with Theorem 2.2.24, this will allow us to define a Radon measure on $\mathrm{Sp}(\mathcal{A})$. We are grateful to Georges Skandalis for pointing this fact out to us.

Proposition 2.2.29. *Let $\mathcal{A} = C_0(X)$, where X is a locally compact Polish space. Let also $\mathcal{I}$ be an ideal of $\mathcal{A}$, dense in $\mathcal{A}$ with regard to the sup norm $\|\cdot\|_\infty$. Then, $C_c(X) \subseteq \mathcal{I}$, where $C_c(X)$ denotes the space of compactly supported continuous functions on X.*

Proof. Let us fix a non-zero function f in $C_c(X)$, supported in a compact subset K.

The Tietze extension theorem grants us a function g in $C_0(X)$ such that $g = 1$ on K. By density, we have a function $h \in \mathcal{I}$ such that $\|g - h\|_\infty < \frac{1}{2}$. In particular

$$\operatorname{supp} f \subseteq K \subseteq \left\{x \in X \colon |h(x)| > \frac{1}{2}\right\}.$$

Applying the Tietze extension theorem again, we find $j \in C_0(Y)$ such that $j = \frac{1}{h}$ on K. We conclude the proof by noting that $hjf = f$ is in $\mathcal{I}$, as $\mathcal{I}$ is an ideal. $\square$

2.2.6 Density for C^* subalgebras of L^∞

In what follows, given a set of functions $\mathcal{F} \subseteq \mathrm{L}^\infty(X, \lambda)$, we denote by $(\mathcal{F})_1$ the intersection of $\mathcal{F}$ with the unit ball of $\mathrm{L}^\infty(X, \mu)$ for the norm $\|\cdot\|_\infty$.

Proposition 2.2.30. *Let λ be a σ-finite measure on a standard Borel space X. Let $\mathcal{G}$ be a unital C^*-subalgebra of $\mathrm{L}^\infty(X, \lambda)$, i.e. a $\|\cdot\|_\infty$-closed unital $*$-subalgebra of $\mathrm{L}^\infty(X, \lambda)$. The following four conditions are equivalent*

(1) $(\mathcal{G})_1 \cap \mathrm{L}^2(X, \lambda)$ is $\|\cdot\|_2$-dense in $(\mathrm{L}^\infty(X, \lambda))_1 \cap \mathrm{L}^2(X, \lambda)$;

(2) $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X, \lambda)$;

(3) the set of finite measure supported elements of $\mathcal{G}$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X, \lambda)$;

(4) $\mathcal{G} \cap \mathrm{L}^1(X, \lambda)$ is $\|\cdot\|_1$-dense in $\mathrm{L}^1(X, \lambda)$.

Moreover, these equivalent conditions imply:

(5) $\mathcal{G}$ is strongly dense in $\mathrm{L}^\infty(X, \lambda)$.

Finally, if λ is finite, (5) implies the four above conditions.

Proof. We clearly have that (3) implies (2). To see the converse, it suffices to show that finite measure supported elements of $\mathcal{G}$ are $\|\cdot\|_2$-dense in $\mathcal{G} \cap \mathrm{L}^2(X, \lambda)$.

To this end, let $f \in \mathcal{G} \cap \mathrm{L}^2(X, \lambda)$. For every $\delta > 0$ and $z \in \mathbb{C}$, let

$$p_\delta(z) = \begin{cases} 0 & \text{if } |z| \leqslant \delta, \\ z \times \frac{|z| - \delta}{|z|} & \text{otherwise.} \end{cases}$$

Since $\mathcal{G}$ is a C^*-algebra and p_δ is continuous, we have $p_\delta \circ f \in \mathcal{G}$. Figure 2.1 illustrates the effect of p_δ on the modulus of the image of f.

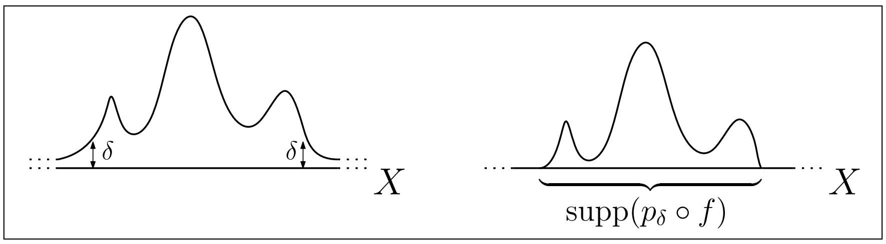

Figure 2.1: Illustration of the argument on an example, showing the graph of $x \mapsto |f(x)|$ (on the left) and the graph of $x \mapsto |p_\delta \circ f(x)|$ (on the right).

Moreover, $p_\delta \circ f$ has finite measure support because $f \in \mathrm{L}^2(X, \lambda)$, and the Lebesgue dominated convergence theorem ensures us that

$$\lim_{\delta \to 0} p_\delta \circ f = f$$

in the L^2-norm, so we conclude that the set of finite measure supported elements of $\mathcal{G}$ is $\|\cdot\|_2$-dense in $\mathcal{G}\cap\mathrm{L}^2(X,\lambda)$, which shows that (2) implies (3) as wanted.

Now since bounded elements are dense in $\mathrm{L}^2(X,\lambda)$, it is clear that (1) implies (2). Conversely, if $\mathcal{G}\cap\mathrm{L}^2(X,\lambda)$ is $\|\cdot\|_2$ dense in $\mathrm{L}^2(X,\lambda)$, let $f\in(\mathrm{L}^\infty(X,\lambda))_1\cap\mathrm{L}^2(X,\lambda)$. Let $f_n\to f$ in $\|\cdot\|_2$ with $f_n\in\mathcal{G}$. Define the continuous function $q:\mathbb{C}\to\mathbb{C}$ by

$$q(z)=\begin{cases} z & \text{if } |z|\leqslant 1,\\ \frac{z}{|z|} & \text{otherwise.}\end{cases}$$

Since $\mathcal{G}$ is a C^*-algebra, for all $n\in\mathbb{N}$ we have $q\circ f_n\in(\mathcal{G})_1$. Moreover for all $x\in X$ we have $|q\circ f_n(x)-f(x)|\leqslant|f_n(x)-f(x)|$, so we also have $qf_n\to f$ in $\|\cdot\|_2$, thus finishing the proof that (2) implies (1). So conditions (1), (2) and (3) are all equivalent.

We now prove that (4) and (2) are equivalent using the square and root functions on complex moduli:

$$\begin{aligned} r(z)=r(|z|\,e^{i\theta_z})&:=\sqrt{|z|}e^{i\theta_z}\\ s(z)=s(|z|\,e^{i\theta_z})&:=|z|^2\,e^{i\theta_z}.\end{aligned}$$

Note that r and s are homeomorphisms of $\mathbb{C}$, inverse of each other. Moreover given $f\in\mathrm{L}^1(X,\lambda)$ the function $s\circ f$ is in $\mathrm{L}^2(X,\lambda)$ and vice-versa. Furthermore, if f is in $\mathcal{G}$, $r\circ f$ and $s\circ f$ remain in $\mathcal{G}$ as it is a C^* algebra. We prove that (4) implies (2).

Let then $g\in\mathrm{L}^2(X,\lambda)$, and let $f=s\circ g$ so that $g=r\circ f$ with $f\in\mathrm{L}^1(X,\lambda)$. By assumption, we have a sequence (f_n) converging to f for the L^1 norm, with $f\in\mathcal{G}$. By a classical theorem attributed to Riesz–Fischer (see e.g. [Bou04, IV §3 Thm. 3]), we can extract a subsequence (f_{n_k}) converging pointwise almost everywhere to f so that there exists $h\in\mathrm{L}^1(X,\lambda)$ verifying $|f_{n_k}|\leqslant h$ almost everywhere for any k. Thus, the sequence $(r\circ f_{n_k})$ is in L^2 and $r\circ f_{n_k}\to r\circ f$ pointwise almost everywhere. Moreover, we have $|r\circ f_{n_k}|\leqslant r\circ h\in\mathrm{L}^2(X,\lambda)$ for any k. By the L^2 version of the Lebesgue dominated convergence theorem, we have $r\circ f_{n_k}\to r\circ f$ in L^2 norm. In other words $g=r\circ f$ can be approximated for $\|\cdot\|_2$ by functions in $\mathcal{G}\cap\mathrm{L}^2(X,\lambda)$ as wanted, thus showing that (4) implies (2). The symmetric argument gives the reverse implication, so conditions (1), (2), (3) and (4) are all equivalent.

We now connect them to (5) by showing first that (1) implies strong density of $\mathcal{G}$ in $\mathrm{L}^\infty(X,\lambda)$. To this end, first note that $\mathrm{L}^\infty(X,\lambda)\cap\mathrm{L}^2(X,\lambda)$ is strongly dense in $\mathrm{L}^\infty(X,\lambda)$: if $f\in\mathrm{L}^\infty(X,\lambda)$ and (X_n) is an increasing sequence of subsets of finite measure subsets of X such that $X=\bigcup_{n\in\mathbb{N}}X_n$, then for all $\xi\in\mathrm{L}^2(X,\lambda)$

$$\|f\xi-f\mathbb{1}_{X_n}\xi\|_2^2=\int_{X\setminus X_n}|f(x)\xi(x)|^2\,d\lambda(x)\longrightarrow 0$$

by the Lebesgue dominated convergence theorem, so $f\mathbb{1}_{X_n}\to f$ strongly. Towards showing the desired implication, assume (1), it now suffices to strongly approximate any $f\in\mathrm{L}^\infty(X,\lambda)\cap\mathrm{L}^2(X,\lambda)$ by a sequence of elements of $\mathcal{G}$. Replacing f by $f/\|f\|_\infty$ if need be, we may as well assume $f\in(\mathrm{L}^\infty(X,\lambda))_1$. We then have $a_n\to f$ in $\|\cdot\|_2$ with $a_n\in(\mathcal{G})_1$. By density of step functions and the fact that $\|a_n\|_\infty\leqslant 1$, it suffices to show that if B is a finite measure subset, then $a_n\mathbb{1}_B\to f\mathbb{1}_B$ for $\|\cdot\|_2$, which is an immediate consequence of the fact that

$$\|a_n\mathbb{1}_B-f\mathbb{1}_B\|_2^2=\int_B|a_n(x)-f_n(x)|^2\,d\lambda(x)\leqslant\|a_n-f\|_2^2\longrightarrow 0.$$

Finally, if λ is finite, let us show (5) implies (2). Take any generalized sequence (f_i) in $\mathcal{G}$ converging strongly to $f\in\mathrm{L}^\infty(X,\lambda)$, then $f_i\mathbb{1}_X\to f\mathbb{1}_X$, which means exactly that $\|f_i-f\|_2\to 0$. This concludes the proof, as the finiteness of λ yields that $\mathrm{L}^\infty(X,\lambda)$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X,\lambda)$. □

Remark 2.2.31. Similar arguments show that we can add to the list of the first 4 equivalent conditions the L^1 versions of condition (1) and (3), namely

(1') $(\mathcal{G})_1 \cap \mathrm{L}^1(X,\lambda)$ is $\|\cdot\|_1$-dense in $(\mathrm{L}^\infty(X,\lambda))_1 \cap \mathrm{L}^1(X,\lambda)$;

(3') the set of finite measure supported elements of $\mathcal{G}$ is $\|\cdot\|_1$-dense in $\mathrm{L}^1(X,\lambda)$.

We did not include them in the statement for brevity.

As a concrete example of the above situation, we have the following well-known lemma, which will be useful in our characterization of spatial realizations of boolean actions via Gelfand's theorem.

Lemma 2.2.32 (see [Coh13, Prop. 7.4.3])**.** *Let Y be a locally compact Hausdorff space equipped with a Radon measure η, and consider the C^*-algebra $C_0(Y)$ of continuous functions that vanish at infinity on Y. Then $C_0(Y) \cap \mathrm{L}^2(Y,\eta)$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(Y,\eta)$. In particular, the strong closure of $C_0(Y)$ in $\mathcal{B}(\mathrm{L}^2(Y,\eta))$ is equal to $\mathrm{L}^\infty(Y,\eta)$.* □

2.3 G-continuity and spatial models

2.3.1 G-continuity

We recall the definition of G-continuity, which has notably been used in [KS11] and in [GW05] to discuss the existence of spatial models for boolean actions of Polish groups.

Definition 2.3.1. Let G be a Polish group, and let α be a measure-preserving boolean G-action on a standard σ-finite space (X,λ). We say that f in $\mathrm{L}^\infty(X,\lambda)$ is **G-continuous** if $\left\|f - f\circ\alpha(g_n^{-1},\cdot)\right\|_\infty \to 0$ whenever $g_n \to e_G$. The space of G-continuous functions will thereafter be denoted by $\mathcal{G}$.

The set $\mathcal{G}$ of G-continuous functions is easily seen to be a unital $*$-subalgebra of $\mathrm{L}^\infty(X,\lambda)$. We will now see that it is actually a C^*-algebra, i.e. that it is $\|\cdot\|_\infty$-closed. This will be a direct consequence of the following well-known proposition.

Proposition 2.3.2. *Let (M,d) be a metric space on which a topological group G acts by isometries. Denote by M_G the set of $x \in M$ such that $d(g_n\cdot x, x) \to 0$ whenever $g_n \to e_G$. Then M_G is a closed subset of M, and it is the largest subset of M onto which G acts continuously.*

Proof. Let (x_k) be a sequence of elements of M_G converging to $x \in M$. Fix $\varepsilon > 0$. Towards showing $x \in M_G$, take a sequence (g_n) tending to e_G. By the triangle inequality, we have the following for all $k,n \in \mathbb{N}$:

$$d(x, g_n\cdot x) \leqslant d(x,x_k) + d(x_k, g_n\cdot x_k) + d(g_n\cdot x_k, g_n\cdot x).$$

Fix k large enough so that $d(x,x_k) < \varepsilon$. Since g_n is an isometry, we have $d(g_n\cdot x_k, g_n\cdot x) = d(x,x_k)$, and so both the first and third terms in the above sum are less than ε. For the second term, since x_k is a point of continuity for the G-action, it is smaller than ε for n big enough, which concludes the proof that M_G is closed.

Now by the definition of M_G, the restriction of the action of G on a set which intersects the complement of M_G cannot be continuous. We thus only need to show that the action on M_G is continuous. To this end, let $x_n \to x$ with $x_n, x \in M_G$ and let $g_n \to g$. Then

$$\begin{aligned} d(g_n\cdot x_n, g\cdot x) &\leqslant d(g_n\cdot x_n, g_n\cdot x) + d(g_n\cdot x, g\cdot x) \\ &= d(x_n,x) + d(g^{-1}g_n\cdot x, x). \end{aligned}$$

Since $g^{-1}g_n \to e_G$ and $x \in M_G$, we have $d(g^{-1}g_n\cdot x, x) \to 0$, and since $x_n \to x$, we also have $d(x_n,x) \to 0$. So $d(g_n\cdot x_n, g\cdot x) \to 0$ as wanted, which finishes the proof. □

Corollary 2.3.3. *Let $\pi : G \to \mathrm{Aut}(X, \lambda)$ be a boolean action. Then the space $\mathcal{G}$ of G-continuous functions is actually a unital C^*-subalgebra of $\mathrm{L}^\infty(X, \lambda)$ onto which G acts continuously.*

Proof. We have already observed that the G-action on $(M, d) = (\mathrm{L}^\infty(X, \lambda), \|\cdot\|_\infty)$ associated to π (given by $g \cdot f = f \circ \pi(g)^{-1}$) is by isometries, and by definition $\mathcal{G} = M_G$ so the conclusion directly follows from the previous proposition. □

We finally observe that we have the following natural source of G-continuous functions.

Lemma 2.3.4. *Let X be a locally compact Polish space, let $\alpha : G \times X \to X$ be a continuous G-action, consider the action on $C_0(X)$ by precomposition by the inverse, then every element of $C_0(X)$ is G-continuous, namely $\left\|f - f \circ \alpha(g^{-1}, \cdot)\right\|_\infty \to 0$ when $g \to e_G$, where $\|\cdot\|_\infty$ is the supremum norm.*

Proof. Let $f \in C_0(X)$, let $\varepsilon > 0$. Let K be a compact set such that for all $x \notin K$ we have $|f(x)| < \varepsilon$. By continuity of the action and of f, for every $x \in K$ there is a neighborhood U_x of x and a symmetric neighborhood $\mathcal{N}_x$ of e_G such that $|f(\alpha(g, x')) - f(x)| < \varepsilon$ for all $g \in \mathcal{N}_x$ and all $x' \in U_x$. Take a finite subcover $(U_{x_i})_{i=1}^n$ of K, let $\mathcal{N} = \bigcap_{i=1}^n \mathcal{N}_{x_i}$, then if $x \in U_{x_i}$, the triangle inequality yields

$$|f(\alpha(g, x)) - f(x)| \leqslant |f(\alpha(g, x)) - f(x_i)| + |f(x_i) - f(x)| < 2\varepsilon.$$

Now let $x \in X$ be arbitrary and take $g \in \mathcal{N}$, we have three cases to consider:

- if $x \in K$ then $|f(\alpha(g, x)) - f(x)| < 2\varepsilon$ by what we just did;
- if $x \notin K$ and $\alpha(g, x) \in K$ by symmetry of $\mathcal{N}$ $\left|f(\alpha(g^{-1}, \alpha(g, x))) - f(\alpha(g, x))\right| < 2\varepsilon$ so $|f(x) - f(\alpha(g, x))| < 2\varepsilon$;
- if $x \notin K$ and $\alpha(g, x) \notin K$, then we both have $|f(x)| < \epsilon$ and $|f(\alpha(g, x))| < \varepsilon$, so again $|f(x) - f(\alpha(g, x))| < 2\varepsilon$.

We conclude that $\|f - f \circ \alpha(g, \cdot)\|_\infty < 2\varepsilon$ for all $g \in \mathcal{N}$, which finishes the proof that f is G-continuous. □

2.3.2 Continuous Radon models for infinite measure-preserving boolean actions

Following the terminology of Glasner-Tsirelson-Weiss, a **continuous spatial model** for a boolean measure-preserving action α of a Polish group G on (X, λ) is a continuous G-action on a Polish measured space (Y, η) which is booleanly isomorphic to α. Moreover, if Y is a locally compact Polish space and η is Radon, we call the G-action on (Y, η) a **continuous Radon model** for α. We can now state and prove our version of the Glasner-Tsirelson-Weiss result in the context of possibly infinite measures, namely Theorem C.

Theorem 2.3.5. *Let G be a Polish group, and let α be a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The following are equivalent:*

(1) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^1(X, \lambda)}^{\|\cdot\|_1} = \mathrm{L}^1(X, \lambda),$$

(2) the algebra $\mathcal{G}$ of G-continuous functions satisfies

$$\overline{\mathcal{G} \cap \mathrm{L}^2(X, \lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X, \lambda),$$

(3) the action α admits a continuous Radon model.

Proof. In the whole argument, we systematically view boolean G-actions as continuous actions by integral preserving $*$-automorphisms on L^∞, as per Proposition 2.2.13 and Proposition 2.2.15. We also recall Proposition 2.2.16, which allows us to understand boolean isomorphism of action at the level of L^∞.

By Proposition 2.2.30, (2) and (1) are equivalent, since Corollary 2.3.3 ensures that $\mathcal{G}$ is a unital C^*-subalgebra of $\mathrm{L}^\infty(X,\lambda)$.

The implication (3)$\Rightarrow$(2) is a consequence of the fact that if β is the action on the continuous Radon model (Y,η) as in (3), then viewed as an action on $\mathrm{L}^\infty(Y,\eta)$, the restriction of β to $C_0(Y)$ is continuous by Lemma 2.3.4. Since $C_0(Y)$ satisfies $\overline{C_0(Y)\cap \mathrm{L}^2(Y,\eta)}^{\|\cdot\|_2} \subseteq \mathrm{L}^\infty(Y,\eta)$, we obtain through the boolean isomorphism between α and β that $\mathcal{G}$ satisfies the desired density condition $\overline{\mathcal{G}\cap \mathrm{L}^2(X,\lambda)}^{\|\cdot\|_2} = \mathrm{L}^2(X,\lambda)$.

The converse (2)$\Rightarrow$(3) requires more work, as in [GTW05]. We fix a boolean measure-preserving G-action α, viewed as a continuous G-action by $*$-automorphisms on $\mathrm{L}^\infty(X,\lambda)$ which preserves the integral. We assume that its space $\mathcal{G}$ of G-continuous functions (which is a C^*-algebra by Corollary 2.3.3) satisfies condition (2).

Step 1. Choosing a suitable separable α-invariant C^*-subalgebra $\mathcal{A}\subseteq\mathcal{G}$. By Proposition 2.2.30, the set of finite measure-supported elements of $\mathcal{G}$ is $\|\cdot\|_2$-dense in $\mathcal{G}$. Since $\mathrm{L}^2(X,\lambda)$ is separable for the L^2 norm, we may fix a countable subset $\mathcal{D}\subseteq\mathcal{G}$ consisting of functions whose supports have finite measure, such that $\mathcal{D}$ is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X,\lambda)$.

Let $\mathcal{E}=\alpha(G)\mathcal{D}$, then we claim that $\mathcal{E}$ is $\|\cdot\|_\infty$-separable: if Γ is a countable dense subset of G then by G-continuity the countable set $\alpha(\Gamma)\mathcal{D}$ is dense in $\mathcal{E}$. If we finally let $\mathcal{A}$ denote the C^*-algebra generated by $\mathcal{E}$, then $\mathcal{A}$ is still separable: the countable set of finite $\mathbb{Q}[i]$-linear combinations of finite products of elements of $\alpha(\Gamma)\mathcal{D}\cup(\alpha(\Gamma)\mathcal{D})^*$ is dense therein. By construction $\mathcal{A}\subseteq\mathcal{G}$ and $\mathcal{A}$ is $\|\cdot\|_\infty$-separable. Moreover, it contains a $\|\cdot\|_2$-dense subset $\mathcal{D}$ of $\mathrm{L}^2(X,\lambda)$ consisting of functions whose supports have finite measure.

Step 2. Building the space and the measure. By the theorem of Gelfand-Naimark (Theorem 2.2.28), the map

$$\rho \;:\; \mathcal{A} \longrightarrow C_0(\mathrm{Sp}(\mathcal{A}))$$

given by $\rho(a)(\chi)=\chi(a)$ for any χ in $\mathrm{Sp}(\mathcal{A})$, is an isometric $*$-isomorphism. Let $Y:=\mathrm{Sp}\,\mathcal{A}$, then since $\mathcal{A}$ is separable we obtain from Proposition 2.2.27 that Y is locally compact Polish.

Now let

$$\mathcal{I}:=\{a\in\mathcal{A}\colon \lambda(\operatorname{supp} a)<+\infty\}\,.$$

Then $\mathcal{I}$ is and ideal of $\mathcal{A}$ which is $\|\cdot\|_2$-dense in $\mathrm{L}^2(X,\lambda)$ since it contains $\mathcal{D}$.

Functions in $\mathcal{I}$ are essentially bounded and have supports of finite measure, therefore they are integrable, so the measure λ defines a positive linear functional

$$\begin{array}{rcccc}\Psi_\lambda & : & \mathcal{I} & \longrightarrow & \mathbb{C}\\ & & f & \longmapsto & \int_X f\,d\lambda.\end{array}$$

The linear functional Ψ_λ can be transported through the Gelfand isomorphism, yielding the positive linear functional $\Psi_\lambda\circ\rho^{-1}:\rho(\mathcal{I})\to\mathbb{C}$. Proposition 2.2.29 ensures that $C_c(Y)\subseteq\rho(\mathcal{I})$, where $C_c(Y)$ denotes the space of compactly supported continuous functions on Y. We then restrict $\Psi_\lambda \circ \rho^{-1}$ to $C_c(Y)$, and by Theorem 2.2.24 we obtain a unique Radon measure η on Y such that for all $f \in C_c(Y)$,

$$\int_Y f d\eta = \int_X \rho^{-1}(f) d\lambda.$$

Step 3. Extending ρ to the whole $\mathrm{L}^\infty(X,\lambda)$. We will now extend the definition of ρ to functions in $\mathrm{L}^\infty(X,\lambda)$. We are essentially reformulating the uniqueness of the GNS construction for weights in our restricted setup. While the construction is natural, it is quite long to set up and we thus encourage the reader to take this step for granted at first reading. The following diagram summarises the situation and the notations that we will use throughout this step.

$$\begin{tikzcd}
\mathrm{L}^2(X,\lambda) \arrow[r, "\widetilde{\rho}"] & \mathrm{L}^2(Y,\eta) \\
\mathcal{I} \arrow[u, hook, "\|\cdot\|_2\text{-cls}"] \arrow[r, "\rho_{\restriction\mathcal{I}}"] \arrow[d, hook, "\|\cdot\|_\infty\text{-cls}"] & \rho(\mathcal{I}) \arrow[u, hook, "\|\cdot\|_2\text{-cls}"'] \arrow[d, hook, "\|\cdot\|_\infty\text{-cls}"'] \\
\mathcal{A} \arrow[loop left, "\alpha"] \arrow[r, "\rho"] \arrow[d, hook, "\text{str-cls}"] & C_0(Y) \arrow[loop right, "\beta"] \arrow[d, hook, "\text{str-cls}"'] \\
\mathrm{L}^\infty(X,\lambda) \arrow[r, "\overline{\rho}"] \arrow[d, hook, "M"] & \mathrm{L}^\infty(Y,\eta) \arrow[d, hook, "M"'] \\
\mathcal{B}(\mathrm{L}^2(X,\lambda)) \arrow[r, "\widehat{\rho}"] & \mathcal{B}(\mathrm{L}^2(Y,\eta))
\end{tikzcd}$$

By definition of η, for any f in $\mathcal{I}$ we have $\int_X f d\lambda = \int_Y \rho(f) d\eta$. In other words, $\rho_{\restriction\mathcal{I}}$ preserves integrals, hence it takes the inner product of $\mathrm{L}^2(X,\lambda)$ to that of $\mathrm{L}^2(Y,\eta)$. Therefore ρ induces a surjective isometry $\widetilde{\rho}$ between the $\|\cdot\|_2$-closures of $\mathcal{I}$ and $\rho(\mathcal{I})$, which are $\mathrm{L}^2(X,\lambda)$ and $\mathrm{L}^2(Y,\eta)$ respectively, by density of $\mathcal{I}$ (this is where the fact that $\mathcal{D} \subseteq \mathcal{I}$ is crucially used) and by Lemma 2.2.32.

We then have a natural $*$-isomorphism $\widehat{\rho} : \mathcal{B}(\mathrm{L}^2(X,\lambda)) \to \mathcal{B}(\mathrm{L}^2(Y,\eta))$ given by the conjugacy by $\widetilde{\rho}$, namely for all $f \in \mathcal{B}(\mathrm{L}^2(X,\lambda))$

$$\widehat{\rho}(f) = \widetilde{\rho} f \widetilde{\rho}^{-1}.$$

Being the conjugation by a surjective isometry, $\widehat{\rho}$ takes the strong topology on $\mathcal{B}(\mathrm{L}^2(X,\lambda))$ to the strong topology on $\mathcal{B}(\mathrm{L}^2(Y,\eta))$.

Recall that we defined M as the multiplication embedding $f \in \mathrm{L}^\infty(X,\lambda) \mapsto M_f \in \mathcal{B}(\mathrm{L}^2(X,\lambda))$. We use the same notation for the multiplication embedding $M : \mathrm{L}^\infty(Y,\eta) \to \mathcal{B}(\mathrm{L}^2(Y,\eta))$. For any functions a in $\mathcal{A}$ and ξ in $\mathcal{I}$, we have

$$\rho(M_a\xi) = \rho(a\xi) = \rho(a)\rho(\xi) = M_{\rho(a)}\rho(\xi).$$

By density of $\mathcal{I}$ and by continuity of the multiplication, this still holds if we take ξ to be in $\mathrm{L}^2(X,\lambda)$, *i.e.* for any a in $\mathcal{A}$ and any ξ in $\mathrm{L}^2(X,\lambda)$, we have

$$\widetilde{\rho}(M_a\xi) = M_{\rho(a)}\widetilde{\rho}(\xi).$$

This is equivalent to saying that $\widetilde{\rho} M_a \widetilde{\rho}^{-1} = M_{\rho(a)}$, for any a in $\mathcal{A}$, which can be reformulated as $\widehat{\rho}(M_a) = M_{\rho(a)}$ by definition of $\widehat{\rho}$. Since $\mathcal{A} \cap \mathrm{L}^2(X,\lambda)$ is dense in $\mathrm{L}^2(X,\lambda)$, by Proposition 2.2.30 the strong-closure of $M(\mathcal{A})$ in $\mathcal{B}(\mathrm{L}^2(X,\lambda))$ is equal to $M(\mathrm{L}^\infty(X,\lambda))$.

Moreover, by Lemma 2.2.32 the strong closure of $M(\rho(\mathcal{A})) = M(C_0(Y))$ in $\mathcal{B}(\mathrm{L}^2(Y,\eta))$ is equal to $M(\mathrm{L}^\infty(Y,\eta))$.

Therefore the isomorphism $\widehat{\rho}$ restricts to an isomorphism of von Neumann algebras between $M(\mathrm{L}^\infty(X,\lambda))$ and $M(\mathrm{L}^\infty(Y,\eta))$ sending the measure λ to η. As the homomorphisms M are isomorphisms on their images, defining $\overline{\rho}$ by

$$\overline{\rho}(f) = M^{-1}\widehat{\rho}M(f)$$

for any f in $\mathrm{L}^\infty(X,\lambda)$ allows us to extend ρ to $\overline{\rho} : \mathrm{L}^\infty(X,\lambda) \to \mathrm{L}^\infty(Y,\eta)$.

Step 4: Defining β and showing that it is the action we want. We now show that G acts on (Y,η) in a spatial manner, and that this action is booleanly isomorphic to the original boolean G-action on (X,λ).

Since $\overline{\rho}$ is an isomorphism $\mathrm{L}^\infty(X,\lambda) \to \mathrm{L}^\infty(Y,\eta)$, we already have our candidate boolean action β on $\mathrm{L}^\infty(Y,\eta)$, uniquely defined by letting, for all $f \in \mathrm{L}^\infty(X,\lambda)$,

$$\beta(g,\overline{\rho}(f)) = \overline{\rho}(\alpha(g,f)).$$

Now G acts on $\mathcal{A}$ via α, yielding a spatial action $\hat{\alpha}$ on $Y = \mathrm{Sp}\,\mathcal{A}$ defined by: for any $\chi \in Y$ and any $a \in \mathcal{A}$, $\hat{\alpha}(g,\chi)(a) = \chi(\alpha(g^{-1},a))$.

Since $\overline{\rho}$ extends ρ, we have in particular that for every $a \in \mathcal{A}$, $\beta(g,\rho(a)) = \rho(\alpha(g,a))$. So $\beta(g,\rho(a))(\chi) = \chi(\alpha(g,a))$ for any χ in Y. We can rewrite this as:

$$\beta(g,\rho(a))(\chi) = \chi(\alpha(g,a)) = \hat{\alpha}(g^{-1},\chi)(a) = \rho(a)(\hat{\alpha}(g^{-1},\chi))$$

In other words, when restricted to $\rho(\mathcal{A}) = C_0(Y)$, the action β coincides with the precomposition by the inverse associated to the spatial action $\hat{\alpha}$. By strong density of $M(C_0(Y))$ in $M(\mathrm{L}^\infty(Y,\eta))$ and strong continuity of $\overline{\rho}$, the same conclusion holds on $\mathrm{L}^\infty(Y,\eta)$, namely β coincides with the precomposition by the inverse associated with the spatial action $\hat{\alpha}$.

In other words β is the boolean action associated with $\hat{\alpha}$, and since β was integral preserving we have that $\hat{\alpha}$ is measure-preserving (this could also be checked directly using the Riesz-Markov-Kakutani theorem).

We finally need to check is that $\hat{\alpha}$ is continuous, using the fact that $\mathcal{A}$ consists solely of G-continuous functions. The topology on $\mathrm{Sp}(\mathcal{A})$ is the topology of pointwise convergence, so we fix a in $\mathcal{A}$. Let (g_n) be a sequence of elements of G converging to e_G and let (χ_n) be converging to χ in $Y = \mathrm{Sp}\,\mathcal{A}$. We have

$$\begin{aligned}
|\hat{\alpha}(g_n,\chi_n)(a) - \chi(a)| &= |\widehat{a}(\hat{\alpha}(g_n,\chi_n)) - \chi(a)| \\
&\leqslant |\widehat{a}(\hat{\alpha}(g_n,\chi_n)) - \widehat{a}(\chi_n)| + |\widehat{a}(\chi_n) - \widehat{a}(\chi)| \\
&\leqslant \left\|\widehat{\alpha(g_n^{-1},a)} - \widehat{a}\right\|_{C_0(Y)} + |\chi_n(a) - \chi(a)| \\
&\leqslant \left\|\alpha(g_n^{-1},a) - a\right\|_\infty + |\chi_n(a) - \chi(a)|\,.
\end{aligned}$$

By convergence of (χ_n) the second term tends to zero as n grows, and the first term tends to zero as n grows by G-continuity of a. Therefore the G-action on Y is continuous as wanted. □

2.4 Applications and examples

2.4.1 Construction of G-continuous functions when G is locally compact

In the whole section, we work with a fixed locally compact Polish group G, and denote by m its left Haar measure. We start by recalling the definition of convolution between an essentially bounded function on a measure space and a function that is integrable with regard to the Haar measure of the group acting on said space.

Definition 2.4.1. Consider a boolean measure-preserving G-action α on (X,λ) as in Definition 2.2.2. Let $f \in \mathrm{L}^\infty(X,\lambda)$ and $\delta \in \mathrm{L}^1(G,m)$. For any x in X, the **convolution product** $\delta * f : X \to \mathbb{C}$ is defined by: for all $x \in X$,

$$\delta * f(x) = \int_G \delta(g) f(\alpha(g^{-1},x))dm(g).$$

Using for δ an approximation of the identity, we will see that G-continuous functions are always dense for boolean measure-preserving actions. We start by proving the following well-known lemma.

Lemma 2.4.2. *Consider a locally compact Polish group G, and a boolean measure-preserving action α of G on a standard σ-finite space (X,λ). Let $\delta : G \to \mathbb{R}_+$ be continuous of integral 1 and compactly supported. Then, for any function f in $\mathrm{L}^\infty(X,\lambda) \cap \mathrm{L}^1(X,\lambda)$, $\delta * f$ is in $\mathrm{L}^\infty(X,\lambda) \cap \mathrm{L}^1(X,\lambda)$ with $\|\delta * f\|_1 \leqslant \|f\|_1$, and is G-continuous.*

Proof. We first check that $\delta * f$ is in $\mathrm{L}^1(X,\lambda)$, as it is clear that it is in $\mathrm{L}^\infty(X,\lambda)$. Using Fubini's theorem and the fact that δ takes nonnegative values we have

$$\begin{aligned}
\int_X |\delta * f(x)|\, d\lambda(x) &= \int_X \left| \int_G \delta(g) f(\alpha(g^{-1},x))dm(g) \right| d\lambda(x) \\
&\leqslant \int_X \int_G \left|\delta(g) f(\alpha(g^{-1},x))\right| dm(g)d\lambda(x) \\
&= \int_G \delta(g) \int_X \left|f(\alpha(g^{-1},x))\right| d\lambda(x)dm(g) \\
&= \|f\|_1 ,
\end{aligned}$$

the last equality being a consequence of the fact that $\int_G \delta(g)dm(g) = 1$ and g preserves the measure. In particular we have $\delta * f \in \mathrm{L}^1(X,\lambda)$ and moreover $\|\delta * f\|_1 \leqslant \|f\|_1$.

We finally prove that it is G-continuous. Let us take $\varepsilon > 0$. For any h in G we have

$$\delta * f(\alpha(h,x)) = \int_G \delta(g) f(\alpha(g^{-1}h,x))dm(g) = \int_G \delta(hg) f(\alpha(g^{-1},x))dm(g).$$

Therefore, we have

$$|\delta * f(x) - \delta * f(\alpha(h,x))| \leqslant \int_G |\delta(g) - \delta(hg)| \left|f(\alpha(g^{-1},x))\right| dm(g). \tag{2.1}$$

By Lemma 2.3.4 for the G-action on itself by left translation, we can fix a neighborhood $\mathcal{N}$ of e_G such that for all $h \in \mathcal{N}$ and $g \in G$, $|\delta(g) - \delta(hg)| < \varepsilon$. Also, note that if $K = \operatorname{supp}\delta$, then $\delta(g) - \delta(hg) = 0$ for all $g \notin K \cup h^{-1}K$, which has measure at most $2m(K)$ since m is a left Haar measure. It follows that

$$\int_G |\delta(g) - \delta(hg)|\, dm(g) < 2\varepsilon \times m(K)$$

Therefore we can control the left hand side of Equation (2.1) by the L^∞ norm of f:

$$|\delta * f(x) - \delta * f(\alpha(h,x))| < 2\varepsilon \times m(K) \times \|f\|_\infty$$

which concludes the proof. □

We can now prove Theorem D in the special case that the acting group is locally compact.

Theorem 2.4.3. *Let G be a locally compact Polish group, and consider a boolean measure-preserving G-action on a standard σ-finite space (X, λ). Then the action admits a continuous spatial model on a locally compact Polish space (Y, η), where η is a Radon measure.*

Proof. We want to apply condition (1) from Theorem 2.3.5 in order to conclude.

Since bounded functions are dense in $\mathrm{L}^1(X, \lambda)$, we fix some $f \in \mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^1(X, \lambda)$, some $\varepsilon > 0$, and our aim is to find a G-continuous function which is ε-close to f for $\|\cdot\|_1$.

For any $\delta \in \mathrm{L}^1(G, m)$ compactly supported of integral 1 taking nonnegative values, we know from Lemma 2.4.2 that $\delta * f$ is in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^1(X, \lambda)$ and is G-continuous. By Fubini's Theorem, we also have

$$\begin{aligned}
\|f - \delta * f\|_1 &= \int_X \left| f(x) - \int_G \delta(g) f(\alpha(g^{-1}, x)) dm(g) \right| d\lambda(x) \\
&= \int_X \left| \int_G \delta(g) f(x) dm(g) - \int_G \delta(g) f(\alpha(g^{-1}, x)) dm(g) \right| d\lambda(x) \\
&\leqslant \int_X \int_G \delta(g) \left| f(x) - f(\alpha(g^{-1}, x)) \right| dm(g) d\lambda(x) \\
&= \int_G \delta(g) \int_X \left| f(x) - f(\alpha(g^{-1}, x)) \right| d\lambda(x) dm(g) \\
&= \int_G \delta(g) \|f - f \circ \alpha(g^{-1}, \cdot)\|_1 dm(g)
\end{aligned}$$

But as noted in Section 2.2.3, the G-action on $\mathrm{L}^1(X, \lambda)$ is continuous. Therefore there exists a neighborhood $\mathcal{N}_\varepsilon$ of e_G such that $\|f - f \circ \alpha(g^{-1}, \cdot)\|_1 < \varepsilon$ for any g in $\mathcal{N}_\varepsilon$. Taking now δ as above whose support is moreover contained in $\mathcal{N}_\varepsilon$, we obtain $\|f - \delta * f\|_1 < \varepsilon$, which concludes the proof. □

Combining Mackey's theorem (Theorem 2.2.7), and Theorem 2.4.3, we obtain Theorem A as a corollary.

Corollary 2.4.4. *Let G be a locally compact Polish group and (X, λ) be a standard σ-finite space. Let also α be a measure-preserving G-action on (X, λ). Then α is spatially isomorphic to a continuous measure-preserving G-action on a locally compact Polish space Y endowed with a Radon measure η.*

Proof. By Theorem 2.4.3 the boolean action associated with α admits a Polish model β on (Y, η), where Y is locally compact and Polish, and η is a Radon measure. By Mackey's theorem (item (2) of Theorem 2.2.7), since α and β are booleanly isomorphic, they are in fact spatially isomorphic. □

In Section 2.5.2 we give another way to obtain Corollary 2.4.4 without relying on the use of Mackey's theorem. By working directly with measurable bounded functions and the genuine supremum norm instead of equivalence classes of functions up to measure zero, we explicitly construct the desired spatial isomorphism.

2.4.2 Actions of isometry groups of locally compact separable spaces

In this section, building upon Theorem 2.4.3 and Theorem 2.3.5, we extend the result of Kwiatkowska-Solecki on the existence of continuous spatial models for isometry groups of locally compact separable spaces to our σ-finite setup. We quickly recall the following result from Gao and Kechris ([GK03, Thm. 7]): the isometry groups of locally compact separable spaces are exactly the closed subgroups of groups of the form

$$\prod_{n \in \mathbb{N}} \mathfrak{S}_\infty \ltimes H_n^{\mathbb{N}},$$

where H_n is a lcsc group for any n, and $\mathfrak{S}_\infty = \mathrm{Sym}(\mathbb{N})$ acts by permutation of the coordinates of $H_n^{\mathbb{N}}$. As in Kwiatkowska and Solecki's paper, we rely on the following characterization of this class of groups (endowed with the topology of pointwise convergence), itself established from the previously mentioned one.

Theorem 2.4.5 (Kwiatkowska-Solecki, see [KS11, Thm. 1.2])**.** *Let G be a Polish group. Then G is topologically isomorphic to the group of isometries of a locally compact separable metric space if and only if it satisfies the following property:*

($*$) for every open neighborhood of the identity $U \subseteq G$ there is a closed subgroup $H < G$ such that $H \subseteq U$, $N(H)$ is open, and the Polish space G/H is locally compact.

Let us remark that in ($*$), the normalizer subgroup $N(H)$ is open, hence closed, in particular $N(H)/H$ is a closed subset of the locally compact space G/H. Since it is moreover a topological group, we conclude that $N(H)/H$ is a locally compact Polish group.

Proposition 2.4.6. *Let G be the isometry group of a locally compact separable metric space, and consider α a boolean measure-preserving G-action on a standard σ-finite space (X, λ). The set of square integrable G-continuous functions with regard to this action is dense in $\mathrm{L}^2(X, \lambda)$.*

Proof. Consider an element f of $\mathrm{L}^2(X, \lambda)$, and fix $\varepsilon > 0$. As G acts continuously on $\mathrm{L}^2(X, \lambda)$ there exists an open set U such that for any h in U we have $\left\|f - f \circ \alpha(h^{-1}, \cdot)\right\|_2 < \varepsilon$. As G satisfies Condition ($*$), we can fix a closed subgroup $H < G$ such that $H \subseteq U$, the quotient space G/H is locally compact and the normalizer $N(H)$ is open.

Consider now
$$\mathcal{C} = \overline{\mathrm{Conv}\{(f \circ \alpha(h^{-1}, \cdot)) \mid h \in H\}}^{\|\cdot\|_2},$$
the L^2-closure of the convex hull of the orbit of f under the H-action. Note that $\mathcal{C}$ is contained in the closed ball centered in f and of radius ε.

By the projection onto a closed convex set theorem (see e.g. [Rud87, Thm. 4.10.]), there is a unique element of minimal norm $\widehat{\xi}$ in $\mathcal{C}$, satisfying in particular $\left\|f - \widehat{\xi}\right\|_2 \leqslant \varepsilon$. Since the group H acts by unitaries on $\mathcal{C}$, the vector $\widehat{\xi}$ is fixed by the H-action. In particular, $\widehat{\xi}$ is H-continuous, although it might be unbounded and have support of infinite measure. To conclude the first part of the proof we approximate $\widehat{\xi}$ by functions in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^2(X, \lambda)$ whose support has finite measure while retaining the H-invariance. Let
$$\xi_n := \mathbb{1}_{\{\frac{1}{n} \leqslant |\widehat{\xi}| \leqslant n\}} \widehat{\xi}.$$
For any n the function ξ_n is in $\mathrm{L}^\infty(X, \lambda) \cap \mathrm{L}^2(X, \lambda)$, has support of finite measure, and is H-invariant, in particular it is H-continuous. As $\xi_n \to \widehat{\xi}$ in $\|\cdot\|_2$, we can fix, for some n large enough, a function $\xi := \xi_n$ that is ε-close to $\widehat{\xi}$, and hence 2ε-close to f, while being H-invariant and having a support of finite measure.

Let now $\mathcal{M}$ be the von Neumann algebra generated by the $N(H)$-translates of ξ, i.e. the strong closure in $\mathrm{L}^\infty(X, \lambda)$ of the linear span of the $N(H)$-translates of ξ and of the constant function 1. We claim that any element of $\mathcal{M}$ is fixed by H. Indeed, any $N(H)$-translates of ξ is easily checked to be H-invariant, so the linear span of the $N(H)$-translates of ξ consists of H-invariant vectors. Finally, since each $h \in H$ acts continuously on $\mathrm{L}^\infty(Y, \lambda)$ for the strong operator topology, its set of fixed points is closed, so $\mathcal{M}$ consists of H-invariant functions.

We now argue, using a simpler version of the arguments in the proof of Theorem 2.3.5, that the $N(H)$-action on $(\mathcal{M}, \lambda)$ can be seen as a boolean action on a σ-finite space, possibly with atoms. It will then be easy to see that it descends to a boolean action of the locally compact Polish group $N(H)/H$, allowing us to use Theorem 2.4.3.

Denote by $\mathcal{J}$ the ideal in $\mathrm{L}^\infty(X,\lambda)$ of functions whose support has finite measure. First observe that since ξ belongs to $\mathcal{J}$ and the $N(H)$-action preserves the measure λ, we have that $\mathcal{J}\cap\mathcal{M}$ is strongly dense in $\mathcal{M}$. In a similar fashion to the first step of the proof of Theorem 2.3.5, we let $\mathcal{D}$ be a countable dense subset of $\mathcal{J}\cap\mathcal{M}\cap\mathrm{L}^2(X,\lambda)$ for the L^2 norm, and let $\mathcal{A}$ be the $\|\cdot\|_\infty$-closure of the linear span of $\mathcal{D}$. Then $\mathcal{A}$ is a separable C^*-subalgebra of $\mathcal{M}$ which is strongly dense in $\mathcal{M}$, and such that the ideal $\mathcal{I} := \mathcal{A}\cap\mathcal{J}$ is $\|\cdot\|_\infty$-dense in $\mathcal{A}$. Using now the Gelfand theorem exactly as in the second step of the proof of Theorem 2.3.5, we have a locally compact Polish space Y endowed with a Radon measure η so that after identification through the Gelfand isomorphism, $\mathcal{A} = C_0(Y)$ and hence by strong density $\mathcal{M} = \mathrm{L}^\infty(Y,\eta)$. By construction η and λ coincide on $\mathcal{I}$, in particular they define the same measure on Y. So the $N(H)$-action on $\mathcal{M} = \mathrm{L}^\infty(Y,\eta)$ is continuous in the sense that we gave right before Proposition 2.2.16, namely for every $A\in\mathrm{MAlg}_f(Y,\eta)$, and every $g_n\to e_G$ in $N(H)$,

$$\int_Y |g_n\cdot\mathbb{1}_A - \mathbb{1}_A|\,d\eta \to 0.$$

We thus now have a boolean action of $N(H)$ on (Y,η), where (Y,η) is a standard σ-finite space, possibly with atoms, and moreover $\mathcal{M} = \mathrm{L}^\infty(Y,\eta)$. But since every element of $\mathcal{M}$ is fixed by H, the restriction of this boolean action to H is trivial and the boolean action thus descends to a boolean action of the quotient group $N(H)/H$, which is a locally compact Polish group. Applying Theorem 2.4.3 along with Theorem 2.3.5, we now find $\xi'\in\mathrm{L}^2(Y,\eta)\subseteq\mathrm{L}^2(X,\lambda)$ which is ε-close to ξ and $N(H)/H$-continuous, and hence $N(H)$-continuous when we view again this $N(H)/H$-action as an $N(H)$-action.

Since $N(H)$ is open in G, we have in particular that ξ', seen as an element of $\mathrm{L}^2(X,\lambda)$, is G-continuous, and by the triangle inequality it is 3ε-close to the function f we started with, which finishes the proof. □

We can now directly prove Theorem D.

Theorem 2.4.7. *Let G be the isometry group of a separable locally compact space. Then every boolean measure-preserving G-action on a standard σ-finite space (X,λ) admits a continuous Radon model.*

Proof. By Proposition 2.4.6, the G-continuous functions are dense in $\mathrm{L}^2(X,\lambda)$, and we can then apply Theorem 2.3.5 to conclude the proof. □

Remark 2.4.8. By the above theorem, it is always possible to find a continuous Radon model for any boolean measure-preserving action of the non-archimedean Polish group $\mathfrak{S}_\infty$. However, as we explain in Section 2.5.4, it is not always possible to turn boolean isomorphisms between $\mathfrak{S}_\infty$ spatial actions into spatial isomorphisms, as opposed to what happens for locally compact groups (see the second item of Theorem 2.2.7).

Remark 2.4.9. It is a result of Nessonov that for every boolean *non-singular* $\mathfrak{S}_\infty$-action α on (X,λ), there is an equivalent σ-finite measure $\lambda'\in[\lambda]$ such that α becomes measure-preserving [Nes19, Thm. 1.1]. So our result yields that every boolean non-singular $\mathfrak{S}_\infty$-action admits a continuous Radon model.

2.4.3 An ergodic boolean action with two distinct spatial models

In this section we present an example of an ergodic boolean action with two distinct spatial models on probability spaces, which was explained to us by Todor Tsankov. The action is that of $\mathfrak{S}_\infty$ on the countable product of the unit interval endowed with the Lebesgue measure $\left([0,1]^{\mathbb{N}},\lambda^{\otimes\mathbb{N}}\right)$. The first spatial action α is the natural action of $\mathfrak{S}_\infty$ by permutation of the coordinates. For the second action β, we first define the space $\mathrm{LO}(\mathbb{N})$ of linear orderings of $\mathbb{N}$:

$$\mathrm{LO}(\mathbb{N}) = \left\{x \subseteq \mathbb{N}^2 \mid x \text{ is a linear order on } \mathbb{N}\right\},$$

and for any element x (that we will denote in a more usual way by $n <_x m \iff (n,m) \in x$) of $\mathrm{LO}(\mathbb{N})$, the $\mathfrak{S}_\infty$-action β is defined by

$$(n,m) \in \beta(\sigma, x) \iff n(\sigma\cdot <_x)m \iff \sigma^{-1}(n) <_x \sigma^{-1}(m)$$

for any σ in $\mathfrak{S}_\infty$. We define on $\mathrm{LO}(\mathbb{N})$ the uniform measure μ_u by

$$\mu_u\left([k_1 <_x k_2 <_x \ldots <_x k_n]\right) = \frac{1}{n!}$$

for any distinct $k_1, \ldots, k_n$ in $\mathbb{N}$, where

$$[k_1 <_x k_2 <_x \ldots <_x k_n] = \{x \in \mathrm{LO}(\mathbb{N}) \mid k_1 <_x k_2 <_x \ldots <_x k_n\}.$$

This probability measure is invariant under the $\mathfrak{S}_\infty$-action β, and do note that it is the only such measure by construction. Do recall also that, as $\mathrm{LO}(\mathbb{N})$ is a compact space as a closed subset of $\{0,1\}^{\mathbb{N}^2}$, $\mathfrak{S}_\infty$ is not extremely amenable. Indeed, $\mathfrak{S}_\infty$ contains transpositions, and in particular it cannot preserve a linear ordering of $\mathbb{N}$, hence the previous assertion.

The correspondence between the two actions is defined as follows:

$$\begin{array}{cccc} \Phi & : & [0,1]^{\mathbb{N}} & \longrightarrow & \mathrm{LO}(\mathbb{N}) \\ & & (x_i)_{i\in\mathbb{N}} & \longmapsto & <_x, \end{array}$$

where $<_x$ is defined by $n <_x m \iff x_n < x_m$ (the order $<$ on the right-hand side being the natural order on $[0,1]$). Note that for $\lambda^{\otimes\mathbb{N}}$-almost all x in $[0,1]^{\mathbb{N}}$, the x_n are distinct, so Φ is well defined. We then show that the images by Φ of two distinct elements are isomorphic as orderings of $\mathbb{N}$.

Definition 2.4.10. An order $<$ on a set A is **dense on** A if for any two elements n, m in A such that $n < m$, there exists a third element k in A such that $n < k < m$.

Definition 2.4.11. An order $<$ on a set A is **unbounded on** A if for any element k in A, there exists two elements n and m in A such that $n < k < m$.

Proposition 2.4.12. *An order $<$ in $\mathrm{LO}(\mathbb{N})$ is dense and unbounded on $\mathbb{N}$, μ_u-a.e.*

Proof. Fix n, m and k in $\mathbb{N}$. We have

$$\mathbb{P}_{\mu_u}\left([n<k<m] \mid [n<m]\right) = \frac{1}{3},$$

therefore by Borel-Cantelli's Lemma, as n and m vary, $n < k < m$ happens infinitely often, and thus $<$ is unbounded μ_u-a.e. The same argument used with k varying instead also ensures that $<$ is dense μ_u-a.e. which concludes the proof. $\square$

The following back-and-forth argument is well-known and classical, but we give it for the sake of completeness.

Proposition 2.4.13. *Any two dense and unbounded orders on infinite countable sets are isomorphic, i.e. there exists an order preserving bijection between the respective spaces.*

Proof. Let $A = \{a_i \mid i \in \mathbb{N}\}$ and $B = \{b_i \mid i \in \mathbb{N}\}$ be two countable infinite sets, and let $<_A$ and $<_B$ be two dense, unbounded linear orderings on A and B respectively. We inductively construct an embedding $\varphi : (A, <_A) \hookrightarrow (B, <_B)$.
We arbitrarily send a_0 to b_0. We then send a_1 to $\varphi(a_1)$ such that $(b_0, \varphi(a_1))$ is ordered like (a_0, a_1). At rank n, a_n is sent to $\varphi(a_n)$, an element that respects the order of $(b_0, \dots, \varphi(a_{n-1}))$, which necessarily exists because $<_B$ is dense and unbounded.

In other words, for any finite subset F of $(A, <_A)$, and for any a in $A \setminus F$, it is possible at any rank to extend the existing embedding $F \hookrightarrow (B, <_B)$ into $F \cup \{a\} \hookrightarrow (B, <_B)$.
By doing the same construction with $(B, <_B)$, and given that $<_B$ is also dense and unbounded, it is possible to define by induction

$$\varphi_n : F_n \subseteq (A, <_A) \hookrightarrow (B, <_B)$$

such that F_n is a finite subset of A, and such that for any n we have

$$\begin{cases} \{a_0, \dots, a_n\} \subseteq F_{2n} \\ \{b_0, \dots, b_n\} \subseteq \varphi_{2n+1}(F_{2n+1}) \end{cases}$$

The limit function of the φ_n is the order-preserving bijection we sought. □

Proposition 2.4.13 ensures that μ_u-a.e. there is only one $\mathfrak{S}_\infty$-orbit on $\mathrm{LO}(\mathbb{N})$. On the other hand, the natural $\mathfrak{S}_\infty$-action on $\left([0,1]^{\mathbb{N}}, \lambda^{\otimes\mathbb{N}}\right)$ only admits orbits of measure 0. We then have the following proposition.

Proposition 2.4.14. *The application $\Phi : \left([0,1]^{\mathbb{N}}, \lambda^{\otimes\mathbb{N}}\right) \longrightarrow (\mathrm{LO}(\mathbb{N}), \mu_u)$, sending (x_i) to $<_x$, where $<_x$ is defined by $n <_x m \iff x_n < x_m$, is $\mathfrak{S}_\infty$-equivariant, measure-preserving and $\lambda^{\otimes\mathbb{N}}$-a.e. injective.*

Proof. Checking the $\mathfrak{S}_\infty$-equivariance is straightforward, and so is checking the fact that Φ is measure-preserving. For the $\lambda^{\otimes\mathbb{N}}$-a.e. injectivity we first fix q in $\mathbb{Q} \cap [0,1]$. The Law of Large Numbers ensures that for $\lambda^{\otimes\mathbb{N}}$-almost any sequence (x_i) we have

$$\lim_{n \to +\infty} \frac{|\{i \in \{0, \dots, n-1\} : x_i < q\}|}{n} = q. \tag{2.2}$$

Taking a countable intersection of conull sets, we can therefore find a conull subset X_0 of $[0,1]^{\mathbb{N}}$ such that whenever (x_i) is in X_0, Equation (2.2) holds for any q in $\mathbb{Q} \cap [0,1]$. A straightforward density argument now shows that Equation (2.2) holds for any q in $[0,1]$ and any $(x_i) \in X_0$.

Let us conclude by showing that $\Phi_{\restriction X_0}$ is injective: take distinct elements $x = (x_i)$ and $y = (y_i)$ in X_0, there is an index k such that $x_k \neq y_k$. Then taking $q = x_k$ and $q = y_k$ for (x_i) and (y_i) respectively in (2.2), we obtain that

$$\lim_{n \to +\infty} \frac{|\{i \in \{0, \dots, n-1\} : x_i < x_k\}|}{n} \neq \lim_{n \to +\infty} \frac{|\{i \in \{0, \dots, n-1\} : y_i < y_k\}|}{n}.$$

This implies that $<_x \neq <_y$, as their respective proportions of integers smaller than the integer k are different. □

The previous proposition gives an example of a negative answer to the boolean to spatial isomorphism problem for the case of the non-locally compact group $\mathfrak{S}_\infty$, for actions on probability spaces:

Proposition 2.4.15. *Let $G = \mathfrak{S}_\infty$. The G-action α on $\left([0,1]^{\mathbb{N}}, \lambda^{\otimes\mathbb{N}}\right)$ and the G-action β on $(\mathrm{LO}(\mathbb{N}), \mu_u)$ are booleanly isomorphic, but not spatially isomorphic.*

Proof. We define Φ as in Proposition 2.4.14. This application is (well-defined and) injective on a conull subset $X_0 \subseteq [0,1]^{\mathbb{N}}$. As it is also G-equivariant, setting $X_g = X_0 \cap g \cdot X_0$ for any g in G defines a boolean isomorphism between α and β. They are not spatially isomorphic however, since α only has orbits of measure 0, but Proposition 2.4.13 ensures that β has a conull orbit. □

2.5 Spatial actions and continuous Radon models

In this section, we are interested in a related question on Borel G-actions on infinite measured spaces: given such an action, can it be Borel embedded G-equivariantly into a continuous G-action on a locally compact Polish space so that the pushforward measure is a Radon measure?

We will first give an abstract characterization of this, and then show that the answer is always positive for locally compact groups, thus proving Theorem B. After that, we discuss the related question of continuous realizations of Borel infinite measure-preserving actions. Finally, we give a simple example showing our embedding theorem fails for the non locally compact group $G = \mathfrak{S}_\infty$.

2.5.1 Characterization

In this section, we will work in the space $\mathcal{L}^\infty(X)$ which we define as the C^*algebra of all bounded Borel functions $X \to \mathbb{C}$, endowed with the norm of uniform convergence $\|f\|_\infty = \sup_{x\in X} |f(x)|$ (no essential supremum here!). Observe that if G acts on X, the notion of G-continuous element of $\mathcal{L}^\infty(X)$ still makes sense for the norm $\|\cdot\|_\infty$ that we just defined. We also denote by $\mathcal{L}^1(X, \lambda)$ the space of integrable complex-valued functions on X.

Theorem 2.5.1. *Let G be a Polish group, suppose $\alpha : G \curvearrowright (X, \lambda)$ is a Borel action by measure-preserving bijections on a standard σ-finite space (X, λ). The following are equivalent:*

(i) There is a continuous G-action β on a Polish locally compact space Y and a Borel injective map $\Phi : X \to Y$ such that $\Phi_*\lambda$ is Radon and $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$ for all $x \in X$ and all $g \in G$;

(ii) The algebra of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ contains a countable subset which separates points.

Proof. Let us begin by the easier implication (i)⇒(ii), and assume we have $\alpha : G \times X \to X$ Borel preserving a σ-finite measure λ, $\beta : G \times Y \to Y$ continuous and $\Phi : X \to Y$ injective such that for all $x \in X$, $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$.

By compactness, every compactly supported continuous function on Y is G-continuous with respect to the action β. Since Y is locally compact second-countable, it admits a compatible proper metric d. Let $\{y_n : n \in \mathbb{N}\}$ be a countable dense subset of Y, and define for each $n \in \mathbb{N}$ a continuous compactly supported function f_n by

$$f_n(x) = \max\left(1 - d(y_n, x), 0\right).$$

The density of (y_n) implies that this family of G-continuous functions separates the points of Y. It then follows from the injectivity of Φ that $(f_n \circ \Phi)_{n\in\mathbb{N}}$ separates the points of X, and by the equivariance of Φ it consists of G-continuous functions with respect to α as wanted.

We now prove the converse implication (ii)⇒(i). Let us fix a countable set of functions $\mathcal{D} \subseteq \mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ that separates points. We then consider Γ a countable dense subgroup of G and give a similar argument to the one in Step 1 of the proof of Theorem 2.3.5. By G-continuity the $\|\cdot\|_\infty$-closure of $\mathcal{E}_\Gamma := \{f \circ \alpha(\Gamma, \cdot) \mid f \in \mathcal{D}\}$ is equal to $\mathcal{E}_G := \{f \circ \alpha(G, \cdot) \mid f \in \mathcal{D}\}$. We denote by $\mathcal{A}$ the C^*-algebra generated by $\mathcal{E}_G$. The countable set of finite $\mathbb{Q}[i]$-linear combinations of finite products of elements of $\mathcal{E}_\Gamma \cup (\mathcal{E}_\Gamma)^*$ is dense in $\mathcal{A}$. By construction $\mathcal{A}$ is then separable, as the $\|\cdot\|_\infty$-closure of a countable set (of integrable functions). Moreover every function in $\mathcal{A}$ is G-continuous and $\mathcal{A}$ separates the points of X.

Step 1. Building the space and the measure. By Theorem 2.2.28, we have the following isometric $*$-isomorphism:

$$\begin{array}{cccc} \rho & : & \mathcal{A} & \longrightarrow & C_0(\mathrm{Sp}(\mathcal{A})) \\ & & a & \longmapsto & \widehat{a} \end{array}$$

with $\widehat{a}$ being the evaluation on a. By separability of $\mathcal{A}$ and Proposition 2.2.27, the space $Y := \mathrm{Sp}(\mathcal{A})$ is locally compact Polish.

This time, we consider

$$\mathcal{I} = \mathcal{A} \cap \mathcal{L}^1(X, \lambda).$$

It is immediate to check that $\mathcal{I}$ is an ideal of $\mathcal{A}$, and it contains $\mathcal{D}$ by construction. Since $\mathcal{I}$ consists of integrable functions, λ defines a positive linear functional

$$\begin{array}{cccc} \Psi_\lambda & : & \mathcal{I} & \longrightarrow & \mathbb{C} \\ & & f & \longmapsto & \int_X f d\lambda. \end{array}$$

We then use ρ to get a positive linear functional $\Psi_\lambda \circ \rho^{-1} : \rho(\mathcal{I}) \to \mathbb{C}$. The ideal $\mathcal{I}$ is $\|\cdot\|_\infty$-dense in $\mathcal{A}$, so by Proposition 2.2.29 $C_c(Y) \subseteq \rho(\mathcal{I})$, where $C_c(Y)$ denotes the space of compactly supported continuous functions on Y. By Theorem 2.2.24, restricting $\Psi_\lambda \circ \rho^{-1}$ to $C_c(Y)$ gives a unique Radon measure η on Y such that for all $f \in C_c(Y)$,

$$\int_Y f d\eta = \int_X \rho^{-1}(f) d\lambda. \tag{2.3}$$

Step 2. Borel embedding of X into Y. Consider the following map

$$\begin{array}{cccc} \Phi & : & X & \longrightarrow & Y \\ & & x & \longmapsto & \mathrm{ev}_x \end{array}$$

where $\mathrm{ev}_x(a) = a(x)$. Since the algebra $\mathcal{A}$ separates the points of X, this map is injective. Let us prove that it is also Borel. Sets of the form

$$\{\mathrm{ev}_x \colon |\mathrm{ev}_x(a) - \mathrm{ev}_{x_0}(a)| < \varepsilon\},$$

where $x_0 \in X$, $a \in \mathcal{A}$ and $\varepsilon > 0$ form a subbasis for the topology of pointwise convergence on $\mathrm{Im}(\Phi)$. The preimage by Φ of such a set is

$$\{x \in X \colon |a(x) - a(x_0)| < \varepsilon\}$$

which is Borel since a is a Borel map. Therefore Φ is Borel and injective.

In order to check that Φ is measure-preserving, i.e. that $\Phi_* \lambda = \eta$, observe that for any $a \in \mathcal{A}$ and $x \in X$ we have

$$\rho(a) \circ \Phi(x) = \rho(a)(\mathrm{ev}_x) = \mathrm{ev}_x(a) = a(x).$$

It follows that for $a \in \mathcal{I}$ and $f = \rho(a)$ we have

$$\int_X a d\lambda = \int_X \rho(a) \circ \Phi \, d\lambda = \int_Y \rho(a) d\Phi_*\lambda = \int_Y f d\Phi_*\lambda.$$

On the other hand, if we further assume $f \in C_c(Y)$, Equation (2.3) yields the following:

$$\int_X a d\lambda = \int_Y \rho(a) d\eta = \int_Y f d\eta.$$

We conclude that for all $f \in C_c(Y)$, $\int_Y f d\Phi_*\lambda = \int_Y f d\eta$, which by uniqueness in Theorem 2.2.24 yields $\Phi_*\lambda = \eta$ as wanted.

We now define the G-action $\widetilde{\alpha}$ on $\mathcal{L}^\infty(X)$, which is the precomposition by the inverse: $\widetilde{\alpha}(g, a)(x) = a(\alpha(g^{-1}, x))$. This allows us to define the desired G-action β on $Y = \mathrm{Sp}(\mathcal{A})$ by

$$\beta(g, y)(a) \coloneqq y(\widetilde{\alpha}(g^{-1}, a)).$$

We easily check that it is a left action on Y: for any character $y \in Y$ and any two elements g and h in G we have

$$\begin{aligned}
\beta(g, \beta(h, y))(a) &= \beta(h, y)(\widetilde{\alpha}(g^{-1}, a)) \\
&= y(\widetilde{\alpha}(h^{-1}, \widetilde{\alpha}(g^{-1}, a))) \\
&= y(\widetilde{\alpha}((gh)^{-1}, a)) \\
&= \beta(gh, y)(a).
\end{aligned}$$

Moreover, for $y = \mathrm{ev}_x$ we have $\beta(g, y)(a) = \mathrm{ev}_x(\widetilde{\alpha}(g^{-1}, a)) = a(\alpha(g, x)) = \mathrm{ev}_{\alpha(g,x)}(a)$, that is to say that, Φ is G-equivariant: $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$.

In order to conclude, we have to check that β is continuous and preserves η. The fact that it is measure-preserving is a direct consequence of the uniqueness in Theorem 2.2.24. Indeed, $\mathcal{I}$ is G-invariant and for any function f in $\mathcal{I}$,

$$\int_X \widetilde{\alpha}(g, f)(x) d\lambda(x) = \int_X f(\alpha(g^{-1}, x)) d\lambda(x) = \int_X f(x) d\lambda(x).$$

This means that G preserves the integral of elements of $\mathcal{I}$, and by (Equation (2.3)) this implies that it preserves the integral of elements of $C_c(Y)$ (with regard to η). By uniqueness, the pushforward of η by the action of any element of G is equal to η. The G-action β is therefore measure-preserving.

Finally, the continuity of β is obtained in the exact same way as in Step 4 of the proof of Theorem 2.3.5. $\square$

Remark 2.5.2. It is tempting to try to add the following third condition to the above theorem:

(iii) There is a continuous G-action on a Polish space Y and a Borel injective map $\Phi : X \to Y$ such that $\Phi_*\lambda$ is locally finite and $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$ for all $x \in X$ and all $g \in G$.

Clearly *(i)* implies *(iii)*, but we don't know if the converse is true, although we suspect it is not. As we will see in Section 2.5.4, the only examples of actions not satisfying *(i)* that we have actually fail *(iii)*.

2.5.2 The case of locally compact groups

As in the previous section, we use the space $\mathcal{L}^\infty(X)$ of all bounded Borel functions $X \to \mathbb{C}$, endowed with the norm of uniform convergence $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We also use convolution as defined in Section 2.4.1, noting that the proof of Lemma 2.4.2 yields the following statement for everywhere defined functions.

Lemma 2.5.3. *Consider a locally compact Polish group G, and a spatial measure-preserving action of G on a standard σ-finite space (X, λ). Let $\delta : G \to \mathbb{R}_+$ be continuous of integral 1 and compactly supported. Then, for any $f \in \mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$, the function $\delta * f$ is in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ and is G-continuous.* □

We can now prove Theorem B after first recalling its statement.

Theorem 2.5.4. *Let G be a locally compact Polish group, and denote by α a spatial measure-preserving G-action on a standard σ-finite space (X, λ). Then there exists a locally compact Polish space Y endowed with a continuous action β and a Borel injection $\theta : X \to Y$ such that the pushforward measure $\theta_*\lambda$ is Radon and for all $x \in X$ and all $g \in G$,*

$$\theta(\alpha(g, x)) = \beta(g, \theta(x)).$$

Proof. By Item (1) of Theorem 2.1.1 we may as well assume that X is compact and α is continuous (this is also a consequence of an earlier and easier result of Varadarajan, see [Var63, Thm. 3.2]). We also denote by d a compatible metric on X and fix a sequence (x_n) enumerating a dense subset of X. By Theorem 2.5.1, it is sufficient to prove that there exists a countable set $\mathcal{D}$ of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ that separates points. We will construct this set $\mathcal{D}$ using convolution with regard to the Haar measure m on G.

We begin by choosing for every $\varepsilon > 0$ an open neighborhood $\mathcal{N}_\varepsilon$ of e_G in G such that for all $g \in \mathcal{N}_\varepsilon$ and all $x \in X$ we have $d(\alpha(g^{-1}, x), x) < \varepsilon$. Such a neighborhood exists by continuity of α and compactness of X. We then fix a continuous function $\delta_\varepsilon : G \to \mathbb{R}_+$ of integral 1 with a compact support included in $\mathcal{N}_\varepsilon$.

Let us fix $n \in \mathbb{N}$ and $\varepsilon > 0$, our first aim is to define countably many integrable functions separating the points of the open ball $B(x_n, \varepsilon)$ from the points in $X \setminus B(x_n, 3\varepsilon)$. By the choice of δ_ε, for any $x \in B(x_n, \varepsilon)$ and any $y \in X \setminus B(x_n, 3\varepsilon)$, we have

$$\begin{cases} \delta_\varepsilon * \mathbb{1}_{B(x_n,2\varepsilon)}(x) = \displaystyle\int_G \delta_\varepsilon(g)\mathbb{1}_{B(x_n,2\varepsilon)}(\alpha(g^{-1}, x))dm(g) = \int_G \delta_\varepsilon(g)dm(g) = 1 \\ \delta_\varepsilon * \mathbb{1}_{B(x_n,2\varepsilon)}(y) = \displaystyle\int_G \delta_\varepsilon(g)\mathbb{1}_{B(x_n,2\epsilon)}(\alpha(g^{-1}, y))dm(g) = 0 \end{cases} \tag{2.4}$$

However the above function $\delta_\varepsilon * \mathbb{1}_{B(x_n,2\varepsilon)}$ could very well fail to be integrable. We therefore set $X = \sqcup_{k\in\mathbb{N}} X_k$, with $\lambda(X_k) = 1$ for any k, and define

$$A_{n,k,\varepsilon} = X_k \cap B(x_n, 2\varepsilon),$$

so that $\mathbb{1}_{A_{n,k,\varepsilon}} \in \mathcal{L}^1(X, \lambda)$, for any positive integer k, and hence $\delta_\varepsilon * \mathbb{1}_{A_{n,k,\varepsilon}} \in \mathcal{L}^1(X, \lambda)$ for all $\varepsilon > 0$ by Lemma 2.5.3. On Figure 2.2 we can see in orange the set $A_{n,k,\varepsilon}$, as well as the image of $B(x_n, 2\varepsilon)$ under the G-action by elements of $\mathcal{N}_\varepsilon$ in blue.

In order to obtain a countable set of functions, we enumerate $\mathbb{Q}_{>0}$ as $\mathbb{Q}_{>0} = \{\varepsilon_i : i \in \mathbb{N}\}$. We can finally define our countable set of functions:

$$\mathcal{D} := \left\{ \delta_{\varepsilon_i} * \left(\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \right) : i, n, N \in \mathbb{N} \right\}.$$

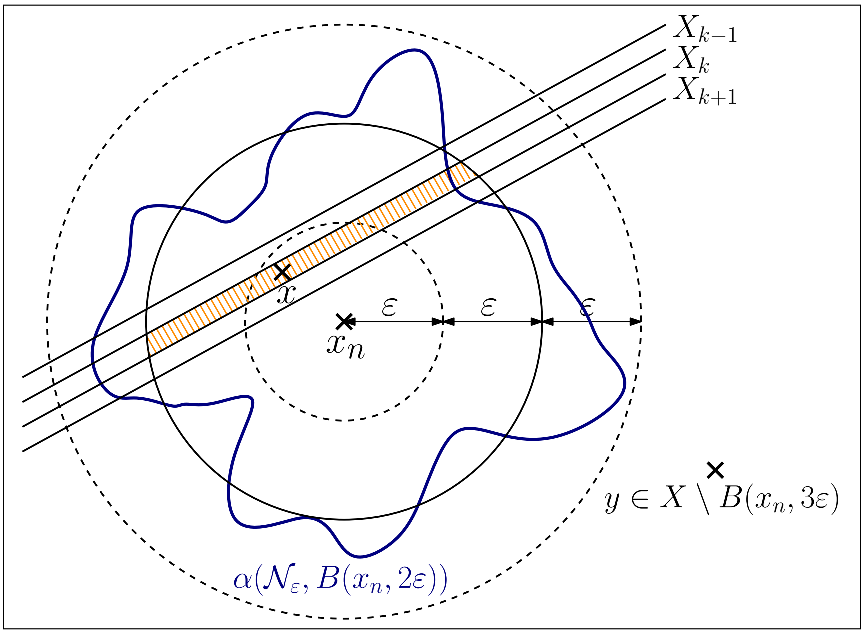

Figure 2.2: Visualisation of the construction.

By Lemma 2.5.3 we have $\mathcal{D} \subseteq \mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ and $\mathcal{D}$ only contains G-continuous functions.

We now prove that $\mathcal{D}$ separates the points of X. To this end, let x and y be in X such that for all $f \in \mathcal{D}$ we have $f(x) = f(y)$. Fix i and n in $\mathbb{N}$. We have $B(x_n, 2\varepsilon_i) = \bigsqcup_{k \in \mathbb{N}} A_{n,k,\varepsilon_i}$, hence we have the pointwise convergence

$$\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \underset{N \to +\infty}{\longrightarrow} \mathbb{1}_{B(x_n, 2\varepsilon_i)}.$$

Therefore by dominated convergence (using δ_{ε_i} as a dominating function) we have:

$$\begin{aligned} \delta_{\varepsilon_i} * \left(\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \right)(x) &= \int_G \delta_{\varepsilon_i}(g) \left(\sum_{k=0}^{N} \mathbb{1}_{A_{n,k,\varepsilon_i}} \right) (\alpha(g^{-1}, x)) dm(g) \\ &\underset{N \to \infty}{\longrightarrow} \int_G \delta_{\varepsilon_i}(g) \mathbb{1}_{B(x_n, 2\varepsilon_i)}(\alpha(g^{-1}, x)) dm(g) = \delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}(x). \end{aligned}$$

The same holds for y, and thus we have

$$\delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}(x) = \delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}(y).$$

By (2.4), the function $\delta_{\varepsilon_i} * \mathbb{1}_{B(x_n, 2\varepsilon_i)}$ takes the value 1 on $B(x_n, \varepsilon_i)$ and the value 0 on $X \setminus B(x_n, 3\varepsilon_i)$, so the previous equality ensures that for any $i, n \in \mathbb{N}$, we can never have simultaneously $d(x_n, x) < \varepsilon_i$ and $d(x_n, y) > 3\varepsilon_i$. By density of (x_n) and the fact that (ε_i) takes arbitrarily small values, we have $x = y$, which concludes the proof. $\square$

2.5.3 Local finiteness on X itself

We now present a nice consequence of Theorem 2.5.4 which was pointed out to us by Nachi Avraham-Re'em, allowing us to answer Question 2 positively when the acting group G is locally compact Polish.

Corollary 2.5.5. *Let G be a locally compact Polish group and $\alpha : G \times X \to X$ be a spatial measure-preserving G-action on a standard σ-finite space (X, λ). Then there is a Polish topology τ_X on X inducing its standard Borel structure such that α is a continuous action with respect to τ_X and λ is locally finite.*

Proof. Identifying X to its image $\Phi(X)$ via the map Φ provided by Theorem 2.5.4, we can assume that X is an α-invariant Borel subset of a locally compact Polish space (Y, τ_Y) on which λ extends to a Radon measure η, that α extends to a continuous η-preserving action β, and that the standard Borel structure on X is induced by the Borel σ-algebra of τ_Y. By [BK96, Thm. 5.1.5], there is a Polish topology τ'_Y on Y which refines τ_Y such that X is τ'_Y-open, β is τ'_Y-continuous and τ'_Y generates the same Borel σ-algebra as τ_Y.

Let us denote by τ_X the Polish topology induced by τ'_Y on X and check that τ_X is as desired. First, since X is τ'_Y-open and τ'_Y is Polish, we have that τ_X is Polish as well by [Kec95, Thm. 3.11]. Moreover τ_Y and τ'_Y both induce the Borel σ-algebra of Y, and since τ_Y induces the standard Borel space structure of X, so does τ_X. Since β is τ'_Y-continuous and α is the restriction of β to X, α is τ_X-continuous. Finally since η is locally finite, its restriction λ to X is locally finite for the topology induced by τ_Y on X, so λ is locally finite for the finer Polish topology τ_X induced by τ'_Y as well. □

In the above proof, we had to give up the local compactness of the ambient space Y, and it is natural to ask whether this can be circumvented. We now observe that even for countable discrete groups, local compactness of the standard Borel space onto which they act may be impossible to obtain.

Proposition 2.5.6. *There is a countable discrete group Γ and a Borel measure-preserving Γ-action on a standard σ-finite space (X, λ) such that X cannot be endowed with a locally compact Polish topology so that the action becomes continuous.*

Proof. Consider, for every pair of rationals (q, r) with $r > 0$, the Borel involutive bijection $I_{q,r}$ of the standard Borel space $\mathbb{R} \setminus \mathbb{Q}$ given by: for all $x \in \mathbb{R} \setminus \mathbb{Q}$,

$$I_{q,r}(x) = \begin{cases} x + r & \text{if } x \in (q - r, q) \\ x - r & \text{if } x \in (q, q + r) \\ x & \text{otherwise.} \end{cases}$$

Let Γ be the group generated by all involutions $I_{q,r}$, which is naturally acting on $X = \mathbb{R} \setminus \mathbb{Q}$ in a Borel manner, we claim this is the action we seek. First, this action preserves the measure λ induced by the Lebesgue measure on $\mathbb{R} \setminus \mathbb{Q}$ since it has the same orbits as the measure-preserving $\mathbb{Q}$-action by translation on $\mathbb{R} \setminus \mathbb{Q}$.

Now suppose by contradiction that τ is a locally compact Polish topology such that the Γ-action on X is continuous. For every pair (q, r) of rationals with $r > 0$, notice that the support of $I_{q,r}$ (which is by definition the set of points not fixed by $I_{q,r}$) is equal to $(q - r, q + r) \cap \mathbb{R} \setminus \mathbb{Q}$, and has to be τ-open by continuity of the Γ-action. This shows that τ refines the topology induced by $\mathbb{R}$ on $\mathbb{R} \setminus \mathbb{Q}$. Since τ is locally compact Polish, we can write $X = \bigcup_n K_n$ where each K_n is compact, and since τ refines the Polish topology of $\mathbb{R} \setminus \mathbb{Q}$, we deduce that $\mathbb{R} \setminus \mathbb{Q}$ is σ-compact, a contradiction since every compact subset thereof has empty interior. □

Remark 2.5.7. The key property of the action of Γ in the above proof is that the supports generate some non locally compact Polish topology. Actions whose supports generate a Hausdorff topology have to be totally non free, meaning that the map taking a point to its stabilizer subgroup is injective. As observed by Vershik (see [Ver12, Sec. 2.3]), any two totally non-free actions are abstractly isomorphic if and only if their stabilizer maps have the same range. In particular, if we remove from the action of the theorem an additional orbit $\mathbb{Q}+x$, we obtain for every $x\in\mathbb{R}$ a Borel action α_x on $\mathbb{R}\setminus(\mathbb{Q}\cup\mathbb{Q}+x)$, and α_x is abstractly isomorphic to α_y if and only if they are equal (which is of course equivalent to $x\in\mathbb{Q}+y$).

This argument cannot work in the setup of free actions : all the orbits of free actions are the same, and a Hilbert hotel argument shows that every free Borel action of a countable group is Borel isomorphic to any of its restrictions to the complement of countably many orbits. We refer the reader to [FKSV23, Sec. 3.3] for some positive results on Borel free actions of countable groups.

2.5.4 A counterexample when $G=\mathfrak{S}_\infty$

Let us take $G=\mathfrak{S}_\infty$ for the whole section. Our aim is to show that Theorem A is false for such a group, in particular Theorem 2.5.4 that we just proved does not hold for such a group as well.

The action α that we consider is the $\mathfrak{S}_\infty$-action on $X=\{0,1\}^{\mathbb{N}}$ by permutation of the coordinates. Let us also fix a sequence (p_n) of weights, with $p_n\in\,]0,1[$ for all n in $\mathbb{N}$. We also require that $p_n\neq p_m$ for $n\neq m$, and that $p_n\to\frac{1}{2}$. Consider the measures μ_n defined by :

$$\mu_n := (p_n\delta_1+(1-p_n)\delta_0)^{\otimes\mathbb{N}},$$

and define the measure λ on X by

$$\lambda := \sum_{n\in\mathbb{N}}\mu_n.$$

Let X_∞ be the subspace of X consisting of sequences that take infinitely many times the value 0, and infinitely many times the value 1. First note that $\mathfrak{S}_\infty$ acts transitively on X_∞. We have $\lambda(X\setminus X_\infty)=0$, as for any n in $\mathbb{N}$ we have $\mu_n(X\setminus X_\infty)=0$, and thus, up to a null set, we can restrict to X_∞.

Each μ_n is a probability measure on X, so λ is an infinite measure. We now verify that λ is σ-finite. Indeed, the strong law of large numbers ensures us that if we define a family (X_n) of Borel sets by

$$X_n := \left\{x\in X\colon \frac{|\{k\in\{0,\dots,m-1\}\colon x_k=1\}|}{m}\underset{m\to+\infty}{\longrightarrow}p_n\right\},$$

that is to say that X_n is the set of the sequences of X with a proportion p_n of 1, then for all n in $\mathbb{N}$ we have $\mu_n(X_n)=1$. We thus have

$$\lambda\left(X\setminus\bigsqcup_{n\in\mathbb{N}}X_n\right)=0.$$

The condition $p_n\neq p_m$, for $n\neq m$, ensures us that the X_n are disjoint sets, and therefore λ is σ-finite.

The following lemma provides an obstruction to having a continuous locally finite model.

Lemma 2.5.8. *Let $\mathcal{N}$ be a neighborhood of the identity in $\mathfrak{S}_\infty$, let $x\in X_\infty$. Then we have $\lambda(\alpha(\mathcal{N},x))=+\infty$.*

Proof. For every n let H_n denote the open subgroup of $\mathfrak{S}_\infty$ given by

$$H_n=\{\sigma\in\mathfrak{S}_\infty\colon \forall i\in\{0,\cdots n\},\,\sigma(i)=i\}.$$

Then the sequence (H_n) is a neighborhood basis of the identity in $\mathfrak{S}_\infty$, so we can find some H_N contained in our neighborhood $\mathcal{N}$. For any x_0 in X_∞, we have

$$\alpha(H_N, x) = \{y \in X_\infty : \forall i \in \{0, \dots, N\},\, y_i = x_i\}$$

so since $p_n \to \frac{1}{2}$ we have

$$\mu_n(\alpha(H_N, x)) \underset{n \to +\infty}{\longrightarrow} \left(\frac{1}{2}\right)^{N+1}.$$

Therefore,

$$\lambda(\alpha(H_N, x)) = \sum_{n \in \mathbb{N}} \mu_n(\alpha(H_N, x)) = +\infty,$$

which concludes the proof. □

Remark 2.5.9. By continuity of the action, for any x in X_∞, for any neighborhood $\mathcal{V}$ of x there exists N in $\mathbb{N}$ such that $\alpha(H_N, x) \subseteq \mathcal{V}$. So by Lemma 2.5.8 and the density of X_∞, all nonempty open subsets of X have infinite measure.

We have all the tools to show that Theorem A can fail badly for non locally compact Polish groups.

Proposition 2.5.10. *The spatial $\mathfrak{S}_\infty$-action α on (X, λ) defined above cannot be spatially isomorphic to a continuous action on a Polish space Y endowed with a locally finite measure η.*

Proof. Suppose that we have a continuous action β on (Y, η) with η locally finite and Y Polish, and that $\Phi : X_0 \to Y$ is a spatial isomorphism between α and β. Let us take some $x \in X_0$, then there is an open set U containing y such that $\eta(U) < \infty$. By continuity of β, there exists a neighborhood of the identity $\mathcal{N} \subseteq G$ satisfying $\beta(\mathcal{N}, \Phi(x)) \subseteq U$. However by Lemma 2.5.8, $\lambda(\alpha(\mathcal{N}, x)) = +\infty$. In particular, $\lambda(\alpha(\mathcal{N}, x) \cap X_0) = +\infty$. By the equivariance condition satisfied by Φ, the set $\beta(\mathcal{N}, \Phi(x))$ contains the infinite measure set $\Phi(\alpha(\mathcal{N}, x) \cap X_0)$, so it has infinite measure, contradicting the fact that it is contained in the finite measure set U. □

We now define a second $\mathfrak{S}_\infty$-action β as the action on $\bigsqcup_n (\{0,1\}^{\mathbb{N}}, \mu_n)$ by permutation of the coordinates, but this time in different distinct copies of $\{0,1\}^{\mathbb{N}}$. This will yield an interesting infinite measure-preserving example of non spatial boolean isomorphism (see Section 2.4.3 for Tsankov's example in the finite measure case).

Proposition 2.5.11. *The two $\mathfrak{S}_\infty$-actions α on $(\{0,1\}^{\mathbb{N}}, \lambda)$ and β on $\bigsqcup_n (\{0,1\}^{\mathbb{N}}, \mu_n)$ are booleanly isomorphic, but not spatially isomorphic.*

Proof. As X_n denotes the μ_n-conull set of sequences with a proportion p_n of coordinates equal to 1, it is possible to send X_n to a copy of itself in the n-th copy of $\{0,1\}^{\mathbb{N}}$ by defining

$$\Phi : \bigsqcup_{n \in \mathbb{N}} X_n \subseteq \{0,1\}^{\mathbb{N}} \longrightarrow \bigsqcup_{n \in \mathbb{N}} X_n \subseteq \bigsqcup_{n \in \mathbb{N}} \{0,1\}^{\mathbb{N}}.$$

For a fixed permutation σ, λ-almost any sequence x in $X_\sigma = \bigsqcup_n X_n$ satisfies $x \in X_k \Rightarrow \alpha(\sigma, x) \in X_k$, and therefore $\Phi(\alpha(\sigma, x)) = \beta(\sigma, \Phi(x)) \in \Phi(X_k)$. The actions α and β are not spatially isomorphic however, as α only has one orbit in $\{0,1\}^{\mathbb{N}}$, but β cannot send an element of $\Phi(X_n)$ to $\Phi(X_m)$ whenever $n \neq m$, which is illustrated by Figure 2.3. □

Remark 2.5.12. More generally, by Proposition 2.5.10, α cannot admit a continuous Radon model, so any boolean isomorphism between α and another continuous action on a space endowed with a Radon measure cannot be a spatial isomorphism.

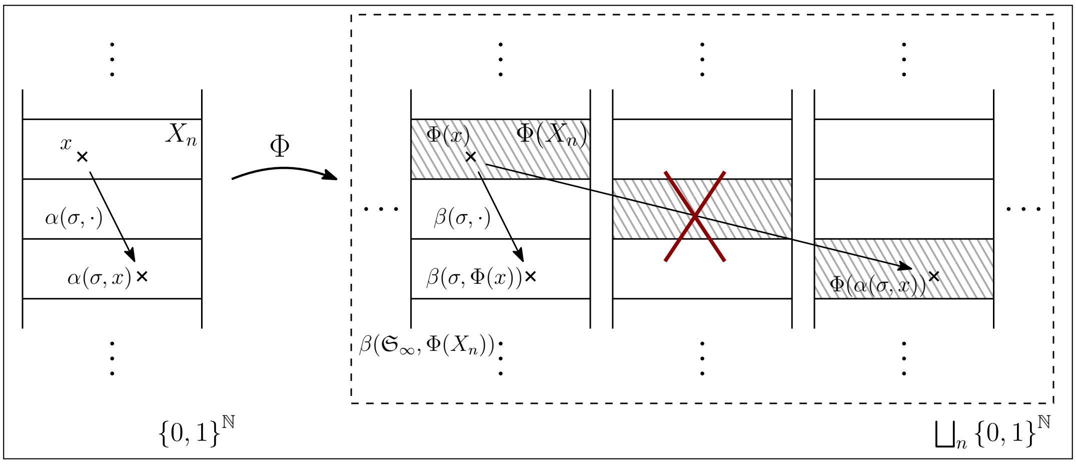

Figure 2.3: Spatial actions α and β on $\{0,1\}^{\mathbb{N}}$ and $\bigsqcup_n \{0,1\}^{\mathbb{N}}$ respectively. The hatched part is $\operatorname{rng}(\Phi)$.

Remark 2.5.13. Similar ideas work for instance when $G = \mathrm{Aut}(\mathbb{Q},<)$, identifying $\mathbb{N}$ to $\mathbb{Q}$ and replacing X_∞ by the set of all sequences $(x_q) \in \{0,1\}^{\mathbb{Q}}$ such that the set $\{q \in \mathbb{Q} \colon x_q = 1\}$ is both dense and codense in $\mathbb{Q}$ (meaning that for all rationals $q_1 < q_2$ there are q and q' in the interval (q_1, q_2) such that $x_q = 1$ and $x_{q'} = 0$). Indeed, this new set has μ_n measure 1, and a back-and-forth argument shows that it consists of a single $\mathrm{Aut}(\mathbb{Q},<)$-orbit. It would be nice to identify more precisely the class of non-archimedean groups for which there exists a spatial infinite measure-preserving action which cannot be spatially isomorphic to a continuous action on a locally finite Polish measured space.

The example we built in Proposition 2.5.11 is clearly non ergodic, hence the following natural question.

Question 3. Does $\mathfrak{S}_\infty$ admit two *ergodic* infinite measure-preserving actions which are booleanly isomorphic, but not spatially isomorphic?

2.6 Poisson point processes and Lévy groups

2.6.1 Construction of the Poisson point process

Let X be a Polish space, we can endow its space of closed subsets $\mathcal{F}(X)$ with the lower Vietoris topology (also called lower semi-finite topology in [Mic51]), which is the second-countable T_0 topology generated by declaring, for every $U \subseteq X$ open, that the set

$$\mathcal{V}_U = \{F \in \mathcal{F}(X) \colon F \cap U \neq \emptyset\}$$

is open. The associated Borel σ-algebra is called the Effros σ-algebra, and it turns $\mathcal{F}(X)$ into a standard Borel space, see for instance [Che21, Cor. 9.3].

Lemma 2.6.1. *Let $U \subseteq X$. The function which takes $F \in \mathcal{F}(X)$ to $|U \cap F|$ is lower semi-continuous if we endow $\mathcal{F}(X)$ with the lower Vietoris topology, in particular it is Borel.*

Proof. Suppose $|U \cap F_0| \geqslant k$, then we find U_1,...,U_k disjoint opens subsets of U such that $F_0 \cap U_i$ is non empty, and this will be true of any $F \in \bigcap_{i=1}^k \mathcal{V}_{U_i}$, hence the result. □

Remark 2.6.2. In particular, the set of all $F \in \mathcal{F}(X)$ such that $F \cap U$ is finite is Borel.

Let us now recall that the Poisson law of intensity $0 < \lambda < \infty$ is the probability measure Pois^{λ} on $\mathbb{N}$ given by

$$\mathrm{Pois}^{\lambda}(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$$

and that its expected value is equal to λ. We also define the degenerate Poisson law of intensity 0 as the Dirac measure at 0.

We now define on any Polish space endowed with an atomless locally finite Borel measure λ *the* **Poisson point process** of intensity λ as the probability measure whose existence and uniqueness are granted by the following elementary version of [AR25, Thm. 3].

Theorem 2.6.3. *Let λ be a locally finite atomless Borel measure on a Polish space X. There is a unique probability measure $\mathbb{P}^X_\lambda$ on $\mathcal{F}(X)$ such that for all $U \subseteq X$ open of finite measure, if the random variable F has law $\mathbb{P}^X_\lambda$, then the random variable $|F \cap U|$ follows the Poisson law of parameter $\lambda(U)$.*

Remark 2.6.4. The theorem does not yield what is usually called a Poisson point process per se (see [AR25, Def. 1.1]). In order to have one, we would need to further check that for all $A \subseteq X$ Borel, the map $F \in \mathcal{F}(X) \mapsto |A \cap F|$ is $\mathbb{P}^X_\lambda$-Lebesgue measurable, and that it follows a Poisson law of parameter $\lambda(A)$. Since we don't need this stronger property, we do not prove it and refer the reader to the proof of [AR25, Thm. 3].

Proof. In order to see the uniqueness, we first remark that closed subsets of the form $\mathcal{C}_U = \{F \in \mathcal{F}(X) \colon F \cap U = \emptyset\} = \mathcal{F}(X) \setminus \mathcal{V}_U$ form a π-system (because $\mathcal{C}_{U_1} \cap \mathcal{C}_{U_2} = \mathcal{C}_{U_1 \cup U_2}$) which generates the σ-algebra. We can then compute the probability of $\mathcal{C}_U$ as follows.

- If $\lambda(U)$ is finite, then by construction $\mathcal{C}_U$ has probability $e^{-\lambda(U)}$.
- Otherwise, by local finiteness we may write $U = \bigcup_n U_n$ where $U_n \subseteq U_{n+1}$ and each U_n has finite measure. We thus have $\lim_n \lambda(U_n) = \lambda(U) = +\infty$, moreover $\mathcal{C}_U$ is contained in each $\mathcal{C}_{U_n}$, which has probability $e^{-\lambda(U_n)} \to 0$. We conclude that $\mathcal{C}_U$ has probability 0.

By the monotone class theorem, this proves the uniqueness of the probability measure.

For the existence, let us first prove it when λ is finite. In this case, we may as well restrict ourselves to defining the measure on the subset $\mathcal{P}_f(X)$ of finite subsets of X, which is Borel by Lemma 2.6.1.

The map $\Phi : \mathbb{N} \times X^{\mathbb{N}} \to \mathcal{P}_f(X)$ which takes $(k, (x_n)_{n \geqslant 0})$ to $\{x_n \colon n < k\}$ is easily seen to be continuous if we put on $\mathcal{P}_f(X)$ the topology induced by the lower Vietoris topology on $\mathcal{F}(X)$, in particular it is Borel. Renormalize λ to a probability measure $\widetilde{\lambda} = \frac{\lambda}{\lambda(X)}$. Endow $\mathbb{N} \times X^{\mathbb{N}}$ with the probability measure $\mu = \mathrm{Pois}^{\lambda(X)} \otimes \widetilde{\lambda}^{\otimes \mathbb{N}}$. We claim that the pushforward measure $\Phi_*\mu$ is the desired Poisson Point Process.

In order to see this, first note that since λ is atomless, μ-almost all $(k, (x_n))$ satisfy that $x_n \neq x_m$ whenever $n \neq m$. It follows that given $U \subseteq X$ open and $l \in \mathbb{N}$, if for $k \geqslant l$ we denote by $\mathcal{P}_l(\{0, \ldots, k-1\})$ the set of subsets of $\{0, \ldots, k-1\}$ of cardinality l, the event $|F \cap U| = l$ has probability

$$\Phi_*\mu(|F \cap U| = l) = e^{-\lambda(X)} \sum_{k \geqslant l} \left(\frac{\lambda(X)^k}{k!} \sum_{K \in \mathcal{P}_l(\{0,\ldots,k-1\})} \left(\frac{\lambda(U)}{\lambda(X)}\right)^l \left(\frac{\lambda(X \setminus U)}{\lambda(X)}\right)^{k-l} \right).$$

Letting $x = \lambda(X)$ and $u = \lambda(U)$, we rewrite this as

$$\begin{aligned} e^{-x}\sum_{k\geqslant l}\frac{1}{k!}\binom{k}{l}u^l(x-u)^{k-l} &= e^{-x}\sum_{k\geqslant l}\frac{u^l(x-u)^{k-l}}{l!(k-l)!} \\ &= e^{-x}\frac{u^l}{l!}e^{x-u} \\ &= e^{-u}\frac{u^l}{l!}. \end{aligned}$$

Since $u = \lambda(U)$, our probability measure $\Phi_*\mu$ has the required distribution and hence the existence is proven when λ is a finite measure.

Let us now deal with the case where λ is infinite. By local finiteness write $X = \bigcup_n V_n$ where each V_n is open of positive finite measure and satisfies $V_n \subseteq V_{n+1}$. Each V_n is Polish for the induced topology, and if we denote by λ_n the restriction of λ to V_n, what we have done so far grants us a unique Borel probability measure $\mathbb{P}_\lambda^{V_n}$ on $\mathcal{P}_f(V_n)$ such that for all $U \subseteq V_n$ open, if F has law $\mathbb{P}_\lambda^{V_n}$ then the random variable $|F \cap U|$ has law $\mathrm{Pois}^{\lambda(U)}$.

For $n \in \mathbb{N}$, consider the continuous hence Borel map $\pi_n : \mathcal{P}_f(V_{n+1}) \to \mathcal{P}_f(V_n)$ which maps F to $F \cap V_n$. Form the projective limit

$$Y = \varprojlim \mathcal{P}_f(V_n) = \left\{(F_n) \in \prod_{n\in\mathbb{N}} \mathcal{P}_f(V_n) \colon \forall n \in \mathbb{N},\ F_n = F_{n+1} \cap V_n\right\},$$

which is a Borel subset of the standard Borel space $\prod_n \mathcal{P}_f(V_n)$ by [Par05, Thm. V.2.5].

For every $n \in \mathbb{N}$, it is a straightforward consequence of uniqueness that $\pi_{n*}\mathbb{P}_\lambda^{V_{n+1}} = \mathbb{P}_\lambda^{V_n}$, so by Kolmogorov's consistency theorem (see e.g. [Par05, Thm. V.3.2]) we have a unique Borel probability measure μ on Y such that for all $n \in \mathbb{N}$, $p_{n*}\mu = \mathbb{P}_\lambda^{V_n}$, where $p_n : Y \to \mathcal{P}_f(V_n)$ is the projection on coordinate n.

Let us finally consider the map $\Psi : Y \to \mathcal{F}(X)$ which takes (F_n) to the increasing union $\bigcup_n F_n$. First note that Ψ is well defined since $\bigcup_n F_n$ has finite intersection in each V_n, and the V_n's exhaust X so $\bigcup_n F_n$ is a discrete subset of X without accumulation points, hence closed (and countable). Next, Ψ is a Borel (actually continuous) map since $\bigcup_n F_n \cap U \neq \emptyset$ if and only if there exists n such that $F_n \cap (V_n \cap U) \neq \emptyset$. We claim that $\mathbb{P}_\lambda^X := \Psi_*\mu$ is the desired Poisson Point Process. Indeed, if U is an arbitrary open subset of X of finite measure, we have that $|F \cap U| = k$ if and only if there exists N such that for all $n \geqslant N$, $|F \cap (U \cap V_n)| = k$. For a fixed N, the event that for all $n \geqslant N$, $|F \cap (U \cap V_n)| = k$ is the intersection of a decreasing sequence of events which has probability

$$\mathbb{P}_\lambda^X\left(\forall n \geqslant N,\ |F \cap (U \cap V_n)| = k\right) = \lim_{n\to+\infty} e^{-\lambda(U\cap V_n)}\frac{\lambda(U \cap V_n)^k}{k!} = e^{-\lambda(U)}\frac{\lambda(U)^k}{k!}.$$

It follows that the event $|F \cap U| = k$ also has probability $e^{-\lambda(U)}\frac{\lambda(U)^k}{k!}$ as wanted. □

Remark 2.6.5. The proof of uniqueness is actually a version of Rényi's theorem (see for instance [LP17, Thm. 6.10]), namely it shows that $\mathbb{P}_\lambda^X$ is uniquely defined by the fact that for every finite measure open subset $U \subseteq X$, we have

$$\mathbb{P}_\lambda^X(F \cap U = \emptyset) = e^{-\lambda(U)}.$$

Remark 2.6.6. Since the measure is locally finite, $\mathbb{P}_\lambda^X$-almost every $F \in \mathcal{F}(X)$ is locally finite, that is to say that every point of F admits a neighborhood whose intersection with F is a singleton.

2.6.2 Application to actions of Lévy groups

Definition 2.6.7. A sequence (X_n, d_n, m_n) of metric spaces with probability measure is a **Lévy family** if it satisfies the following condition: if (A_n) is a sequence of measurable subsets of (X_n) such that $\liminf m_n(A_n) > 0$, then for any $\varepsilon > 0$, $\lim_n m_n((A_n)_\varepsilon) = 1$, where $(A_n)_\varepsilon \coloneqq \{x \in X_n : d_n(x, A_n) < \varepsilon\}$.

A Polish group G is a **Lévy group** if there is a sequence of compact subgroups (K_n) with $K_n \subseteq K_{n+1}$, such that $\bigcup_n K_n$ is dense in G, and such that the (K_n, m_n), where m_n is the normalized Haar measure, is a Lévy family when equipped with a right-invariant compatible metric d on G (the choice of such a d does not matter).

We refer the reader to [Pes06] for more about Lévy groups. Glasner, Tsirelson and Weiss proved the following fundamental result.

Theorem 2.6.8 ([GTW05, Thm. 1.1])**.** *Every measure-preserving spatial action of a Lévy group G on a possibly atomic standard probability space (X, μ) is trivial, i.e. the set of fixed points $\{x \in X : \forall g \in G, g \cdot x = x\}$ is conull.*

We obtain an analogous result for continuous actions of Lévy groups on Polish spaces with a infinite atomless σ-finite measure λ, under the assumption that the measure is locally finite. This result was first obtained by Avraham-Re'em and Roy, see [AR25, Thm. 5], and was stated in our introduction as Theorem E.

Theorem 2.6.9. *Every continuous measure-preserving spatial action of a Lévy group G on a Polish space X endowed with a σ-finite atomless locally finite measure λ is trivial, i.e. the set of fixed points $\{x \in X \mid \forall g \in G, g \cdot x = x\}$ is conull.*

Proof. Suppose by contradiction that we have an action as above on (X, λ), but that the action is not trivial. Consider the standard Borel space $\mathcal{F}(X)$ endowed with the Poisson Point Process on (X, λ) of intensity λ, that we denoted by $\mathbb{P}_\lambda^X$. Then $(\mathcal{F}(X), \mathbb{P}_\lambda^X)$ is a standard probability space, and we now show that G acts spatially on it.

Since every element of G defines a homeomorphism of X, it sends any closed subset of X to a closed subset of X, and in particular G acts on $\mathcal{F}(X)$. Since $g \cdot \mathcal{V}_U = \mathcal{V}_{g^{-1}U}$ and G acts by homeomorphism on X, we have that G acts by homeomorphisms on $\mathcal{F}(X)$ for the lower Vietoris topology.

Let us show that this action is continuous for the lower Vietoris topology. Given $U \subseteq X$ open, we need to show that the set of couples (g, F) such that $g \cdot F \cap U \neq \emptyset$ is open. So take (g_0, F_0) such that $g_0 \cdot F_0 \cap U \neq \emptyset$, then $F_0 \cap g_0^{-1} \cdot U \neq \emptyset$, then let $x_0 \in F_0 \cap g_0^{-1} \cdot U$. In particular $g_0 \cdot x \in U$, so by continuity of the action of G on X, there is a neighborhood V of g_0 and W of x_0 such that $V \cdot W \subseteq U$. Consider the neighborhood $V \times \mathcal{V}_W$ of (g_0, F_0), then any element (g, F) in the neighborhood $V \times \mathcal{V}_W$ of (g_0, F_0) satisfies $F \cap W \neq \emptyset$, and hence $g \cdot F \cap g \cdot W \neq \emptyset$. Since $g \in v$ we have $g \cdot W \subseteq U$, so we conclude that $gF \cap U \neq \emptyset$ as wanted. In particular, the G-action on $\mathcal{F}(X)$ is Borel.

This action preserves $\mathbb{P}_\lambda^X$ by uniqueness. Indeed consider the random variable F of law $\mathbb{P}_\lambda^X$, for any finite measure open subset U of X and any g in G we have

$$\begin{aligned}
\mathbb{P}_\lambda^X(|F \cap U| = k) &= e^{-\lambda(U)} \frac{\lambda(U)^k}{k!} \\
&= e^{-\lambda(g^{-1} \cdot U)} \frac{\lambda(g^{-1} \cdot U)^k}{k!} \\
&= \mathbb{P}_\lambda^X(\left|F \cap g^{-1} \cdot U\right| = k) \\
&= \mathbb{P}_\lambda^X(|g \cdot F \cap U| = k).
\end{aligned}$$

Being a spatial measure-preserving action on a standard probability space, the G-action on $\mathcal{F}(X)$ is trivial by Theorem 2.6.8. We will now see why this cannot be.

We first claim that we can find an open subset U of X with $0 < \lambda(U) < \infty$ and $g \in G$ which satisfies $U \cap g \cdot U = \emptyset$. Indeed, since G acts continuously non-trivially on (X, λ) the set of non fixed points $X_0 \coloneqq \{x \in X \mid \exists g \in G, g \cdot x \neq x\}$ has positive measure. By continuity of the G-action on X, for every $x \in X_0$ we find $g_x \in G$ and an open subset $U_x \subseteq X$ such that $U_x \cap g_x \cdot U_x = \emptyset$. By Lindelöf's theorem, the cover $(U_x)_{x \in X_0}$ admits a countable subcover $(U_{x_i})_{i \in \mathbb{N}}$. Since $\lambda(X_0) > 0$, we conclude that some U_{x_i} satisfies $\lambda(U_{x_i}) > 0$, and thus taking $g = g_{x_i}$ and $U = U_{x_i}$, we have $\lambda(U) > 0$ and $U \cap g \cdot U = \emptyset$ as desired.

Now remember that if $\mathcal{C}_U = \{F \in \mathcal{F}(X) \colon F \cap U = \emptyset\}$, we have $\mathbb{P}_\lambda^X(\mathcal{C}_U) = e^{-\lambda(U)} < 1$. We then have

$$\mathbb{P}_\lambda^X(\mathcal{C}_U \cap g \cdot \mathcal{C}_U) = \mathbb{P}_\lambda^X(\mathcal{C}_U \cap \mathcal{C}_{g^{-1} \cdot U}) = \mathbb{P}_\lambda^X(\mathcal{C}_{U \sqcup g^{-1} \cdot U}) = e^{-\lambda(U \sqcup g^{-1} \cdot U)} = \mathbb{P}_\lambda^X(\mathcal{C}_U)^2,$$

so $\mathbb{P}_\lambda^X(\mathcal{C}_U \cap g \cdot \mathcal{C}_U) < \mathbb{P}_\lambda^X(\mathcal{C}_U)$, yielding the non-triviality of the action and thus the desired contradiction. □

Remark 2.6.10. The hypothesis that λ has no atoms is not important: if it does, then G acts by permutation on these atoms, but the associated Bernoulli shift must be trivial by the Glasner-Tsirelson-Weiss theorem, so G acts trivially on the atomic part of λ.

2.7 The natural action of $\mathrm{Aut}(X, \lambda)$ admits no spatial model

Whirly actions were introduced in the probability measure-preserving context by Glasner, Tsirelson and Weiss, who showed such actions cannot have spatial models. In this last section we record the natural extension of this definition to the infinite measure-preserving setup, and show that whirly actions admit no spatial models. As an example, we prove that the natural boolean action of $\mathrm{Aut}(X, \lambda)$ is whirly and hence does not admit a spatial model.

Definition 2.7.1 ([GTW05, Sec. 3])**.** A boolean measure-preserving action α of a Polish group G on (X, λ) is **whirly** if for all sets A and B of positive measure, for every neighborhood $\mathcal{N}$ of e_G, there exists a element g in $\mathcal{N}$ such that $\lambda(A \cap \alpha(g, B)) > 0$.

Proposition 2.7.2. *Let $G = \mathrm{Aut}(X, \lambda)$. The natural boolean action of G on (X, λ) is whirly.*

Proof. Let A and B be two Borel subsets of X of positive measure, and fix $\varepsilon > 0$. We consider $A_0 \subseteq A$ and $B_0 \subseteq B$ two disjoint Borel subsets, with $\lambda(A_0) = \lambda(B_0) < \varepsilon/2$. We define an element S of $\mathrm{Aut}(X, \lambda)$ as follows:

$$\begin{cases} S_{\restriction X \setminus (A_0 \cup B_0)} = \mathrm{id}_{X \setminus (A_0 \cup B_0)} \\ S(A_0) = B_0 \\ S(B_0) = A_0, \end{cases}$$

and see that for ε small, S is close to the identity on X, for the weak topology. This concludes the proof, as $\lambda(A \cap SB) > 0$. □

Let us now explain the link between whirly actions and G-continuity. We have the following infinite measure version of [GTW05, Prop. 3.3]. The proof is actually the same, but we provide it for the reader's convenience.

Proposition 2.7.3. *Consider a boolean measure preserving action α of a Polish group G on (X, λ). If the action is whirly, then all G-continuous functions are constant. Moreover, such an action cannot admit a spatial model.*

Proof. We first explain why the existence of G-continuous non-constant functions taking values in $\mathbb{R}$ contradicts the fact that the action is whirly. .

Let f be a G-continuous non-constant function taking values in $\mathbb{R}$. There exists $a < b$ in $\mathbb{R}$ such that $A := f^{-1}(]-\infty, a[)$ and $B := f^{-1}(]b, +\infty[)$ have positive measure. By G-continuity, there exists a neighborhood $\mathcal{N}$ of e_G such that any g in $\mathcal{N}$ satisfies $\left\| f \circ \alpha(g_n^{-1}, \cdot) - f \right\|_\infty < b - a$. This implies that $\lambda(A \cap \alpha(g, B)) = 0$, contradicting the fact that the action is whirly.

Now for a complex valued function, one simply has to consider the real and imaginary parts, as they are G-continuous whenever f is. Assuming that f is non-constant implies that at least one or the other is non-constant, thus implying that the action cannot be whirly.

For the second part of the statement, suppose the action has a spatial model on (Y, η). By The Becker-Kechris theorem (Theorem 2.1.1), it is possible to assume that the G-action on Y is continuous, with Y compact Polish. The support of η is closed, thus compact, call it K. Since G acts continuously on Y, any function in $C_c(K)$ is G-continuous by Lemma 2.3.4, and thus any non-constant function in $C_c(K)$ gives a non-constant G-continuous function, a contradiction. □

Combining Proposition 2.7.2 and Proposition 2.7.3, we obtain the following.

Proposition 2.7.4. *Let (X, λ) be a standard σ-finite space. The natural boolean action of* $\mathrm{Aut}(X, \lambda)$ *on* (X, λ) *cannot admit a spatial model.*

Remark 2.7.5. As pointed out by the first referee, the above proposition can also be proven using Glasner, Tsirelson and Weiss' result which says that the natural boolean action of the group of probability measure-preserving transformation $\mathrm{Aut}(X, \mu)$ does not have spatial realizations [GTW05, Sect. 3]. Indeed, assume the natural $\mathrm{Aut}(X, \lambda)$ boolean action on (X, λ) admits a spatial model. By Remark 2.2.9 we find a Borel measure-preserving $\mathrm{Aut}(X, \lambda)$-action α on (X, λ) which induces its natural boolean action. Let X_0 be a Borel subset of measure 1 and consider the group G of all elements of $\mathrm{Aut}(X, \lambda)$ whose support is contained in X_0. Denote by μ the probability measure which is 0 outside of X_0 and coincides with λ on X_0. Observe that G naturally identifies to $\mathrm{Aut}(X, \mu)$ and that under this identification, $\alpha_{\restriction G}$ is a spatial realization of the natural boolean action of $\mathrm{Aut}(X, \mu)$, a contradiction.

Remark 2.7.6. Theorem 2.6.9 applies to $\mathrm{Aut}(X, \lambda)$ when λ is locally finite and the action is continuous. Indeed, Giordano and Pestov have proved in [GP07, Thm. 4.2] that $\mathrm{Aut}(X, \lambda)$ is a Lévy group when endowed with its weak topology. However, Proposition 2.7.4 does not follow from Theorem 2.6.9 as it does not require any topological conditions on the spatial model.

Chapter 3

Polish full groups preserving an infinite measure

This chapter corresponds to the article [Hoa25].

Abstract

We extend some results of Carderi and Le Maître on full groups in the probability context to the infinite measure one: there exists at most one Polish group topology (refining the weak topology and coarser than the uniform topology) on an ergodic full group, and the orbit full group of a locally compact group acting in a Borel manner can be endowed with a Polish group topology. Moreover, orbit full groups are complete invariants for orbit equivalence. We then generalize a result from Le Maître on non-Polishability of the group of finitely supported bijections: the finitely supported elements of an ergodic full group carries a Polish group topology if and only if the associated full group comes from a countable equivalence relation. We finish with algebraic and topological results on ergodic (orbit) full groups concerning normal subgroups, contractibility and genericity of aperiodic elements.

The paper is very heavy going, and I should never have read it had I not written it myself.

A Mathematician's Miscellany, J.E. Littlewood

Contents

3.1 Introduction

In his pioneering work ([Dye59], [Dye63]), Dye introduced the full groups as subgroups of the group $\mathrm{Aut}(X,\mu)$ of probability measure-preserving (pmp) bijections of a standard probabilty space (X,μ), where two such bijections are identified if they coincide on a conull set. This definition extends to the broader setup of non-singular Borel bijections of (X,μ), *i.e.* bijections $T : X \to X$ such that $T_*\mu$ is equivalent to μ. We denote by $\mathrm{Aut}(X,[\mu])$ the group they form.

Definition 3.1.1. A **full group** $\mathbb{G}$ is a subgroup of $\mathrm{Aut}(X,[\mu])$ which is stable under **cutting and pasting**, where a bijection T is obtained by cutting and pasting (T_n) if there exists a countable partition (A_n) of the space such that $T_{\restriction A_n} = T_{n\restriction A_n}$ for any $n \in \mathbb{N}$.

At this level of generality, countably generated full groups were first studied by Krieger in [Kri69]. The study of these groups is enriched when we see them as Polish groups, and as such we recall the following definitions.

Definition 3.1.2. The **uniform topology** on $\mathrm{Aut}(X,[\mu])$ is the group topology induced by the **uniform metric** d_μ defined by $d_\mu(S,T) = \mu(\{x \in X \mid T(x) \neq S(x)\})$. We also define the **bi-uniform metric** $\widetilde{d}$ by $\widetilde{d}(S,T) = d_\mu(S,T) + d_\mu(S^{-1},T^{-1})$.

The group $\mathrm{Aut}(X,[\mu])$ is complete for the bi-uniform metric. Moreover it can be shown that a full group is separable for the uniform topology if and only if it is the full group of a countable equivalence relation (see Definition 3.5.1), in which case it is a Polish group.

In the more specific context of pmp bijections, Carderi and Le Maître took a step out of the countable world and developed in [CLM16] a framework providing many new Polish full groups. Our first order of business is to generalize this construction to the setup of bijections preserving a σ-finite infinite measure. The group they form is denoted by $\mathrm{Aut}(X,\lambda)$, where (X,λ) is a standard σ-finite space. We give the definition of orbit full groups in this context.

Definition 3.1.3. Let G be a Polish group acting on a standard σ-finite space (X,λ) in a Borel manner. The **orbit full group** $[\mathcal{R}_G]$ is defined by

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X,\lambda) \mid \forall x \in X : T(x) \in G\cdot x\}.$$

If (X,λ) is a standard probability space instead, we recover the definition from [CLM16]. However, from a dynamical point of view, the world of σ-finite infinite measures is very different from the world of finite measures. Indeed, the phenomenon of dissipativity appears, exemplified by the map $x \mapsto x+1$ on $(\mathbb{R},\mathrm{Leb})$. The space is cleaved by the *Hopf decomposition* into the conservative (or recurrent) and dissipative parts, and even for well-understood systems such as odometers, Eigen, Hajian and Halverson proved in [EHH98] that recurrence behaves differently: odometers can fail to be multiply recurrent. Furthermore, the group $\mathrm{Aut}(X,\lambda)$ is not a simple group, as opposed to its probability counterpart $\mathrm{Aut}(X,\mu)$ (see [Fat78]). Our first result is the following (for the definition of the *topology of orbital convergence in measure*, see Section 3.5.1).

Theorem F (see Thm. 3.5.9)**.** *let G be a locally compact Polish group acting in a measure-preserving manner on a standard σ-finite space (X,λ). The orbit full group $[\mathcal{R}_G]$ endowed with the topology of orbital convergence in measure is a Polish group.*

Our result is actually more general than this, and expands on a subtle point specific to infinite measures, namely local finiteness. Recall that a Borel measure on a topological space is **locally finite** when every point admits a neighbourhood of finite measure. Under this additional condition, we obtain the conclusion of Theorem F for continuous actions of general Polish groups. In fact this conclusion may fail to hold when the measure is not locally finite and the acting group is not locally compact (see Proposition 3.5.16).

Theorem G (see Thm. 3.5.7, Cor. 3.5.12). *Let G be a Polish group acting in a continuous manner on a Polish space (X, τ_X) equipped with an atomless σ-finite measure λ which is locally finite on τ_X. The orbit full group $[\mathcal{R}_G]$, equipped with the topology of orbital convergence in measure, is a Polish group. Moreover, if the action is measure-preserving and essentially free, G topologically embeds in $[\mathcal{R}_G]$.*

Theorem F follows from Theorem G via Theorem 2.5.4, which allows us to turn any Borel infinite measure-preserving action of a locally compact Polish group into a continuous one, where the measure is locally finite.

We now restrict our study to the much more rigid class of ergodic full groups. Recall that a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \lambda)$ is **ergodic** if for every $A \subseteq X$ satisfying $\lambda(T(A)\Delta A) = 0$ for every $T \in \mathbb{G}$, we have that A is either null or conull. Moreover, any measure-preserving action of a Polish group G yields a group homomorphism $\pi : G \to \mathrm{Aut}(X, \lambda)$, and we say that the G-action is ergodic when $\pi(G)$ is ergodic as a subgroup of $\mathrm{Aut}(X, \lambda)$. Our next result is a reconstruction-type theorem in the vein of [Dye63, Thm. 2], building upon Fremlin's generalization to groups with *many involutions* (see Theorem 3.5.28), which is the appropriate condition to consider and is satisfied by all ergodic full groups.

Theorem H (see Thm. 3.5.31). *Let G and H be two locally compact Polish groups acting in a measure-preserving ergodic manner on a standard σ-finite space (X, λ). Then any group isomorphism between the orbit full groups $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$ is the conjugation by an orbit equivalence $S : X \to X$ which preserves the measure λ up to multiplication by an element of $\mathbb{R}_+^*$.*

Apart from the uniform topology, one can also endow $\mathrm{Aut}(X, \lambda)$ with the weak topology (see Definition 3.3.5), which makes it a Polish group. In the pmp setup, Carderi and Le Maître proved that the topology of orbital convergence in measure brings the weak and the uniform topologies under the same umbrella. It is moreover the only Polish group topology that an ergodic full group can carry ([CLM16, Thm. 4.7]). We establish the following, extending this result to our setup.

Theorem I (see Thm. 3.6.4 and Thm. 3.6.19). *Let $\mathbb{G}$ be an ergodic full group on a standard σ-finite space. Then $\mathbb{G}$ carries at most one Polish group topology, and such a topology is necessarily coarser than the uniform topology, but refines the weak topology, seen as topologies inherited from those on* $\mathrm{Aut}(X, \lambda)$.

Proving that our Polish topology is coarser than the uniform one uses automatic continuity, which has already been well-studied. For instance Sabok has developed a framework giving a proof that $\mathrm{Aut}(X, \mu)$ has automatic continuity in [Sab19], generalized to infinite measure by Le Maître in [LM22]. An indispensable tool for establishing automatic continuity is the *Steinhaus Property*, introduced in [RS07], and used by Kittrell and Tsankov in [KT10] to show that ergodic full groups of countable equivalence relations have automatic continuity. We expand on these techniques, and following an unpublished result from Fremlin, we show that any infinite measure-preserving ergodic full group is Steinhaus when endowed with the uniform topology.

Our result relies on a new factorisation of any infinite-measure preserving bijection (see Proposition 3.6.12), allowing us to decrease the Steinhaus exponent obtained in Fremlin's proof, from 228 to 114.

Next we focus on a group that has no probability measure-preserving analogue: the group $\mathrm{Aut}_f(X,\lambda)$ of bijections with supports of finite measure. In [LM22], Le Maître proved that this group does not admit any Polish group topology, a result in the spirit of the work of Rosendal (see [Ros05]). In the infinite measure-preserving context, we already know that the full group of a countable equivalence relation is a Polish group when endowed with the uniform topology, which has to coincide with the topology of orbital convergence in measure by uniqueness. In that case, the finitely supported bijections of the full group form a group that can be endowed with a Polish group topology, contrasting with the case of $\mathrm{Aut}_f(X,\lambda)$. We show that this is an equivalence.

Theorem J (see Thm. 3.6.29)**.** *Let $\mathbb{G}$ be an ergodic full group on a standard σ-finite space. The group $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda)$ can be endowed with a Polish group topology if and only if $\mathbb{G}$ is the full group of a countable equivalence relation.*

The last part of this work is dedicated to the topological and algebraic properties of ergodic full groups, seen as Polish groups. Contractibility, which has been shown

- for $\mathrm{Aut}(X,\mu)$ by Keane in [Kea70],
- for probability measure-preserving orbit full groups by Carderi and Le Maître in [CLM16],
- for $\mathrm{Aut}(X,\lambda)$ and infinite measure-preserving full groups of countable equivalence relations (in fact even for some type III actions) by Danilenko in [Dan95],

is established for the class of infinite measure-preserving ergodic orbit full groups in Proposition 3.7.5. The strategy used in [Kea70] and then in [CLM16] cannot be directly used, as induced bijections are only defined for conservative bijections. We follow Danilenko's proof from [Dan95, Thm. 2.2], who uses the contractibility of the space of partitions of X into subsets of measure one, to circumvent this issue.

The question of the existence of normal subgroups is also addressed. Eigen proved in [Eig81] that full groups generated by a single ergodic bijection admit a unique non-trivial normal subgroup. We get the following general statement, which in particular uses the decomposition of any bijection into three involutions, a result of Ryzhikov (see Theorem 3.4.18).

Theorem K (see Thm. 3.7.3)**.** *Let $\mathbb{G}$ be an ergodic full group on a standard σ-finite space. The group $\mathbb{G}_f$ is simple, and is the only non-trivial normal subgroup of $\mathbb{G}$.*

Finally, Polishness raises the question of generic properties. Krengel and Sachdeva have proved already that conservative elements and ergodic elements are generic in $\mathrm{Aut}(X,\lambda)$, when endowed with the weak topology (see [Kre67] and [Sac71]). Carderi and Le Maître, on the other hand proved that aperiodic elements are generic in pmp orbit full groups, when the acting group is uncountable. In particular their proof uses a result from [Tör06], which we also generalize to infinite measures. Some results of Choksi and Kakutani from [CK79] about conjugates of ergodic bijections are also used to prove the following.

Theorem L (see Thm. 3.7.16). *Let $\mathbb{G} = [\mathcal{R}_G]$ be an ergodic orbit full group on a standard σ-finite space associated with a Borel action of a Polish group (G, τ_G). If the topology of orbital convergence in measure is Polish on $\mathbb{G}$, then the following are equivalent:*

(1) the aperiodic elements of $\mathbb{G}$ form a dense subset of $\mathbb{G}$;

(2) the ergodic elements of $\mathbb{G}$ form a dense subset of $\mathbb{G}$;

(3) the $\mathbb{G}_f$-conjugacy class of any aperiodic element of $\mathbb{G}$ is dense in $\mathbb{G}$;

(4) for any essentially free measure-preserving action of a countable group $\Gamma \curvearrowright (X, \lambda)$, there is a dense G_δ set of elements of $\mathbb{G}$ inducing a free action of $\Gamma * \mathbb{Z}$.

These equivalent conditions moreover imply the following one:

(5) for any τ_G-neighbourhood V of e_G, the set $\bigcup_{g \in V} \{x \in X \mid g \cdot x \neq x\}$ is conull.

Finally, (5) implies the above conditions if the space X is endowed with a compatible Polish topology, with regard to which the measure is locally finite and the G-action is continuous.

In particular, these equivalent conditions allow us to give another characterization of ergodic full groups coming from a countable equivalence relation (see Corollary 3.7.19).

Given the hypotheses of Theorem 3.5.7 and Theorem 3.7.16, their reliance on Theorem 2.5.4, as well as Proposition 2.5.10 and Proposition 3.5.16, the following question is very natural.

Question. Let G be a Polish group acting on a standard σ-finite space (X, λ) in a measure-preserving manner, and consider the associated orbit full group $[\mathcal{R}_G]$, endowed with its topology τ_m^o of orbital convergence in measure. Are the following equivalent?

- The group $([\mathcal{R}_G], \tau_m^o)$ is a Polish group.
- There exists a locally finite Polish model $G \curvearrowright (Y, \eta)$ for the G-action on (X, λ): *i.e.* a continuous measure-preserving G-action on a Polish space Y endowed with a locally finite measure η, such that the actions are spatially isomorphic.

3.2 Preliminaries of Chapter 3

3.2.1 Infinite measures and measure algebras

We give the definition of measure algebras in the context of σ-finite spaces, and recall a few essential facts.

Definition 3.2.1. We call the **measure algebra** of a standard σ-finite space (X, λ) and denote by $\mathrm{MAlg}(X, \lambda)$ the space of Borel subsets of X, where two such subsets are identified if the measure of their symmetric difference is equal to zero. We will however mostly be interested in $\mathrm{MAlg}_f(X, \lambda)$, the subspace of $\mathrm{MAlg}(X, \lambda)$ comprised of the elements of finite measure, which we will call the **finite measure algebra** of (X, λ). It is equipped with the metric $d_{X,\lambda}$ defined by $d_{X,\lambda}(A, B) := \lambda(A \Delta B)$.

We have the following well-known generalization of Proposition 1.2.25 (see *e.g.* [LM22, Lem. 2.1]).

Proposition 3.2.2. *Let (X, λ) be a standard σ-finite space. Then $(\mathrm{MAlg}_f(X, \lambda), d_{X,\lambda})$ is a separable and complete metric space.*

The existence of a *maximal element* in a measure algebra is very useful. Some authors prove the following via a *measurable Zorn's lemma*, but we can also adapt another proof of Le Maître in the finite measure setting ([LM14a, Appendix A], see also Proposition A.2.6), to our context. This proof works for an abstract measure algebra, with no underlying measured space (see *e.g.* [Fre02, Ch. 32]).

Proposition 3.2.3. *Consider (X,λ) a standard σ-finite space, and $\mathrm{MAlg}_f(X,\lambda)$ the associated finite measure algebra. Let $\mathcal{F} = \{F_i \mid i \in I\}$ be a family of elements of $\mathrm{MAlg}_f(X,\lambda)$.*

(1) If $\mathcal{F}$ is upward directed and satisfies $M = \sup_{A\in\mathcal{F}} \lambda(A) < \infty$, then $\mathcal{F}$ admits a supremum, which has measure M. Moreover, if $\mathcal{F}$ is stable by countable union, then $\sup \mathcal{F}$ is a maximal element. Furthermore, the supremum (resp. maximal element) of the family is obtained as the limit of an increasing sequence of elements of the family.

(2) If $\mathcal{F}$ satisfies $\lambda(\cup_{i\in J} F_i) < R < \infty$ for any finite subset $J \subseteq I$, then $\mathcal{F}$ admits a supremum. Furthermore, the supremum of the family is obtained as the limit of an increasing sequence of finite reunions of elements of the family.

Proof. (1). Let us start by assuming that $\mathcal{F}$ is an upward directed family of elements of $\mathrm{MAlg}_f(X,\lambda)$, in other words for any A, B in $\mathcal{F}$ there exists C in $\mathcal{F}$ such that $A \subseteq C$ and $B \subseteq C$.

Fix a sequence (A_n) of elements of $\mathcal{F}$ with $\lambda(A_n)$ converging to M. As $\mathcal{F}$ is upward directed, by induction we can find $B_n \in \mathcal{F}$ such that $A_n \subseteq B_n$ and $B_{n-1} \subseteq B_n$, which gives us an increasing sequence in $\mathcal{F}$ satisfying $\lambda(B_n) \to M$. We prove that it is a Cauchy sequence. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for any $n \geqslant N$ we have $\lambda(B_n) > M - \varepsilon$. Then for any $n > m \geqslant N$, $d_{X,\lambda}(B_n, B_m) = \lambda(B_n \Delta B_m) = \lambda(B_n \setminus B_m) = \lambda(B_n) - \lambda(B_m) < \varepsilon$. Therefore (B_n) is a Cauchy sequence in $(\mathrm{MAlg}_f(X,\lambda), d_{X,\lambda})$ which is complete, and its limit is $\sup \mathcal{F}$. In particular $\lambda(\sup \mathcal{F}) = \sup_{A\in\mathcal{F}} \lambda(A)$.

We now assume that $\mathcal{F}$ is stable by countable union (do note that this implies being upward directed). By the previous argument, $\sup \mathcal{F}$ is defined as the limit of an increasing sequence (B_n) of elements of $\mathcal{F}$. This means that we have $\sup \mathcal{F} = \lim B_n = \bigcup_{n\in\mathbb{N}} B_n$, which is in $\mathcal{F}$ by hypothesis.

(2) We finally assume that $\mathcal{F}$ is any family of elements of $\mathrm{MAlg}_f(X,\lambda)$ satisfying the measure condition for finite unions. Denote by $\mathcal{G}$ the family of finite reunions of elements of $\mathcal{F}$. It is upward directed, and therefore by the previous argument it admits a supremum $\sup \mathcal{G}$, which is also the supremum of $\mathcal{F}$. ☐

Remark 3.2.4. Let (X,λ) be a standard σ-finite space, and denote by $[\lambda]$ the measure class of λ. There always exists a probability measure μ in $[\lambda]$.

We have the following two classical lemmas about the behaviour of a probability measure in the class of a σ-finite measure.

Lemma 3.2.5 ([Coh13, Lem. 4.2.1])**.** *Let X be a standard Borel space endowed with its natural σ-algebra. Let λ and μ be two Borel measures defined on X, with μ finite. Then μ is absolutely continuous with regard to λ if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that any Borel subset A of X satisfying $\lambda(A) < \delta$ also satisfies $\mu(A) < \varepsilon$.*

Lemma 3.2.6. *Fix μ a probability measure in $[\lambda]$. Let also (A_n) be a sequence of Borel subsets of X. We have the following equivalence.*

$$(\mu(A_n) \to 0) \iff (\forall C \text{ Borel subset of } X \text{ of finite } \lambda\text{-measure}, \ \lambda(A_n \cap C) \to 0).$$

Proof. The converse implication is immediate from Lemma 3.2.5, but we can also explicitly prove it *via* a classical method. Write $X = \bigsqcup_{n\in\mathbb{N}} X_n$, with $\lambda(X_n) = 1$ for all n in $\mathbb{N}$. We define a probability measure μ' in $[\lambda]$ by setting

$$\mu'(B) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \lambda(B \cap X_k)$$

for any Borel subset B of X. Now if $\lambda(A_n \cap X_k) \longrightarrow_n 0$ for all k in $\mathbb{N}$, then $\mu'(A_n) \to 0$. As μ is equivalent to μ', $\mu(A_n) \to 0$, which proves the converse implication.

For the direct implication, consider the Radon-Nikodym derivative f such that $d\lambda = f d\mu$. For any Borel subset C of X of finite measure, we have

$$\lambda(A_n \cap C) = \int_C \mathbb{1}_{A_n} f(x) d\mu(x) \longrightarrow 0$$

by the Lebesgue dominated convergence theorem. □

The interplay between topology and measure is particularly relevant in the case of infinite measures. We start by recalling the definitions of locally finite measures and Radon measures, then recall the link between the two in suitable topological spaces.

Definition 3.2.7. Let λ be a Borel measure on a Polish space X. We say that λ is **locally finite** (with regard to the topology of X) if every $x \in X$ admits an open neighborhood U such that $\lambda(U) < +\infty$.

Definition 3.2.8. A measure λ on the Borel σ-algebra of a Hausdorff topological space X is called **Radon** when it verifies the following:

1. $\lambda(K) < +\infty$ for any compact subset $K \subseteq X$,
2. for each open subset U of X, we have $\lambda(U) = \sup\{\lambda(K) \mid K \subseteq U \text{ and } K \text{ is compact}\}$ (inner regularity on open sets).
3. for each Borel subset A of X, we have $\lambda(A) = \inf\{\lambda(U) \mid A \subseteq U \text{ and } U \text{ is open}\}$ (outer regularity on Borel sets).

The following well-known result explains the link between Radon measure and locally finite measures on locally compact spaces (see *e.g.* [HLM24, Prop. 2.21]).

Proposition 3.2.9. *Let Y be a locally compact Polish space, and let η be a Borel measure on Y. Then η is Radon if and only if it is locally finite.*

3.2.2 Continuous realisations of Borel group actions

One convenient way to obtain information about Borel group actions of topological group on standard spaces is to find a topological space on which the group acts continuously. We need this space to have the same Borel structure as the original one, and to be such that the actions are isomorphic. In this case we talk about a **spatial model**, or a **spatial realisation** of the Borel action. One of the most striking results is the following one.

Theorem 3.2.10 ([BK96, Thm. 5.2.1])**.** *Let G be a Polish group and let α be a Borel G-action on a standard Borel space X. Then there exists a Polish topology on X inducing its Borel structure and such that α is continuous.*

This result does however change the underlying topology of the space on which a group acts, and thus needs not preserve the local finiteness of the measure. As it will be a key hypothesis in our infinite measure context, we need to find a different spatial realisation.

For a measure-preserving action of a locally compact Polish group, the author and Le Maître proved the following:

Theorem 3.2.11 ([HLM24, Thm. 4.4])**.** *Let G be a locally compact Polish group and (X, λ) be a standard σ-finite space. Let also α be a measure-preserving G-action on (X, λ). Then there exists a G-action β on a locally compact Polish space Y endowed with a Radon measure η, and a measure-preserving isomorphism $\Phi : (X, \lambda) \to (Y, \eta)$ with a conull subset $X_0 \subseteq X$ verifying: for all $g \in G$ and all x in X_0 such that $\alpha(g, x) \in X_0$ we have*

$$\Phi(\alpha(g, x)) = \beta(g, \Phi(x)).$$

Definition 3.2.12. Let G be a Polish group acting in a Borel measure-preserving manner on a standard σ-finite space (X, λ). The space (Y, η) satisfying the conditions of Theorem 3.2.11 is called a **continuous Radon model** of the G-action. In other words, Theorem 3.2.11 states that a measure-preserving action of a locally compact Polish group on a standard σ-finite space always admits a continuous Radon model.

This will be enough to provide us with the desired local finiteness under some mild assumptions, thanks to Proposition 3.2.9.

3.3 Topologies on the group of infinite measure-preserving bijections

Let (X, λ) be a standard σ-finite space. The group $\mathrm{Aut}(X, \lambda)$ plays a very important part in our work. It is defined as the group of measure-preserving bijections of (X, λ), *i.e.* bijections $T : X \to X$ satisfying $\lambda(A) = \lambda(T^{-1}(A)) = \lambda(T(A))$ for any Borel subset $A \subseteq X$, where two such bijections are identified when they coincide on a conull set.

3.3.1 Support and separators

We recall that the support of a bijection $T \in \mathrm{Aut}(X, \lambda)$ is defined as

$$\operatorname{supp} T \coloneqq \{x \in X \mid T(x) \neq x\},$$

where the equality is to be understood up to null sets. Notice that for two Borel bijections T, S of X, we have $S(\operatorname{supp} T) = \operatorname{supp}(STS^{-1})$.

When studying measure-preserving bijections of a space with infinite measure, one needs to pay attention to the following group, which does not have any analogue in the probability measure-preserving context.

Definition 3.3.1. The group $\mathrm{Aut}_f(X, \lambda)$ is the normal subgroup of $\mathrm{Aut}(X, \lambda)$ consisting of all T in $\mathrm{Aut}(X, \lambda)$ such that $\lambda(\operatorname{supp} T)$ is finite.

The following notations are very natural, and will be used extensively in the later sections.

Notation 3.3.2. For a family $\mathbb{G}$ of elements of $\mathrm{Aut}(X, \lambda)$ and any two Borel subsets $A, B \subseteq X$, we define

$$\mathbb{G}_A \coloneqq \{T \in \mathbb{G} \mid \operatorname{supp} T \subseteq A\},$$

as well as

$$\mathbb{G}_{(A,B)} \coloneqq \{T \in \mathbb{G} \mid T(A) \subseteq B\}.$$

We finally define

$$\mathbb{G}_f \coloneqq \{T \in \mathbb{G} \mid \lambda(\operatorname{supp} T) < \infty\} = \mathbb{G} \cap \mathrm{Aut}_f(X, \lambda).$$

A separator gives a very useful decomposition of the support, which is more useful than just a non-null Borel subset disjoint from its image by T.

Definition 3.3.3. Let T be an element of $\mathrm{Aut}(X,\lambda)$. A Borel subset A of $\operatorname{supp} T$, such that $\operatorname{supp} T = A \sqcup \big(T(A) \cup T^{-1}(A)\big)$ is called a **separator** for T.

The following well-known lemma is crucial, as it implies that separators always exist for measure-preserving bijections. This proof uses a result from [EG16], a proof of which can be found in Section A.2.2 (see Lemma A.2.9).

Lemma 3.3.4. *Let (X,λ) be a standard σ-finite space, and $T \in \mathrm{Aut}(X,\lambda)$. Let also C be any Borel subset of X. Then there exists a Borel subset $A \subseteq C$ such that $C \cap \operatorname{supp} T = A \sqcup (T(A) \cup T^{-1}(A))$. In particular for $C = X$, this implies that A is a separator for T.*

Proof. The original statement is given in the broader Borel setup, with no measure involved, but here we write the proof with equalities up to null sets.

Using Lemma 5.1 from [EG16], we write $\operatorname{supp} T = \bigsqcup_{n\in\mathbb{N}} B_n$, where (B_n) is a partition of $\operatorname{supp} T$ such that $T(B_n)$ is disjoint from B_n for any $n \in \mathbb{N}$. We then have $C \cap \operatorname{supp} T = \bigsqcup_{n\in\mathbb{N}} C \cap B_n$, and for all $n \in \mathbb{N}$ we have that $T(C \cap B_n)$ is disjoint from $C \cap B_n$.

We now inductively define a sequence (A_n) of Borel subsets of $C \cap \operatorname{supp} T$ as follows. Fix $A_0 = C \cap B_0$, and let A_{i+1} be defined by

$$A_{i+1} = (C \cap B_{i+1}) \setminus \left(\left(\bigsqcup_{j\leqslant i} T(A_j)\right) \bigcup \left(\bigsqcup_{j\leqslant i} T^{-1}(A_j)\right)\right).$$

We conclude the proof by noticing that $A = \bigsqcup_n A_n$ is suitable. Indeed, for any i we have $A_{i+1} \cap \left(\bigsqcup_{j\leqslant i} T(A_j)\right) = \emptyset$ and $A_{i+1} \cap \left(\bigsqcup_{j\leqslant i} T^{-1}(A_j)\right) = \emptyset$ so A is disjoint from both $T(A)$ and $T^{-1}(A)$. We have $C \cap \operatorname{supp} T = A \cup T(A) \cup T^{-1}(A)$, as almost any $x \in (C \cap \operatorname{supp} T) \setminus A$ is either in $T(A)$ or in $T^{-1}(A)$. ☐

3.3.2 Weak topology and topology of convergence in measure

We now recall one of the classical definitions of the weak topology of $\mathrm{Aut}(X,\lambda)$.

Definition 3.3.5. The **weak topology** on $\mathrm{Aut}(X,\lambda)$ is denoted by τ_w and is defined in the following way: $T_n \to T$ if and only if for all $A \subseteq X$ Borel subset of finite measure, one has $\lambda(T_n(A)\Delta T(A)) = d_{X,\lambda}(T_n(A), T(A)) \to 0$.

The following proposition states that it is a Polish group. The proof is given for convenience.

Proposition 3.3.6 ([LM22, Prop. 2.2])**.** *The group $\mathrm{Aut}(X,\lambda)$ is equal to the group of isometries of $(\mathrm{MAlg}_f(X,\lambda), d_{X,\lambda})$ which fix $\emptyset$. As such, it is a Polish group.*

Proof. It is clear that any $T \in \mathrm{Aut}(X,\lambda)$ gives an isometry fixing $\emptyset$. To prove the converse inclusion, let T be an isometry of $(\mathrm{MAlg}_f(X,\lambda), d_{X,\lambda})$ which fixes $\emptyset$ and let us recall the following fact ([Kec10, Section 1 (B)]): if (X,μ) and (Z,ν) are two standard probability spaces, any isometry between $(\mathrm{MAlg}(X,\mu), d_{X,\mu})$ and $(\mathrm{MAlg}(Z,\nu), d_{Z,\nu})$ sending $\emptyset$ to $\emptyset$ comes from a measure-preserving bijection $\phi_{X,Z}$, which is unique up to a null set.

Using the σ-finiteness of λ, we write $X = \bigcup_{n\in\mathbb{N}} X_n$, with $\lambda(X_n) < +\infty$ and $X_n \subseteq X_{n+1}$ for every n in $\mathbb{N}$. Thus, for every n in $\mathbb{N}$, the bijection T induces an isometry between $(\mathrm{MAlg}(X_n, \mu_{\restriction X_n}), d_{X_n,\mu})$ and $(\mathrm{MAlg}(T(X_n), \mu_{\restriction T(X_n)}), d_{T(X_n),\mu})$, which in turn gives us a measure-preserving bijection $\phi_n := \phi_{X_n,T(X_n)}$. By construction, and using the fact that these measure-preserving bijections are unique up to a null set, we can see that ϕ_{n+1} extends ϕ_n. It is then possible to define a measure-preserving bijection ϕ on X, such that $\phi_{\restriction X_n} = \phi_n$, for every $n \in \mathbb{N}$.

Finally, $\mathrm{Aut}(X,\lambda)$ is a Polish group, as a closed subgroup of the isometry group of a complete separable metric space (see *e.g.* [GK03, Thm. 3.1]). ☐

Remark 3.3.7. Equivalently, one can define τ_w in the following manner: $T_n \to T$ if and only if for all $A \subseteq X$ Borel subset of finite measure $\lambda(T_n(A) \setminus T(A)) \to 0$. Indeed, if A and B are of finite measure such that $\lambda(A) = \lambda(B)$, then $A = (A \setminus B) \sqcup (A \cap B)$ and therefore $\lambda(A) = \lambda(A \setminus B) + \lambda(A \cap B)$. As we can write $\lambda(B)$ in the same manner, we have $\lambda(A \setminus B) = \lambda(B \setminus A) = \frac{1}{2}\lambda(A \Delta B)$.

Moreover, if μ is a probability measure in $[\lambda]$ and $T_n \to T$ for τ_w, for any $A \subseteq X$ we have $\mu(T_n(A)\Delta T(A)) \to 0$. This is related to the topology on the group on non-singular bijections of (X, μ), of which $\mathrm{Aut}(X, \lambda)$ is a closed subgroup. We refer to [DS22, Sec. 4] for more on this. We actually have the following.

Lemma 3.3.8. *Let (X, λ) be a standard σ-finite space, and μ be a probability measure in $[\lambda]$. Then for any Borel subset $A \subseteq X$ the application $T \in \mathrm{Aut}(X, \lambda) \mapsto \mu(T(A)\Delta A)$ is τ_w-continuous.*

Proof. Let (T_n) be a sequence of measure-preserving bijections τ_w-converging to the identity in $\mathrm{Aut}(X, \lambda)$. By Lemma 3.2.6 it is enough to prove that for any Borel subset C of finite λ-measure we have $\lambda(C \cap (T_n(A)\Delta A)) \to 0$. We have

$$\begin{aligned}
\lambda(C \cap (T_n(A)\Delta A)) &= \lambda((C \cap T_n(A))\Delta(C \cap A)) \\
&= d_{X,\lambda}(C \cap T_n(A), C \cap A) \\
&\leqslant d_{X,\lambda}(C \cap T_n(A), T_n(C) \cap T_n(A)) + d_{X,\lambda}(T_n(C) \cap T_n(A), C \cap A) \\
&\leqslant d_{X,\lambda}(C, T_n(C)) + d_{X,\lambda}(T_n(C \cap A), C \cap A),
\end{aligned}$$

and both terms tend to 0 by weak convergence of (T_n) to id_X, as $\lambda(C) < +\infty$. □

The second topology we will be interested in is a natural topology that $\mathrm{Aut}(X, \lambda)$ inherits from the space of measurable functions.

Definition 3.3.9. Let (Z, τ_Z) be a Polish space. We denote by $\mathrm{L}^0(X, \lambda, (Z, \tau_Z))$ the space of Lebesgue-measurable maps from X to Z, identified up to measure 0. This notation will often be shortened to $\mathrm{L}^0(X, \lambda, Z)$ if there is no possible confusion on the topology of the range. Let d be a compatible bounded metric on (Z, τ_Z). For any Borel subset $C \subseteq X$ of finite measure, we define $\tilde{d}_C$ by

$$\tilde{d}_C(f, g) \coloneqq \int_C d(f(x), g(x))d\lambda(x).$$

The family of pseudometrics $\tilde{d}_C$ induces the **topology of convergence in measure** denoted by τ_m, where C ranges among the Borel subsets of X of finite measure.

It is of note that τ_m does not depend on the choice of the measure, only on its equivalence class, and also does not depend on the choice of d, but only on the topology it induces. This follows from the following statement.

Proposition 3.3.10 ([Moo76, Prop. 6])**.** *Let (f_n) be a sequence of elements of $\mathrm{L}^0(X, \lambda, (Z, \tau_Z))$, and $f \in \mathrm{L}^0(X, \lambda, (Z, \tau_Z))$, where (Z, τ_Z) is Polish, with d a compatible bounded metric. Then the following are equivalent:*

(a) $\tilde{d}(f_n, f) \longrightarrow 0$,

(b) for any probability measure μ in $[\lambda]$, for all $\varepsilon > 0$, $\mu(\{x \in X \mid d(f_n(x), f(x)) > \varepsilon\}) \longrightarrow 0$,

(c) every subsequence of (f_n) admits a subsequence (f_{n_k}) converging λ-almost everywhere to f, i.e. for λ-almost all x in X we have $f_{n_k}(x) \to f(x)$.

Remark 3.3.11. From the previous statement, we can also define τ_m by the following metric. Let μ be a finite measure in $[\lambda]$. Any compatible bounded metric d on Z defines a metric $\tilde{d}$ on $\mathrm{L}^0(X, \lambda, (Z, \tau_Z))$:

$$\tilde{d}(f, g) := \int_X d(f(x), g(x))\, d\mu(x),$$

then, τ_m is also induced by the metric $\tilde{d}$.

We will need the following classical lemma, proven again for convenience.

Lemma 3.3.12. *Let Z be a Polish space, and (X, λ) be a standard σ-finite space. The space of measurable functions from X to Z that take finitely many values is dense in $\mathrm{L}^0(X, \lambda, Z)$.*

Proof. We start by fixing a compatible metric d on Z. As stated before, we can assume that d is bounded by 1. Let $\varepsilon > 0$ and fix μ a probability measure in $[\lambda]$. As Z is separable, we can write $Z = \bigcup_{n\in\mathbb{N}} B_n$, where $B_n = B(z_n, \varepsilon)$ is the open ball of center z_n and radius ε, and the sequence (z_n) is dense in Z. Let us now consider f an element of $\mathrm{L}^0(X, \lambda, Z)$. We can write $X = f^{-1}(Z) = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n)$, and therefore there exists N such that $\mu\left(\bigcup_{n\leqslant N} f^{-1}(B_n)\right) > 1 - \varepsilon$. Let us also define

$$A_n := f^{-1}(B_n) \setminus \bigcup_{i<n} f^{-1}(B_i)$$

for any $n \leqslant N$. Note that the A_n are disjoint. We can then define

$$\tilde{f}(x) := \begin{cases} z_n & \text{if } x \in A_n \\ z_0 & \text{otherwise.} \end{cases}$$

By construction, for any n smaller than N, for all x in A_n, we have $d(f(x), \tilde{f}(x)) < \varepsilon$. Finally, as

$$\mu\left(\bigsqcup_{n\leqslant N} A_n\right) > 1 - \varepsilon,$$

we have $\mu(\{x \in X \mid d(f(x), \tilde{f}(x)) > \varepsilon\}) < \varepsilon$, which concludes the proof. $\square$

Let us now consider the right action of $\mathrm{Aut}(X, \lambda)$ on $\mathrm{L}^0(X, \lambda, (Z, \tau_Z))$, defined as follows: if $T \in \mathrm{Aut}(X, \lambda)$ and $f \in \mathrm{L}^0(X, \lambda, (Z, \tau_Z))$, then

$$(f \cdot T)(x) := f(T(x))$$

We will often write $T(x) = Tx$ for convenience. Notice that when $Z = X$, we may see $\mathrm{Aut}(X, \lambda)$ as a subset of $\mathrm{L}^0(X, \lambda, (X, \tau_X))$, where we identify a measure-preserving bijection T with the corresponding function $f_T : x \mapsto Tx$.

We then have the following proposition, which will be crucial for the proof of the main result of Section 3.5. The local finiteness hypothesis is what distinguishes this statement from the analogous result in the finite measure context (see [CLM16, Prop. 2.9]). Indeed, when working with infinite measures, one must be careful of the link between topology and measure, and a convenient case happens to be that of a locally finite measure on a Polish space. An even nicer situation is that of a locally compact space equipped with a Radon measure, and by Theorem 3.2.11 we can often assume that it is the case.

Proposition 3.3.13. *Let (Z, τ_Z) be a Polish space and endow $\mathrm{Aut}(X, \lambda)$ with its weak topology τ_w. Consider also τ_X a Polish topology on X, compatible with its Borel structure.*

(1) The action of $\mathrm{Aut}(X,\lambda)$ *on* $\mathrm{L}^0(X,\lambda,(Z,\tau_Z))$ *is continuous. In particular, the inclusion of* $(\mathrm{Aut}(X,\lambda),\tau_w)$ *in* $(\mathrm{L}^0(X,\lambda,(X,\tau_X)),\tau_m)$ *is continuous.*

(2) If λ *is locally finite on* τ_X*, then the inclusion* $(\mathrm{Aut}(X,\lambda),\tau_w)\hookrightarrow(\mathrm{L}^0(X,\lambda,(X,\tau_X)),\tau_m)$ *is a topological embedding.*

(3) If λ *is locally finite on* τ_X*, then* $\mathrm{Aut}(X,\lambda)$ *is* G_δ *in* $(\mathrm{L}^0(X,\lambda,(X,\tau_X)),\tau_m)$*.*

Proof. Note that the second statement of *(1)* (which is actually the part we need) and *(2)* are proved in [Pap68, Thm. 1]. As this result is central in our work, we provide a complete proof. We start off by fixing a compatible metric d on (Z,τ_Z), bounded by 1.
(1). Let (f_n) be a sequence of elements of $\mathrm{L}^0(X,\lambda,(Z,\tau_Z))$ converging to f in measure. Let also (T_n) be a sequence of elements of $\mathrm{Aut}(X,\lambda)$ converging to T weakly. We fix a Borel subset $C\subseteq X$ of finite measure, and we need to establish that $\tilde d_C(f\cdot T,f_n\cdot T_n)\to 0$. We have

$$\begin{aligned}\tilde d_C(f\cdot T,f_n\cdot T_n)&=\int_C d(f(Tx),f_n(T_nx))\,d\lambda(x)\\&=\int_{T(C)} d(f(x),f_n(T_nT^{-1}x))\,d\lambda(x)\\&=\tilde d_{T(C)}(f,f_n\cdot T_nT^{-1}).\end{aligned}$$

Now we fix $\varepsilon>0$, and define $\tilde f$ as an element of $\mathrm{L}^0(X,\lambda,(Z,\tau_Z))$ that takes $k\in\mathbb{N}^*$ different values on $T(C)$, such that $\tilde d_{T(C)}(\tilde f,f)<\varepsilon$, and such that $\tilde f_{\restriction X\setminus T(C)}=f_{\restriction X\setminus T(C)}$. Lemma 3.3.12 ensures the existence of such a function. We have

$$\tilde d_{T(C)}(f,f_n\cdot T_nT^{-1})\leqslant\tilde d_{T(C)}(f,\tilde f)+\tilde d_{T(C)}(\tilde f,\tilde f\cdot T_nT^{-1})+\tilde d_{T(C)}(\tilde f\cdot T_nT^{-1},f_n\cdot T_nT^{-1}).$$

By construction the first term is less than ε.

For the second term, we first define a finite family of sets B_i, $i\in\{1,\dots,k\}$, such that each B_i is the preimage in $T(C)$ of the i-th value taken by $\tilde f$. In particular each B_i has finite measure. We have

$$T(C)=\bigsqcup_{i=1}^k B_i\coloneqq\bigsqcup_{i=1}^k(\tilde f^{-1}(\{b_i\})\cap T(C))$$

where $(b_i)_{i=1}^k$ are the values taken by $\tilde f$ on $T(C)$. The sequence (T_n) converges weakly to T, so TT_n^{-1} converges weakly to id_X, and therefore one has $\lambda(B_i\setminus TT_n^{-1}(B_i))\to 0$ for all $i\leqslant k$, as $\lambda(B_i)=\lambda(TT_n^{-1}(B_i))$. We have

$$\begin{aligned}\{x\in T(C)\mid(\tilde f\cdot T_nT^{-1})(x)\neq\tilde f(x)\}&=\bigsqcup_{i=1}^k\{x\in T(C)\mid\tilde f(x)=b_i\text{ and }(\tilde f\cdot T_nT^{-1})(x)\neq b_i\}\\&=\bigsqcup_{i=1}^k B_i\setminus TT_n^{-1}(B_i)\end{aligned}$$

and so $\lambda(\{x\in T(C)\mid(\tilde f\cdot T_nT^{-1})(x)\neq\tilde f(x)\})\to 0$, as the measure of each $B_i\setminus TT_n^{-1}(B_i)$ converges to 0. This ensures us that the second term can be as small as we need.

Finally, let us notice that the third term is equal to

$$\begin{aligned}\tilde d_{T(C)}(\tilde f\cdot T_nT^{-1},f_n\cdot T_nT^{-1})&=\int_{T(C)}(\tilde f(T_nT^{-1}x),f_n(T_nT^{-1}x))\,d\lambda(x)\\&=\int_{T_n(C)}(\tilde f(x),f_n(x))\,d\lambda(x)\\&=\tilde d_{T_n(C)}(\tilde f,f_n).\end{aligned}$$

Then, for any measurable function $g : X \to [0,1]$ we have

$$\begin{aligned}\left|\int_{T_n(C)} g(x)d\lambda(x) - \int_{T(C)} g(x)d\lambda(x)\right| &= \left|\int_{T_n(C)\cap T(C)} g(x)d\lambda(x) + \int_{T_n(C)\setminus T(C)} g(x)d\lambda(x)\right.\\ &\quad \left. - \left(\int_{T(C)\cap T_n(C)} g(x)d\lambda(x) + \int_{T(C)\setminus T_n(C)} g(x)d\lambda(x)\right)\right|\\ &= \left|\int_{T_n(C)\setminus T(C)} g(x)d\lambda(x) - \int_{T(C)\setminus T_n(C)} g(x)d\lambda(x)\right|\\ &\leqslant \int_{(T_n(C)\setminus T(C))\sqcup(T(C)\setminus T_n(C))} g(x)d\lambda(x)\\ &\leqslant \lambda(T_n(C)\Delta T(C)).\end{aligned}$$

In particular, for $g : x \mapsto d(\tilde{f}(x), f_n(x))$ we obtain that

$$\left|\tilde{d}_{T_n(C)}(\tilde{f}, f_n) - \tilde{d}_{T(C)}(\tilde{f}, f_n)\right| \leqslant \lambda(T_n(C)\Delta T(C)).$$

Since $\lambda(T_n(C)\Delta T(C)) \to 0$, in order to show that the third term tends to zero, it suffices to show that $\tilde{d}_{T(C)}(f_n, \tilde{f}) \to 0$. This is easily done, as $\tilde{d}_{T(C)}(f_n, \tilde{f}) \leqslant \tilde{d}_{T(C)}(f_n, f) + \tilde{d}_{T(C)}(f, \tilde{f})$, and for reasons already stated, both of those terms are at most ε, for n large enough.

To prove that the inclusion $\mathrm{Aut}(X,\lambda) \hookrightarrow \mathrm{L}^0(X,\lambda,(X,\tau_X))$ is continuous, one must simply notice that $\mathrm{Aut}(X,\lambda)$ can be seen as the orbit of id_X in $\mathrm{L}^0(X,\lambda,(X,\tau_X))$: for all $T \in \mathrm{Aut}(X,\lambda)$, we have $\mathrm{id}_X \cdot T = T$.

(2). We just proved in *(1)* that the inclusion of $\mathrm{Aut}(X,\lambda)$ in $\mathrm{L}^0(X,\lambda,(X,\tau_X))$ is continuous. Therefore, the weak topology refines the topology of convergence in measure on $\mathrm{Aut}(X,\lambda)$. To prove that the inclusion is an embedding, let us show that the topology of convergence in measure also refines the weak topology.

Using *(1)* once again, we see that $\mathrm{Aut}(X,\lambda)$ acts on the right by homeomorphisms on $\mathrm{L}^0(X,\lambda,(X,\tau_X))$, and thus for any sequence (T_n) of elements of $\mathrm{Aut}(X,\lambda)$, $T_n \to T$ in measure if and only if $T_nT^{-1} \to \mathrm{id}_X$ in measure. Furthermore, as λ is σ-finite, it is also inner regular, and thus a sequence (T_n) of elements of $\mathrm{Aut}(X,\lambda)$ converges weakly to T if and only if $\lambda(T_n(K)\Delta T(K)) \to 0$ for all Borel compact sets K of finite measure. Our aim is then to show that if a sequence (T_n) converges to id_X in measure, then for all Borel compacts K of finite measure, we have weak convergence of (T_n) to id_X, *i.e.* $\lambda(T_n(K)\Delta K) \to 0$.

Let us fix $\varepsilon > 0$ and a Borel compact set K of finite measure. Using the fact that λ is locally finite, we consider for each point of K an open neighbourhood of that point, which has finite measure. As K is compact, we extract from this cover of K a finite subcover $(\mathcal{N}_i)_{i\in I}$ of K consisting of finite measure subsets. We define

$$K_r \coloneqq \{x \in X \mid \exists y \in K, d(x,y) < r\},$$

where $r > 0$ and d is a compatible metric on X. As K is closed, we have $K = \bigcap_{n\in\mathbb{N}^*} K_{\frac{1}{n}}$, and as $d\left(X \setminus \bigcup_{i\in I}\mathcal{N}_i, K\right) > 0$, by compactness, there exists m in $\mathbb{N}^*$, such that $K_{\frac{1}{m}} \subseteq \bigcup_{i\in I}\mathcal{N}_i$. Thus, $\lambda\left(K_{\frac{1}{m}}\right) \leqslant \lambda\left(\bigcup_{i\in I}\mathcal{N}_i\right) < +\infty$, as I is finite, and the measure of each $\mathcal{N}_i$ is also finite. Therefore, $\lambda(K) = \inf_{n\geqslant 1}\lambda\left(K_{\frac{1}{n}}\right)$, and so there exists a $\delta < \varepsilon$ such that $\lambda(K_\delta) < \lambda(K) + \varepsilon$.

The characterization (b) of Proposition 3.3.10 ensures us that for n large enough,

$$\lambda(\{x \in K \mid d(T_n(x), x) \geqslant \delta\}) < \varepsilon,$$

which, as $\{x \in K \mid T_n(x) \notin K_\delta\} \subseteq \{x \in K \mid d(T_n(x), x) \geqslant \delta\}$, implies that

$$\lambda(\{x \in K \mid T_n(x) \notin K_\delta\}) < \varepsilon.$$

This means that $\lambda(K \setminus T_n^{-1}(K_\delta)) < \varepsilon$, and thus that $\lambda(T_n(K) \setminus K_\delta) < \varepsilon$, as T_n is a measure-preserving bijection. Since $\lambda(K) = \lambda(T_n(K))$, we then have the following:

$$\begin{aligned}\lambda(T_n(K)\Delta K) &= 2\lambda(T_n(K) \setminus K) \\ &\leqslant 2\lambda(T_n(K) \setminus K_\delta) + 2\lambda(K_\delta \setminus K) \\ &\leqslant 2\varepsilon + 2\varepsilon\end{aligned}$$

which proves that (T_n) converges to T weakly. Therefore the topology of convergence in measure and the weak topology both refine each other, and thus $\mathrm{Aut}(X, \lambda)$ equipped with the weak topology embeds into $\mathrm{L}^0(X, \lambda, (X, \tau_X))$ equipped with the topology of convergence in measure.

(3). The proof is immediate by combining *(2)*, Proposition 1.1.4, and Proposition 3.3.6. □

The hypothesis of local finiteness in Proposition 3.3.13 is used to find a measure-controllable set containing the compact subset we approximate, and cannot be dispensed with. We give examples in Section 3.5.2.

We will finally need the following.

Proposition 3.3.14 ([Moo76, Prop. 7], see also [Kec10, Prop. 19.6])**.** *Let G be a Polish group. Then the space $\mathrm{L}^0(X, \lambda, G)$ is a Polish group for the topology of convergence in measure and the pointwise product.*

3.3.3 Uniform topology on $\mathrm{Aut}(X, \lambda)$

In this section we define the uniform topology τ_u on the group $\mathrm{Aut}(X, \lambda)$. In the probability context, Halmos defined in [Hal56] two metrics on the group of measure-preserving automorphisms and showed that they are comparable, and that they both induce τ_u. The generalization of this topology to the group of non-singular bijections was introduced by [Ion65], and studied in the infinite measure context for instance in [CK79]. The aim of this section is to get some adaptability by presenting different equivalent definitions.

Definition 3.3.15. We define the following topologies on $\mathrm{Aut}(X, \lambda)$:

(1) Let μ be a probability measure in $[\lambda]$, and define the metric d_μ by

$$d_\mu : S, T \mapsto \mu\left(\{x \in X \mid S(x) \neq T(x)\}\right).$$

The topology τ_1 is the topology induced by d_μ.

(2) Let μ be a probability measure in $[\lambda]$, and define the metric δ_μ by

$$\delta_\mu : S, T \mapsto \sup\left\{\mu\left(S(B)\Delta T(B)\right) \mid B \subseteq X\right\}.$$

The topology τ_2 is the topology induced by δ_μ.

(3) We define on $\mathrm{Aut}(X, \lambda)$ the topology τ_3 induced by the family of pseudometrics

$$d'_{u,C} : S, T \mapsto \lambda\left(\{x \in C \mid S(x) \neq T(x)\}\right),$$

where C ranges among Borel subset of X of finite measure.

(4) The **uniform topology** τ_u on $\mathrm{Aut}(X,\lambda)$ is the topology induced by the family of pseudometrics

$$d_{u,C} : S,T \mapsto \sup\{\lambda(C\cap(S(B)\Delta T(B))) \mid B\subseteq X \text{ Borel subset}\},$$

where C ranges among Borel subset of X of finite measure.

Remark 3.3.16. Do notice that τ_1 and τ_2 do not depend on the choice of μ in the class of λ, by Lemma 3.2.5.

Before comparing the previously defined metrics, we show that both τ_1 and τ_2 make the group $\mathrm{Aut}(X,\lambda)$ a topological group.

Lemma 3.3.17. *Both* $(\mathrm{Aut}(X,\lambda),\tau_1)$ *and* $(\mathrm{Aut}(X,\lambda),\tau_2)$ *are topological groups.*

Proof. In this whole proof, μ is a fixed probability measure in $[\lambda]$.

We start with τ_1. First notice that d_μ is left-invariant (but not right-invariant!), that is to say that $d_\mu(T_1,T_2)=d_\mu(ST_1,ST_2)$ for T_1, T_2 and S in $\mathrm{Aut}(X,\lambda)$. For the continuity of multiplication, we consider $S_n\to S$ and $T_n\to T$ for τ_1, and we have

$$\begin{aligned} d_\mu(S_nT_n,ST) &\leqslant d_\mu(S_nT_n,S_nT)+d_\mu(S_nT,ST)\\ &\leqslant d_\mu(T_n,T)+d_\mu(S_nT,ST) \end{aligned}$$

by left-invariance. If we define $A_n \coloneqq \{x\in X \mid S_n(x)\neq S(x)\}$, we notice that

$$d_\mu(S_nT,ST)=\mu\left(\{x\in T(X)\mid S_n(x)\neq S(x)\}\right)=T_*\mu(A_n)\longrightarrow 0,$$

as $\mu(A_n)\to 0$ and $T_*\mu$ is a probability measure in $[\mu]$ (by Remark 3.3.16). The first term of the previous inequality also tends to 0 because $T_n\to T$, so group multiplication is τ_1-continuous.

Let us now prove continuity of the inverse, and to that end we consider $T_n\to T$ for τ_1. We have $(T(x)\neq T_n(x)) \iff (T_n^{-1}T(x)\neq x) \iff (T_n^{-1}T(x)\neq T^{-1}T(x))$. Therefore, we have

$$\{x\in X\mid T(x)\neq T_n(x)\}=T\left(\{x\in X\mid T_n^{-1}(x)\neq T^{-1}(x)\}\right),$$

which concludes the proof, as $(T^{-1})_*\mu$ is equivalent to μ.

The proof for τ_2 is similar, using this time that δ_μ is right-invariant (and not left-invariant). Let $S_n\to S$ and $T_n\to T$ be τ_2-convergent sequences. We have

$$\begin{aligned} \delta_\mu(S_nT_n,ST) &\leqslant \delta_\mu(S_nT_n,ST_n)+\delta_\mu(ST_n,ST)\\ &\leqslant \delta_\mu(S_n,S)+\delta_\mu(ST_n,ST) \end{aligned}$$

by right-invariance. We observe that for any Borel subset A of X we have $\mu(ST_n(A)\Delta ST(A)) = \mu(S(T_n(A)\Delta T(A)))$, so $\delta_\mu(ST_n,ST)\to 0$, because $\mu(T_n(A)\Delta T(A))\to 0$ for any Borel subset A, and $(S^{-1})_*\mu$ is equivalent to μ. The first term in the inequality tends to 0 because $S_n\to S$, so we have proved that the group multiplication is continuous.

To prove that the inverse is continuous, take $T_n\to T$, and consider a Borel subset A of X. We have

$$\mu(T^{-1}(T(A)\Delta T_n(A)))=\mu(A\Delta T^{-1}T_n(A))=\mu(T_n^{-1}(A)\Delta T^{-1}(A))$$

by right-invariance again, which concludes the proof, as $T_*\mu$ is equivalent to μ. □

Proposition 3.3.18. *The topologies* τ_1*,* τ_2*, and* τ_3 *are all equal to the uniform topology* τ_u *on* $\mathrm{Aut}(X,\lambda)$*.*

Proof. Lemma 3.2.6 ensures that τ_1 is equal to τ_3.

We now compare τ_1 and τ_2. Fix a probability measure $\mu \in [\lambda]$, and notice that, thanks to Lemma 3.3.17, it is enough to establish metric inequalities when evaluated between a non-trivial element T and the identity element of $\mathrm{Aut}(X,\lambda)$. Fix such a T in $\mathrm{Aut}(X,\lambda)$ and consider $c = d_\mu(T, \mathrm{id}_X) = \mu(\operatorname{supp} T)$. Lemma 3.3.4 ensures the existence of a Borel subset A of $\operatorname{supp} T$ such that $\operatorname{supp} T = A \sqcup \big(T(A) \cup T^{-1}(A)\big)$. Note that up to replacing A with $T^{-1}(A)$ if needed, we necessarily have $\mu(T(A)\Delta A) \geqslant \frac{1}{3}c$. We have

$$\delta_\mu(T, \mathrm{id}_X) \geqslant \mu(T(A)\Delta A) \geqslant \frac{1}{3}c = \frac{1}{3}d_\mu(T, \mathrm{id}_X).$$

For the converse inequality, we notice that for any Borel subset A of X, we have $T(A)\Delta A \subseteq \operatorname{supp} T$, which ensures us that $\delta_\mu(T, \mathrm{id}_X) \leqslant d_\mu(T, \mathrm{id}_X)$. We have just proved that $\tau_1 = \tau_2$.

Let us finally show that this topology is equal to τ_u. We fix S and T in $\mathrm{Aut}(X,\lambda)$. Similarly to what we did in the proof of Lemma 3.2.6, we write $X = \bigsqcup_{n\in\mathbb{N}} X_n$, with $\lambda(X_n) = 1$ for all n in $\mathbb{N}$, and define a probability measure μ in $[\lambda]$ by setting

$$\mu(A) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \lambda(A \cap X_k)$$

for any Borel subset $A \subseteq X$. Evaluating μ on subsets of the form $S(A)\Delta T(A)$, we obtain the following

$$\delta_\mu(S,T) \leqslant \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} d_{u,X_k}(S,T),$$

meaning that τ_u refines τ_2. Finally, we obtain that τ_3 refines τ_u by applying the same argument as before: for any finite measure Borel subset C and any Borel subset A we have $C \cap (S(A)\Delta T(A)) \subseteq \{x \in C \mid S(x) \neq T(x)\}$, so $d_{u,C}(S,T) \leqslant d'_{u,C}(S,T)$, which concludes the proof. $\square$

Remark 3.3.19. From the definition of $d'_{u,C}$, if we set

$$\mathcal{N}_{C,\varepsilon} = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(C \cap \operatorname{supp} T) \leqslant \varepsilon\},$$

then the family $\{\mathcal{N}_{C,\varepsilon} \mid C \text{ Borel subset of } X \text{ of finite measure}, \varepsilon > 0\}$ is a base of neighbourhoods of the identity in $(\mathrm{Aut}(X,\lambda), \tau_u)$. One can also consult [Fre02, Prop. 494Cb].

Remark 3.3.20. Although stated for infinite measure-preserving bijections, most of this section actually holds in the broader context of non-singular bijections.

3.4 Full groups and exchanging involutions

In this section we present full groups and pseudo full groups. We give the historic definition of Dye from his work in [Dye59] (in a more modern version, as commonly found in the literature), as some results hold true even without any group acting on (X,λ). We will give the more specific definition of orbit full groups in Section 3.5. In this whole section, (X,λ) will be a standard σ-finite space as usual.

Definition 3.4.1. Consider (T_n) a sequence of elements of $\mathrm{Aut}(X,\lambda)$. An element T in $\mathrm{Aut}(X,\lambda)$ is obtained by **cutting and pasting** (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have

$$T_{\restriction A_n} = T_{n\restriction A_n}.$$

Definition 3.4.2. A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\lambda)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation 3.4.3. For any family $(T_i)_{i\in I}$ with $T_i \in \mathrm{Aut}(X,\lambda)\ \forall\, i\in I$, we denote by $[(T_i)_{i\in I}]$ the full group generated by $(T_i)_{i\in I}$, that is to say the smallest full group that contains the family. In particular, any $S\in[(T_i)_{i\in I}]$ satisfies $\operatorname{supp} S \subseteq \bigcup_{i\in I}\operatorname{supp} T_i$, up to a null set.

Definition 3.4.4. We say that a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\lambda)$ is **ergodic** if for every $A\subseteq X$ such that $\lambda(T(A)\Delta A)=0$ for every T in $\mathbb{G}$, we have that A is either null or conull.

Ergodicity of a group $\mathbb{G}$ immediately yields ergodicity of any group containing $\mathbb{G}$, but it does not go down to subgroups in general. We do however have the following well known proposition (see *e.g.* [Aar97, Prop. 1.6.7]), about the behaviour of ergodicity with regard to weak-density. The proof is immediate from Lemma 3.3.8.

Proposition 3.4.5. *Let $\mathbb{G}\leqslant \mathrm{Aut}(X,\lambda)$ be an ergodic subgroup, and let Γ be a τ_w-dense subgroup of $\mathbb{G}$. Then Γ is ergodic.*

Example 3.4.6. For any ergodic full group $\mathbb{G}\leqslant \mathrm{Aut}(X,\lambda)$, $\mathbb{G}_f$ is τ_w-dense in $\mathbb{G}$ (see Remark 3.6.26), so $\mathbb{G}$ is ergodic if and only if $\mathbb{G}_f$ is ergodic.

For the uniform topology on full groups, we have the following. Recall that for any probability measure $\mu\in[\lambda]$, $d_\mu(S,T)=\mu(\{x\in X\mid S(x)\neq T(x)\})$. The result is classical even in the broader setting of non-singular full groups, and attributed to Hamachi and Osikawa [HO81, Sec. II.5.1].

Proposition 3.4.7. *Any full group $\mathbb{G}\leqslant \mathrm{Aut}(X,\lambda)$ is complete when equipped with the bi-uniform metric $\widetilde{d}: S,T\mapsto d_\mu(S,T)+d_\mu(S^{-1},T^{-1})$, for any probability measure μ in $[\lambda]$. In particular, $\mathbb{G}$ is closed in $(\mathrm{Aut}(X,\lambda),\tau_u)$.*

Proof. Let $\mathbb{G}$ be a full group and (T_n) be a $\widetilde{d}$-Cauchy sequence in $\mathbb{G}$. By passing to a subsequence if necessary, we can assume that $\widetilde{d}(T_n,T_{n+1})<2^{-n}$. We define $A_n=\{x\in X\mid T_n(x)\neq T_{n+1}(x)\}$, and we have $\mu(A_n)=d_\mu(T_n,T_{n+1})\leqslant \widetilde{d}(T_n,T_{n+1})<2^{-n}$. As $\sum_n \mu(A_n)$ is finite, the Borel-Cantelli lemma states that almost all $x\in X$ are in finitely many A_n. Therefore, if we define (for $N\in\mathbb{N}^*$)

$$B_N=\{x\in X\mid (T_N(x)\neq T_{N-1}(x))\text{ and }(T_N(x)=T_n(x),\forall\, n\geqslant N)\}\,,$$

then the B_N are disjoint, as they sort almost every x in X by the rank of the last A_n they are in. Thus $(B_N)_{N\in\mathbb{N}^*}$ is a partition of X, up to a null set. Moreover, as each T_n is a bijection we have

$$\begin{aligned} T_N(B_N)&=\left\{x\in X\mid (x\neq T_{N-1}(T_N^{-1}(x)))\text{ and }(x=T_n(T_N^{-1}(x)),\forall\, n\geqslant N)\right\}\\ &=\left\{x\in X\mid (T_{N-1}^{-1}(x)\neq T_N^{-1}(x))\text{ and }(T_n^{-1}(x)=T_N^{-1}(x),\forall\, n\geqslant N)\right\}.\end{aligned}$$

Recall now that $d_\mu(T_n^{-1},T_{n+1}^{-1})\leqslant\widetilde{d}(T_n,T_{n+1})<2^{-n}$, which means that $(T_N(B_N))_{N\in\mathbb{N}^*}$ is also a partition of X up to a null set, by the same arguments. Thus, if we define T by $T(x)=T_N(x)$ for $x\in B_N$ (for any $N\in\mathbb{N}^*$), T is invertible almost everywhere. T is in $\mathbb{G}$ by definition, and $(B_N)_{N\in\mathbb{N}^*}$ is a partition of X, so

$$\mu\left(\bigcup_{N>k}B_N\right)\xrightarrow[k\to 0]{}0,$$

and therefore $d_\mu(T_N,T)\to 0$. By continuity of the inverse $T\mapsto T^{-1}$ (see Lemma 3.3.17), $d_\mu(T_N^{-1},T^{-1})\to 0$, so $(T_N)_{N\in\mathbb{N}^*}$ converges to T for $\widetilde{d}$. By Lemma 3.3.17 again we can conclude that $\mathbb{G}$ is closed in $(\mathrm{Aut}(X,\lambda),\tau_u)$, as the topology induced by $\widetilde{d}$ refines τ_u, and is in fact equivalent to it. ∎

Remark 3.4.8. If one were to use d_μ instead of the symmetrized metric $\widetilde{d}$, the default of completeness would appear even at the level of the group $\mathrm{Aut}(X,[\mu])$ of non-singular bijections. Indeed, consider the following example. We define (X,μ) to be $\mathbb{R}_+$ endowed with an atomless probability measure (one can think of the positive half of a centered Gaussian measure, with the appropriate normalization). We define the sequence $(T_n)_{n\geqslant 1}$ of elements of $\mathrm{Aut}(X,[\mu])$ by setting

$$T_n(x) = \begin{cases} x+1 \text{ on } [0,n[, \\ x-n \text{ on } [n,n+1[, \\ x \text{ on } [n+1,+\infty[, \end{cases}$$

for any $n \geqslant 1$ and any $x \in X$. In other words, T_n shifts the beginning of the positive real line by one, until n, and the last unit interval $[n,n+1[$ is moved back to the beginning. Figure 3.1 illustrates the action of T_3 and T_4 on X.

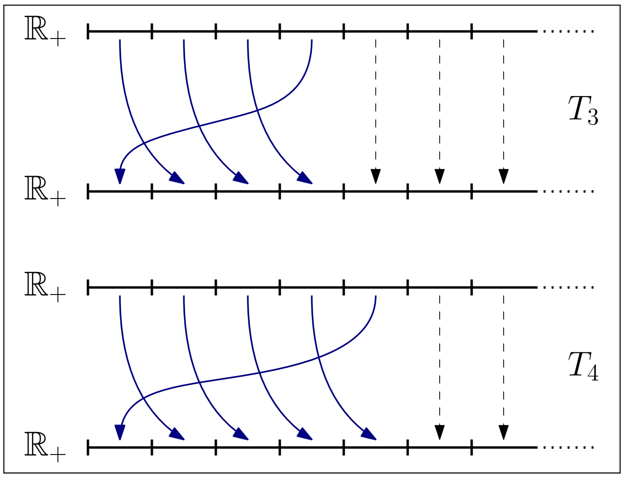

Figure 3.1: Action of T_3 and T_4 on the positive real numbers.

We have that $\mu([n,n+1[) \to 0$ as n goes to infinity, so $d_\mu(T_n,T_{n+1}) \to 0$, but $d_\mu(T_n^{-1},T_{n+1}^{-1}) = \mu([0,1[)$. Accordingly, the limit (for d_μ) of (T_n) is the map $x \mapsto x+1$, which is not a bijection of X.

We now prove that there exists a *partial measure-preserving isomorphism* sending any fixed Borel subset of positive measure to another fixed Borel subset of equal measure. The probability version of this result has many applications, see for instance [CLM16], [Fat78] or [Kec10]. We first give the definitions of partial isomorphisms and pseudo-full groups, in order to give a general version of the main proposition of this section.

Definition 3.4.9. Let $\mathbb{G}$ be a subgroup of $\mathrm{Aut}(X,\lambda)$.

(1) Let A and B be two Borel subsets of X. Saying that $\phi : A \to B$ is a **partial isomorphism** of $\mathbb{G}$ means that there exists a partition $(A_n)_{n\in\mathbb{N}}$ of A, a partition $(B_n)_{n\in\mathbb{N}}$ of B, and a sequence $(T_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{G}$ such that $T_n(A_n) = B_n$ and $T_{n\restriction A_n} = \phi_{\restriction A_n}$, for every n in $\mathbb{N}$. The domain of ϕ is $\mathrm{dom}(\phi) = A$ and its range is $\mathrm{rng}(\phi) = B$, up to null sets. In particular $\lambda(A) = \lambda(B)$.

(2) The set of all partial isomorphisms of $\mathbb{G}$ is called the **pseudo-full group** of $\mathbb{G}$, and is denoted by $[[\mathbb{G}]]$.

(3) The **uniform topology** defined on $\mathbb{G}$ generalizes to $[[\mathbb{G}]]$: it is the topology generated

by the following metric:

$$d(\phi_1,\phi_2) := \mu(\{x \in \mathrm{dom}(\phi_1)\cap\mathrm{dom}(\phi_2) \mid \phi_1(x)\neq\phi_2(x)\}) \\ +\mu(\mathrm{dom}(\phi_1)\Delta\mathrm{dom}(\phi_2)),$$

where $\mu \in [\lambda]$ is a probability measure. We refer to [Dan95] for more on this topology.

Remark 3.4.10. It is tempting to see the partial isomorphisms in $[[\mathbb{G}]]$ as restrictions of elements of $\mathrm{Aut}(X,\lambda)$ (and it is indeed the case in the probability context, see [LM14a, Cor. 1.15]), but it is not the case here. For example, consider $X=\mathbb{R}$ acting on itself by translation. The partial isomorphism defined by

$$\begin{array}{rccc} \varphi & : & \bigsqcup_{n\in\mathbb{Z}} [n,n+1[& \longrightarrow & \bigsqcup_{n\in\mathbb{Z}} [2n,2n+1[\\ & & x\in[n,n+1[& \longmapsto & x+n\in[2n,2n+1[\end{array}$$

has $\mathbb{R}$ as its domain, and $\bigsqcup_{n\in\mathbb{Z}}[2n,2n+1[$ as its range, which is not conull.

The following is a crucial tool for full groups, and is well-known, but we have found no reference for the σ-finite version of the statement. As such, we provide a complete and detailed proof. Do note that there are no conditions on the measure of the complements of A and B in X, as highlighted by Remark 3.4.10.

Proposition 3.4.11. *Let $\mathbb{G}$ be an ergodic subgroup of $\mathrm{Aut}(X,\lambda)$. Consider two Borel subsets A and B of X such that $\lambda(A)=\lambda(B)$. Then we can find a partial automorphism $\phi_{A,B}\in[[\mathbb{G}]]$ satisfying:*

1. $\mathrm{dom}(\phi_{A,B})=A$ *and* $\mathrm{rng}(\phi_{A,B})=B$ *(up to null sets),*
2. *The restriction of Φ to $\left\{(A,B)\in(\mathrm{MAlg}_f(X,\lambda))^2 \mid \lambda(A)=\lambda(B)\right\}$ is continuous for the uniform topology.*

Proof. The proof is roughly the same as the proof of Lemma 7.10 of [KM04], which is stated in the case of a probability measure and a countable acting group, but it easily adapts to our case by considering a countable dense subgroup of $\mathbb{G}$.

Since $(\mathbb{G},\tau_w)$ is a subgroup of the separable and metrizable (thanks to Proposition 3.3.6) group $(\mathrm{Aut}(X,\lambda),\tau_w)$, it is itself separable. As such, we can consider (T_n) a countable dense subset of $\mathbb{G}$ and $\Gamma=(\gamma_n)$ the countable subgroup generated by (T_n).

Let A and B first be two Borel subsets of X, such that $0<\lambda(A)=\lambda(B)<+\infty$. We recursively define a countable family of pairwise disjoint subsets A_n of A as follows:

$$\begin{cases} \widetilde{A}_0=(\gamma_0^{-1}B)\cap A \\ \widetilde{A}_{n+1}=\left(\gamma_{n+1}^{-1}\left(B\setminus\bigsqcup_{m\leqslant n}\gamma_m\widetilde{A}_m\right)\right)\bigcap\left(A\setminus\bigsqcup_{m\leqslant n}\widetilde{A}_m\right). \end{cases}$$

The set $\widetilde{A}_n$ represents the elements of A sent by γ_n to B, after removing the elements previously sent. Now set $\widetilde{A}=\bigsqcup_{n\in\mathbb{N}}\widetilde{A}_n$ and let $\phi:\widetilde{A}\to\bigsqcup_{n\in\mathbb{N}}\gamma_n\widetilde{A}_n$ be the Borel application that sends $x\in\widetilde{A}_n$ to $\gamma_n(x)\in\gamma_n\widetilde{A}_n$. By definition, ϕ is a partial isomorphism between $\widetilde{A}$ and $\bigsqcup_{n\in\mathbb{N}}\gamma_n\widetilde{A}_n$. In particular, $\lambda(\mathrm{dom}(\phi))=\lambda(\mathrm{rng}(\phi))$.

Let us now suppose that either $\lambda(A\setminus\mathrm{dom}(\phi))>0$ or $\lambda(B\setminus\mathrm{rng}(\phi))>0$. As A and B have finite measure, we have $\lambda(A\setminus\mathrm{dom}(\phi))=\lambda(B\setminus\mathrm{rng}(\phi))>0$. Define $\overline{B}=\bigcup_{n\in\mathbb{N}}\gamma_n(A\setminus\mathrm{dom}(\phi))$, and notice that $\overline{B}$ is non null and invariant under the action of Γ. Ergodicity and the fact that Γ is dense in $\mathbb{G}$ ensure that $\overline{B}$ is conull, by Proposition 3.4.5. This coupled with the fact that $\lambda(B\setminus\mathrm{rng}(\phi))>0$ implies that there exists an integer n such that

$$\lambda\left((B\setminus\mathrm{rng}(\phi))\bigcap\gamma_n(A\setminus\mathrm{dom}(\phi))\right)>0.$$

We define n_0 as the smallest such integer. As γ_{n_0} is measure-preserving, we then have

$$\lambda\left(\gamma_{n_0}^{-1}(B \setminus \operatorname{rng}(\phi)) \bigcap (A \setminus \operatorname{dom}(\phi))\right) > 0.$$

Notice now that $(B \setminus \operatorname{rng}(\phi)) \subseteq (B \setminus \bigsqcup_{m<n_0} \gamma_m \widetilde{A}_m)$ and $(A \setminus \operatorname{dom}(\phi)) \subseteq (A \setminus \bigsqcup_{m<n_0} \widetilde{A}_m)$, which means that $\left(\gamma_{n_0}^{-1}(B \setminus \operatorname{rng}(\phi)) \bigcap (A \setminus \operatorname{dom}(\phi))\right)$ is contained in $\widetilde{A}_{n_0}$ by construction, and thus it is contained in $\operatorname{dom}(\phi)$. This is the contradiction we sought, as this set has positive measure and is contained both in $\operatorname{dom}(\phi)$ and in $A \setminus \operatorname{dom}(\phi)$.

Now if $\lambda(A) = \lambda(B) = +\infty$, observe that $(A, \lambda_{\restriction A})$ and $(B, \lambda_{\restriction B})$ are both standard σ-finite spaces. It is then possible to write $A = \bigsqcup A'_i$ and $B = \bigsqcup B'_i$, with $\lambda(A'_i) = \lambda(B'_i) < +\infty$ for every i in $\mathbb{N}$. The previous argument gives us a sequence (ϕ_i) of partial isomorphisms with domains (A'_i) and ranges (B'_i). The partial isomorphism defined on A by $\phi_{\restriction A'_i} = \phi_i$ is in $[[\mathbb{G}]]$ by construction, and is suitable so we take it to be $\phi_{A,B}$.

For the last part of the statement, we notice that by construction, if (A_n) and (B_n) are as in the statement, then

$$\lambda\left(\{x \in (A_n \cap A) \cup (B_n \cap B) \mid \phi_{A_n,B_n}(x) \neq \phi_{A,B}(x)\}\right) \longrightarrow 0,$$

which concludes the proof. □

The following two corollaries are very useful and used extensively throughout the rest of this work. Recall that for any Borel subset $C \subseteq X$, $\mathbb{G}_C = \{T \in \mathbb{G} \mid \operatorname{supp} T \subseteq C\}$.

Corollary 3.4.12. *Let $\mathbb{G}$ be an ergodic full group and let $C \subseteq X$ be any Borel subset. Let A and B be two Borel subsets of C such that $\lambda(A) = \lambda(B)$ and $\lambda(C \setminus A) = \lambda(C \setminus B)$. Then there exists an element T of $\mathbb{G}_C$ such that $T(A) = B$ and $T(C \setminus A) = C \setminus B$.*

Proof. Proposition 3.4.11 yields two partial isomorphisms ϕ_1 and ϕ_2 such that $\operatorname{dom}(\phi_1) = A$, $\operatorname{rng}(\phi_1) = B$, $\operatorname{dom}(\phi_2) = C \setminus A$, and $\operatorname{rng}(\phi_2) = C \setminus B$. We define

$$T := \begin{cases} \phi_1 & \text{on } A \\ \phi_2 & \text{on } C \setminus A \\ \mathrm{id}_X & \text{on } X \setminus C. \end{cases}$$

The bijection T is in $\mathbb{G}$ by virtue of it being a full group, and because gluing the partitions of A and $C \setminus A$ (resp. B and $X \setminus B$) with the complement of C yields a partition of $\operatorname{dom}(T) = X$ (resp. $\operatorname{rng}(T) = X$). □

Corollary 3.4.13. *Let $\mathbb{G}$ be an ergodic full group and let A and B be two Borel subsets of X such that $\lambda(A \setminus B) = \lambda(B \setminus A)$. Then there exists an involution U in $\mathbb{G}_{A\Delta B}$ such that $U(A) = B$. Moreover, $(A, B) \in (\mathrm{MAlg}_f(X, \lambda))^2 \mapsto U$ is τ_u-continuous.*

Proof. Proposition 3.4.11 yields a partial isomorphism ϕ such that $\operatorname{dom}(\phi) = A \setminus B$ and $\operatorname{rng}(\phi) = B \setminus A$. We define

$$U := \begin{cases} \phi & \text{on } A \setminus B \\ \phi^{-1} & \text{on } B \setminus A \\ \mathrm{id}_X & \text{on } X \setminus (A\Delta B), \end{cases}$$

which is in $\mathbb{G}$ and is suitable. Continuity follows from the continuity of Φ in Proposition 3.4.11. □

Remark 3.4.14. There is a clear distinction here between the probability measure-preserving case and our context. Indeed, if (X, μ) is a standard probability space and $\mu(A) = \mu(B)$, then we automatically have $\mu(A \setminus B) = \mu(B \setminus A)$. Thus Corollary 3.4.13 can be seen as a strengthening of Corollary 3.4.12, as we directly get that A can be sent to B by an involution of $\mathbb{G}$.

If (X, λ) is a standard σ-finite space however, one has to be more cautious when manipulating Borel subsets, even those of equal measure with complements of equal measure. For instance set $(X, \lambda) = (\mathbb{R}, \mathrm{Leb})$ and let us consider:

$$A \coloneqq [0, +\infty[\quad \text{and} \quad B \coloneqq \bigsqcup_{n \in \mathbb{N}} [2n, 2n+1[.$$

We have $\lambda(A) = \lambda(B) = \lambda(X \setminus A) = \lambda(X \setminus B) = \lambda(A \setminus B) = +\infty$ but $\lambda(B \setminus A) = 0$. If $\mathbb{G}$ is an ergodic full group, by Corollary 3.4.12 there exists $T \in \mathbb{G}$ such that $T(A) = B$. However, if there existed a measure-preserving bijection U such that $U(A) = B$ and $U(B) = A$ (in particular the involutions given by Corollary 3.4.13 satisfy this property), we would have $U(A \setminus B) = U(A) \setminus U(B) = B \setminus A$, which is not possible because $\lambda(A \setminus B) \neq \lambda(B \setminus A)$.

The involutions given by Corollary 3.4.13 play a crucial role in understanding the structure of ergodic full groups (see Section 3.6 and Section 3.7), and since they are trivial on $A \cap B$, the following definition is very natural.

Definition 3.4.15. Let A and B be two disjoint Borel subsets of X satisfying $\lambda(A) = \lambda(B)$ and $\lambda(X \setminus A) = \lambda(X \setminus B)$. An involution $U \in \mathrm{Aut}(X, \lambda)$ such that $U(A) = B$ and $U_{\restriction X \setminus (A \sqcup B)} = \mathrm{id}_{X \setminus (A \sqcup B)}$ is called a **(A, B)-exchanging involution**, *i.e.* an involution sending A to B.

Remark 3.4.16. Do note that Corollary 3.4.13 has the following straightforward consequence. If $\mathbb{G}$ is an ergodic full group and A is a Borel subset of X of positive measure, then $\mathbb{G}_A$ is ergodic on A (*i.e.* ergodic as a subgroup of $\mathrm{Aut}(A, \lambda_{\restriction A})$). Indeed, if it was not, there would exist a subset $B \subseteq A$ of positive measure satisfying $\lambda(T(B) \Delta B) = 0$ for all $T \in \mathbb{G}_A$. By Corollary 3.4.13, we can find an exchanging involution $U \in \mathbb{G}_A$ exchanging a positive measure subset of B with a subset of $A \setminus B$ of equal measure, a contradiction.

We in fact have that all involutions in $\mathrm{Aut}(X, \lambda)$ are exchanging involutions. The proof is immediate from Lemma 3.3.4.

Proposition 3.4.17 ([Fre02, Cor. 382F])**.** *Let U be an involution in $\mathrm{Aut}(X, \lambda)$. Then there exists two disjoint Borel subsets A and B verifying $\lambda(A) = \lambda(B)$ and $\lambda(X \setminus A) = \lambda(X \setminus B)$, and such that U is an (A, B)-exchanging involution.*

Putting together Proposition 3.4.17 and the following theorem by Ryzhikov, we are able to reconstruct full groups from their exchanging-involutions.

Theorem 3.4.18 ([Ryz85], see also [Mil04, Cor. 5.2])**.** *Let $\mathbb{G}$ be an ergodic full group. For any Borel subset C of X, any element of $\mathbb{G}_C$ can be expressed as a product of at most three involutions in $\mathbb{G}_C$.*

We now use the exchanging involutions to get the following generalization of [CLM16, Prop. 3.12]. The arguments are from [Kec10, Prop. 3.1] and generalize to the infinite measure-preserving setting without trouble thanks to Corollary 3.4.13. The proof adapts almost verbatim. We also use Example 3.4.6 to obtain conditions on the group $\mathbb{G}_f$.

Proposition 3.4.19. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be a full group. The following are equivalent:*

(1) $\mathbb{G}$ is ergodic;

(2) $\mathbb{G}_f$ *is ergodic;*

(3) $\mathbb{G}$ *is dense in* $(\mathrm{Aut}(X,\lambda),\tau_w)$*;*

(4) $\mathbb{G}_f$ *is dense in* $(\mathrm{Aut}(X,\lambda),\tau_w)$.

Proof. Example 3.4.6 establishes the equivalence of *(1)* and *(2)*. The implication *(4)* $\implies$ *(3)* is immediate, and the converse comes from the already mentioned fact that $\mathbb{G}_f$ is τ_w-dense in $\mathbb{G}$, and we in fact have the stronger uniform density, in the case where $\mathbb{G}$ is ergodic (see Remark 3.6.26), so it remains to show that *(1-2)* is equivalent to *(3)*. We follow the proof from [Kec10, Prop. 3.1].

(1) $\implies$ *(3)*. Assume that $\mathbb{G}$ is ergodic, and fix $S \in \mathrm{Aut}(X,\lambda)$. A τ_w-neighborhood basis for S is given by sets of the form $\left\{T \in \mathrm{Aut}(X,\lambda) \;\middle|\; \forall n \in \mathbb{N} : \lambda(T(A_n)\Delta S(A_n)) < \frac{\varepsilon}{2^n}\right\}$, where $\varepsilon > 0$ and (A_n) is a countable partition of X into Borel subsets of finite measure. By Corollary 3.4.13 we can find for each $n \in \mathbb{N}$ an involution $U_n \in \mathbb{G}$ such that $U_n(A_n) = S(A_n)$. Then the bijection T defined on A_n by $U_{n\restriction A_n}$ is in $\mathbb{G}$ and is suitable.

(3) $\implies$ *(1)*. Assume now that $\mathbb{G}$ is not ergodic. This means that we can find $A \in \mathrm{MAlg}(X,\mu)$ (where $\mu \in [\lambda]$ is a probability measure), neither null nor conull, which is $\mathbb{G}$-invariant. If $\mathbb{G}$ was τ_w-dense in $\mathrm{Aut}(X,\lambda)$, then by Lemma 3.3.8, the set A would be $\mathrm{Aut}(X,\lambda)$-invariant, which is not possible. $\square$

From Proposition 3.4.19 and Proposition 1.1.19 we can generalize [CLM16, Cor. 3.13].

Corollary 3.4.20. *The only ergodic full group that is Polish for* τ_w *is* $\mathrm{Aut}(X,\lambda)$.

We give more details on the link between Polish topologies on ergodic full groups and the weak topology in Section 3.6.1, and on Polish topologies on $\mathbb{G}_f$ in Section 3.6.4.

We end this section with the following lemma describing the behaviour of involutions under conjugation.

Lemma 3.4.21. *Let* (X,λ) *be a standard* σ*-finite space and* $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ *be an ergodic full group. For any* $C \subseteq X$*, if* U *and* V *are two involutions in* $\mathbb{G}_C$ *with* $\lambda(\operatorname{supp} U) = \lambda(\operatorname{supp} V)$ *and* $\lambda(C \setminus \operatorname{supp} U) = \lambda(C \setminus \operatorname{supp} V)$*, then* U *and* V *are conjugated in* $\mathbb{G}_C$.

Proof. The construction is classical, and summarized by Figure 3.2.

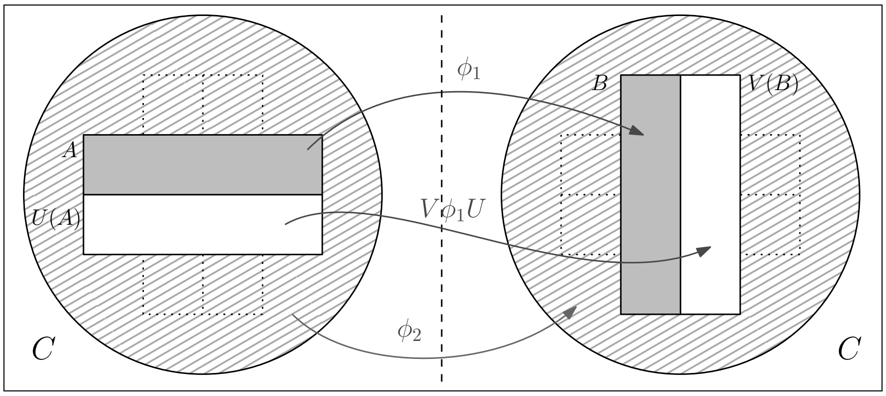

Figure 3.2: Construction of T.

By Proposition 3.4.17 there exists two Borel subsets A and B such that $\operatorname{supp} U = A \sqcup U(A) \subseteq C$ and $\operatorname{supp} V = B \sqcup V(B) \subseteq C$. By Proposition 3.4.11, we can find $\phi_1 \in [[\mathbb{G}]]$ such that $T(A) = B$ and $\phi_2 \in [[\mathbb{G}]]$ such that $\phi_2(C \setminus \operatorname{supp} U) = C \setminus \operatorname{supp} V$. We define T as follows:

$$T := \begin{cases} \phi_1 & \text{on } A \\ V\phi_1U & \text{on } U(A) \\ \phi_2 & \text{on } C \setminus \operatorname{supp} U \\ \mathrm{id}_X & \text{on } X \setminus C. \end{cases}$$

We have $TU = V\phi_1 = VT$ on A, $TU = \phi_1 U = VT$ on $U(A)$ and $TU = \phi_2 = VT$ on $C \setminus \operatorname{supp} U$. Moreover T is in $\mathbb{G}_C$ by construction, so T is the desired conjugation between U and V. $\square$

3.5 Orbit full groups on locally finite spaces

3.5.1 Polish structures on orbit full groups

Let us recall the definition of an orbit full group, which is a special case of a full group of an equivalence relation. They arise from the action of a group on (X, λ) and represent our main examples of full groups, and as such they will be the focus of our work in this section. The whole section follows what was done in the finite measure context in [CLM16]. In particular we give an infinite measure result analogous to their Theorem 3.17. We start off with the main definition.

Definition 3.5.1. Let $\mathcal{R}$ be an equivalence relation on a standard σ-finite space (X, λ). The set of all $T \in \mathrm{Aut}(X, \lambda)$ such that for all $x \in X$, $(x, Tx) \in \mathcal{R}$ is called the **full group of** $\mathcal{R}$, and is denoted by $[\mathcal{R}]$.

Let now G be a group acting on a standard σ-finite space (X, λ) in a Borel manner. This action defines an equivalence relation $\mathcal{R}_G$, where $(x, x') \in \mathcal{R}_G$ if there exists $g \in G$ such that $x' = g \cdot x$. The group $[\mathcal{R}_G]$ is naturally the full group of the equivalence relation $\mathcal{R}_G$, and in other words

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X, \lambda) \mid Tx \in G \cdot x, \text{ for all } x \in X\}.$$

The full group $[\mathcal{R}_G]$ is called the **orbit full group** of the G-action.

Several remarks have to be made in order to comprehend the orbit full groups, as there are a few subtleties at play. Our first remark is from [LM14a, Lem. A.18], and actually holds in the non-singular context.

Remark 3.5.2. As elements of an orbit full group are defined up to measure zero, if an element $T \in [\mathcal{R}]$ satisfies $(x, Tx) \in \mathcal{R}$ almost everywhere, up to modifying T on a null set we can assume that T satisfies this property everywhere. In other words, in the previous definition the "for all $x \in X$" can be replaced by "for almost all $x \in X$".

Indeed, if $T \in [\mathcal{R}]$ and $X_0 := \{x \in X \mid (Tx, x)\}$ we can define

$$\widetilde{T} = \begin{cases} T & \text{on } \bigcap_{n \in \mathbb{Z}} T^n(X_0) \\ \mathrm{id}_X & \text{elsewhere.} \end{cases}$$

As $\bigcap_{n \in \mathbb{Z}} T^n(X_0)$ is conull and T-invariant, $\widetilde{T}$ coincides with T on a conull set, and $(x, \widetilde{T}x) \in \mathcal{R}$ for all $x \in X$.

Remark 3.5.3. We insist on the fact that the orbit full group $[\mathcal{R}_G]$ crucially depends on the spatial G-action. In particular, two booleanly isomorphic spatial actions can yield non isomorphic full groups. Indeed, consider for instance the two $\mathfrak{S}_\infty$-actions described in Section 2.4.3. As explicitly stated in Proposition 2.4.15, the two boolean actions are isomorphic, but the spatial actions are not spatially isomorphic, and from the proof one can gather that the orbit full group of β is the whole group $\mathrm{Aut}(X, \mu)$, while elements in the orbit full group of α can only move points in orbits of measure zero.

Remark 3.5.4. A Polish group G acting in an ergodic manner on (X, λ) yields an ergodic orbit full group $[\mathcal{R}_G]$, but the converse is not necessarily true (see *e.g.* [CLM16, Ex. 3.14]).

Let us now consider a Polish group G acting in a Borel manner on a σ-finite space (X, λ). Let us also consider the space $\mathrm{L}^0(X, \lambda, (G, \tau_G))$, which is Polish when equipped with the topology of convergence in measure by Proposition 3.3.14. We define $\Phi : \mathrm{L}^0(X, \lambda, (G, \tau_G)) \to \mathrm{L}^0(X, \lambda, (X, \tau_X))$ as follows:

$$\Phi(f)(x) \coloneqq f(x) \cdot x.$$

We view $\mathrm{Aut}(X, \lambda)$ as a subspace of $\mathrm{L}^0(X, \lambda, (X, \tau_X))$, and we define $\widetilde{[\mathcal{R}_G]} \coloneqq \Phi^{-1}(\mathrm{Aut}(X, \lambda))$.

The following lemma works exactly the same as the finite measure case. The proof is given for convenience only.

Lemma 3.5.5 ([CLM16, Lem. 3.15])**.** *We have the equality* $\Phi\left(\widetilde{[\mathcal{R}_G]}\right) = [\mathcal{R}_G]$.

Proof. For almost all $x \in X$, $(\Phi(f)(x), x) \in \mathcal{R}_G$, for any $f \in \mathrm{L}^0(X, \lambda, (G, \tau_G))$. In particular $\Phi\left(\widetilde{[\mathcal{R}_G]}\right) \subseteq [\mathcal{R}_G]$.

Conversely, let $T \in [\mathcal{R}_G]$, and let us consider the Borel set

$$A_T \coloneqq \{(x, g) \in X \times G \mid g \cdot x = Tx\}.$$

A_T is Borel, and thus analytic, we can then apply the Jankov-von Neumann Uniformization theorem ([Kec95, Thm. 18.1]). There exists an analytic (in particular it is Lebesgue-measurable, see *e.g.* [Kec95, Thm. 21.10]) function $f : X \to G$ of which A_T is the graph. If a couple $(x, f(x))$ is in A_T, this precisely means that $f(x) \cdot x = Tx$, and so $T = \Phi(f)$, which proves the converse inclusion. $\square$

The space $\widetilde{[\mathcal{R}_G]}$ is a subspace of $\mathrm{L}^0(X, \lambda, (G, \tau_G))$, and as such can be equipped with the topology of convergence in measure.

Definition 3.5.6. The **topology of orbital convergence in measure** τ_m^o on the orbit full group $[\mathcal{R}_G]$ is the quotient topology of the topology on convergence in measure on $\widetilde{[\mathcal{R}_G]}$ by $\mathrm{Ker}(\Phi)$.

The main theorem of this section states that $([\mathcal{R}_G], \tau_m^o)$ is a Polish group, when G acts in a continuous manner on a Polish space endowed with a locally finite measure.

Theorem 3.5.7. *Let G be a Polish group acting in a continuous manner on a Polish space (X, τ_X) equipped with an atomless σ-finite measure λ which is locally finite on τ_X. The orbit full group $[\mathcal{R}_G]$, equipped with the topology of orbital convergence in measure, is a Polish group.*

Proof. Let (f_n) be a sequence of elements of $\mathrm{L}^0(X, \lambda, (G, \tau_G))$ converging to f in measure, and by Proposition 3.3.10, let (f_{n_k}) be a subsequence such that $f_{n_k}(x) \to f(x)$ λ-almost everywhere. Then by continuity of the action

$$\Phi(f_{n_k})(x) = f_{n_k}(x) \cdot x \longrightarrow f(x) \cdot x = \Phi(f)(x)$$

for λ-almost all $x \in X$. This ensures us that Φ is continuous.

We also know that $\mathrm{Aut}(X, \lambda)$ is G_δ in $\mathrm{L}^0(X, \lambda, (X, \tau))$ thanks to Proposition 3.3.13, and so we write $\mathrm{Aut}(X, \lambda) = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n$. We have

$$\widetilde{[\mathcal{R}_G]} = \Phi^{-1}(\mathrm{Aut}(X, \lambda)) = \Phi^{-1}\left(\bigcap_{n \in \mathbb{N}} \mathcal{O}_n\right) = \bigcap_{n \in \mathbb{N}} \Phi^{-1}(\mathcal{O}_n),$$

which ensures us that $\widetilde{[\mathcal{R}_G]}$ is G_δ in $\mathrm{L}^0(X, \lambda, (G, \tau_G))$, by the previously proven continuity. Thanks to Proposition 1.1.4, it is Polish.

Let us now define the group structure on $\widetilde{[\mathcal{R}_G]}$. Let f and g be two elements of $\widetilde{[\mathcal{R}_G]}$. We define the group operation $*$, the inverse and the neutral element as follows:

$$\begin{aligned}
&(f * g)(x) := f(\Phi(g)(x))g(x) \\
&f^{-1}(x) := f(\Phi(f)^{-1}(x))^{-1} \\
&e_{\widetilde{[\mathcal{R}_G]}} : x \in X \mapsto e_G.
\end{aligned}$$

Note that here, $\Phi(f)^{-1}$ is the function in $\mathrm{L}^0(X, \lambda, (X, \tau_X))$ mapping x to the unique element $\overline{x}$ such that $f(\overline{x}) \cdot \overline{x} = x$, and is well defined since $\Phi(f)$ is a bijection whenever $f \in \widetilde{[\mathcal{R}_G]}$. We check that f^{-1} is well defined as the inverse of f:

$$\begin{aligned}
(f * f^{-1})(x) &= f(f^{-1}(x) \cdot x)f^{-1}(x) \\
&= f(f(\Phi(f)^{-1}(x))^{-1} \cdot x)f(\Phi(f)^{-1}(x))^{-1} \\
&= f(f(\overline{x})^{-1} \cdot x)f(\overline{x})^{-1} \\
&= f(\overline{x})f(\overline{x})^{-1} \\
&= e_G,
\end{aligned}$$

Indeed, $f(\overline{x}) \cdot \overline{x} = x$ so $f(\overline{x})^{-1} \cdot x = \overline{x}$. On the other hand, we have that $(f^{-1} * f)(x) = f^{-1}(\Phi(f)(x))f(x) = f(\Phi(f)^{-1}(\Phi(f)(x)))^{-1}f(x) = f(x)^{-1}f(x) = e_G$, so f^{-1} is well defined.

The definition of Φ also ensures us that group multiplication is well defined, and Proposition 3.3.14 and Proposition 3.3.13 along with the continuity of Φ ensure us group operations are continuous. For $x \in X$, we then have

$$\begin{aligned}
\Phi(f * g)(x) &= (f * g)(x) \cdot x \\
&= (f(\Phi(g)(x))g(x)) \cdot x \\
&= f(\Phi(g)(x)) \cdot (g(x) \cdot x) \\
&= f(\Phi(g)(x)) \cdot (\Phi(g)(x)) \\
&= \Phi(f)(\Phi(g)(x))
\end{aligned}$$

which proves that $\Phi_{\restriction \widetilde{[\mathcal{R}_G]}}$ is a group homomorphism between $\widetilde{[\mathcal{R}_G]}$ and $\mathrm{Aut}(X, \lambda)$. We can then write

$$[\mathcal{R}_G] = \Phi\left(\widetilde{[\mathcal{R}_G]}\right) \cong \widetilde{[\mathcal{R}_G]} \big/ \mathrm{Ker}(\Phi).$$

Finally, $\mathrm{Ker}(\Phi)$ is normal and closed by continuity of Φ, so from Proposition 1.1.22 we can deduce that $[\mathcal{R}_G]$ is a Polish group. □

Remark 3.5.8. Note that the multiplication in the previous proof is given by the cocycle relation verified by the lift of an automorphism T by the application Φ. Indeed if $c_T : X \to G$ is such that $\Phi(c_T) = T$, we have $T(x) = \Phi(c_T)(x) = c_T(x) \cdot x$. Therefore, if T and U are elements of $\mathrm{Aut}(X, \lambda)$, we have

$$c_{TU}(x) \cdot x = c_T(c_U(x) \cdot x) \cdot (c_U(x) \cdot x).$$

In the case of a standard probability space rather than a standard σ-finite space, it is possible to weaken the conditions of the previous theorem, by simply asking for a Borel group action, rather than a continuous one. It is indeed possible to do so by using Theorem 3.2.10, as there are no topological properties that need to be preserved. This case is detailed in [CLM16, Thm. 3.17]. For our infinite measure setup, we need to use a different model: by using a continuous Radon model for the action (see Section 3.2.2), we are able to obtain the following for locally compact Polish groups.

Theorem 3.5.9. *Let G be a locally compact Polish group acting in a Borel measure-preserving manner on a standard σ-finite space (X, λ). The orbit full group $([\mathcal{R}_G], \tau_m^o)$ is a Polish group.*

Proof. By Theorem 3.2.11, the Borel measure-preserving G-action on (X, λ) is isomorphic to a continuous measure-preserving G-action on (Y, η), where Y is locally compact Polish, and η is Radon. In particular η is locally finite on τ_Y by Proposition 3.2.9. Theorem 3.5.7 concludes, as the G-action on (Y, η) induces the same full group as the G-action on (X, λ). $\square$

Let us end this section by recalling the definition of an essentially free action, and by stating how it pertains to our result.

Definition 3.5.10. We say that an action of G on (X, λ) is **essentially free** if there exists a conull G-invariant subset $A \subseteq X$, such that for all $g \in G \setminus \{e_G\}$ and every $x \in A$ we have $g \cdot x \neq x$. That is to say that, in restriction to A, the G-action is free.

Remark 3.5.11. If G acts in an essentially free manner on (X, λ), the application Φ is injective, and is therefore a bijection between $\widetilde{[\mathcal{R}_G]}$ and $[\mathcal{R}_G]$. As Φ is a continuous group homomorphism between $\widetilde{[\mathcal{R}_G]}$ and $[\mathcal{R}_G]$, we can assert that those two groups are topologically isomorphic. By identifying G with the set of constant maps $x \mapsto g$ in $\mathrm{L}^0(X, \lambda, (G, \tau_G))$, we have the following.

Corollary 3.5.12. *Let G be a Polish group and (X, λ) be a Polish space equipped with an atomless σ-finite locally finite measure λ. Consider a Borel measure-preserving G-action on (X, λ). If the action is essentially free, then G embeds into $([\mathcal{R}_G], \tau_m^o)$.*

3.5.2 Necessity of the local finiteness

In this section we detail the behaviour of the weak topology and the topology of (orbital) convergence in measure when the hypothesis of local finiteness is not satisfied. We then provide two examples, and in particular the example with the action of $\mathfrak{S}_\infty$ (which is not locally compact) shows that the application of Theorem 3.2.11 to Theorem 3.5.7 cannot be replaced by a more general statement, as noticed in [HLM24, Prop. 5.7].

First, notice that having up to countably many points only admitting neighbourhoods of infinite measure does not constitute an issue, as it is still possible to remove them, since the measure is atomless. The real obstacle arises when there exists a subset of positive measure comprised solely of such points.

We can think of the following as an addition to Proposition 3.3.13.

Proposition 3.5.13. *Let X be a Polish space endowed with an atomless σ-finite measure λ, such that the set of elements of X which only admits neighbourhoods of infinite measure has positive measure. Then the topology of convergence in measure on any ergodic full group $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ does not refine the weak topology.*

Proof. Denote by A the set of elements of X which only admits neighbourhoods of infinite measure:

$$A := \{x \in X \mid \text{for any neighbourhood } \mathcal{N}_x \text{ of } x, \lambda(\mathcal{N}_x) = +\infty\}.$$

First note that we can assume that A has finite measure (if that is not the case, simply consider a Borel subset of A of finite measure for this argument). We fix a compatible metric d on X and $\varepsilon > 0$. We will construct a measure-preserving bijection T of X that sends every element of A ε-close to itself, but such that $d_{X,\lambda}(A, T(A)) > c$, for a c that does not depend on ε.

As X is Polish, A is in particular second countable so we can apply Lindelöf's lemma. Thus we have $A \subseteq \bigcup_{n\in\mathbb{N}} B(x_n, \varepsilon)$, where $B(x_n, \varepsilon)$ is the open ball of center x_n and radius ε, and x_n is in A for every n in $\mathbb{N}$. We set $B_0 := B(x_0, \varepsilon) \cap A$. We have $\lambda(B(x_0, \varepsilon)) = +\infty$, and so there exists $C_0 \subseteq B(x_0, \varepsilon) \setminus A$, such that $\lambda(C_0) = \lambda(B_0) < +\infty$. By Corollary 3.4.12, there exists φ_0 in $\mathbb{G}_{B(x_0,\varepsilon)}$ such that $\varphi_0(B_0) = C_0$. For $n > 0$, we then define

$$B_n := \left(B(x_n, \varepsilon) \setminus \bigcup_{k=0}^{n-1} B_k \right) \cap A,$$

and notice that $\sqcup B_n$ covers A. We then choose

$$C_n \subseteq B(x_n, \varepsilon) \setminus \left(A \cup \bigsqcup_{k=0}^{n-1} C_k \right)$$

such that $\lambda(C_n) = \lambda(B_n) < +\infty$. We then choose, still thanks to Corollary 3.4.12, a element φ_n in $\mathbb{G}_{B(x_n,\varepsilon)}$ such that $\varphi_n(B_n) = C_n$. We finally define T as follows:

$$T = \begin{cases} \varphi_n & \text{on } B_n \ , \ n \in \mathbb{N} \\ \varphi_n^{-1} & \text{on } C_n \ , \ n \in \mathbb{N} \\ \mathrm{id}_X & \text{elsewhere.} \end{cases}$$

For any x in A, x and $T(x)$ are in the same ball of radius ε, so $d(x, T(x)) < 2\varepsilon$, but $\lambda(A \Delta T(A)) = 2\lambda(A)$. As $\lambda(A) > 0$, a sequence (T_n) constructed by considering $\varepsilon_n \to 0$ cannot converge weakly to id_X, but will do so in measure by the characterization given in Remark 3.3.11. ☐

Example 3.5.14. Our first concrete example is in $X = \mathbb{R}^2$, endowed with its usual topology. We define the measure λ as follows:

$$\lambda = c_{\mathbb{Q}} \otimes \mathrm{Leb}_{\mathbb{R}}.$$

We explicitly build a suitable sequence of bijections preserving λ.

The measure space (X, λ) is σ-finite and the measure is atomless by construction, but every point of X only admits open neighbourhoods of infinite measure. In particular λ is not locally finite for the usual $\mathbb{R}^2$ topology. Let (q_n) be a sequence of strictly positive rational numbers converging to 0. We consider the vertical segments $A_n := \{(q_n, y) \mid y \in [0, 1]\}$ and $A_\infty := \{(0, y) \mid y \in [0, 1]\}$ which are of length 1, and the measure-preserving bijections (T_n) defined by

$$T_n(x, y) = \begin{cases} (0, y) & \text{if } x = q_n \\ (q_n, y) & \text{if } x = 0 \\ (x, y) & \text{otherwise.} \end{cases}$$

In other words, T_n exchanges A_n and A_∞, and fixes the rest of the space. The sequence (T_n) converges pointwise, so by Proposition 3.3.10 (T_n) converges to id_X in measure, however $\lambda(T_n(A_\infty) \Delta A_\infty) = \lambda(A_\infty) + \lambda(A_n) = 2$ and thus (T_n) does not converge weakly to id_X.

We now turn to our second example, which provides information on τ_m^o this time. Start by recalling the following.

Proposition 3.5.15 (see *e.g.* [Kec95, I.9])**.** *The group $G = \mathfrak{S}_\infty$ equipped with the induced topology as a subset of $\mathbb{N}^{\mathbb{N}}$ is Polish, but is not locally compact.*

Example of an action of $\mathfrak{S}_\infty$. Let $G = \mathfrak{S}_\infty$, the group of all permutations of $\mathbb{N}$, be acting on $X = \{0,1\}^{\mathbb{N}}$ by permutation of the coordinates. The action is the same as the one described in [HLM24, Sec. 5.4], which can be consulted for additional details. We give the description of the measure and of the action once again.

Let us fix a sequence (p_n) of weights, with $p_n \in \,]0,1[$ for all n in $\mathbb{N}$. We also ask that $p_n \neq p_m$ for $n \neq m$, and that $p_n \to \frac{1}{2}$. Let us also consider the following measure λ on X:

$$\lambda \coloneqq \sum_{n\in\mathbb{N}} \mu_n \coloneqq \sum_{n\in\mathbb{N}} (p_n\delta_1 + (1-p_n)\delta_0)^{\otimes\mathbb{N}}.$$

Let X_∞ be the subspace of X consisting of sequences that take infinitely many times the value 0, and infinitely many times the value 1. Do note that G acts in a transitive manner on X_∞. We have $\lambda(X \setminus X_\infty) = 0$, as for any n in $\mathbb{N}$ we have $\mu_n(X \setminus X_\infty) = 0$, and thus, up to a null set, we need only consider such sequences.

We also define

$$X_n \coloneqq \left\{ x \in X : \frac{|\{k \in \{0,\dots,m-1\} : x_k = 1\}|}{m} \underset{m\to+\infty}{\longrightarrow} p_n \right\},$$

that is to say that X_n contains the sequences of X_∞ with a proportion p_n of 1. Then for all n in $\mathbb{N}$ we have $\mu_n(X_n) = 1$.

The space (X,λ) is σ-finite, but every nonempty open set of X has infinite measure, and in particular it is not locally finite.

For every n let H_n denote the open subgroup of $\mathfrak{S}_\infty$ given by

$$H_n = \{\sigma \in \mathfrak{S}_\infty : \forall i \in \{0,\cdots n\},\, \sigma(i) = i\}.$$

Then the sequence (H_n) is a neighborhood basis of the identity in $\mathfrak{S}_\infty$.

As noted before, the author and Le Maître proved in [HLM24, Sec. 5.4] that such an action cannot admit a continuous spatial model with a locally finite measure.

The action of $\mathfrak{S}_\infty$ on X is essentially transitive, and thus we have $\mathrm{Aut}(X,\lambda) = [\mathcal{R}_{\mathfrak{S}_\infty}]$. As the group $\mathrm{Aut}(X,\lambda)$ has a unique Polish group topology (see [Kal85], [LM22, Cor 2.14], or Section 3.6), $\mathrm{Aut}(X,\lambda)$ equipped with any topology other than the weak topology cannot be Polish. In the next Proposition, we show that the orbital topology of convergence in measure does not coincide with the weak topology.

Proposition 3.5.16. *Consider the natural action of $G = \mathfrak{S}_\infty$ on $X = \{0,1\}^{\mathbb{N}}$ by permutation of the coordinates. Endow X with $\lambda = \sum_{n\in\mathbb{N}} \mu_n = \sum_{n\in\mathbb{N}} (p_n\delta_1 + (1-p_n)\delta_0)^{\otimes\mathbb{N}}$, where (p_n) is an injective sequence of elements of $]0,1[$ converging to any real l in $]0,1[$. There exists a sequence (T_n) of elements of $\mathrm{Aut}(X,\lambda)$ that converges orbitally in measure to the identity, but does not do so weakly. In particular, τ_m^o does not refine τ_w on $\mathrm{Aut}(X,\lambda)$.*

Proof. For τ_m^o, n-cylinder sets of X will be very useful, for n in $\mathbb{N}$. We define such sets as the sets of sequences in which the coordinates 0 to n are fixed. In other words a n-cylinder is a set of the form

$$\mathcal{C}_{n,(\varepsilon_i)} \coloneqq \{x \in X \mid \forall i \in \{0,\dots,n\} : x_i = \varepsilon_i\},$$

where $(\varepsilon_i)_{i\leqslant n}$ is a fixed family with $\varepsilon_j = 0$ or 1 for any $j \leqslant n$. It is immediate to see that for any n, there are 2^{n+1} n-cylinders. Notice that for any n in $\mathbb{N}$, H_n acts transitively on each of the n-cylinders. We fix n, and $\mathcal{C}$ a n-cylinder. We have the following:

$$\lambda(X_0 \cap \mathcal{C}) = \mu_0(X_0 \cap \mathcal{C}) = \mu_0(X_0)\prod_{i=0}^{n} q_i = \prod_{i=0}^{n} q_i$$

where $q_i = p_0$ or $1-p_0$, depending on the value taken by the coordinate of rank i in elements of $\mathcal{C}$. As p_0 is in $]0,1[$, the coefficient $\prod_{i=0}^{n} q_i$ is strictly positive, and therefore $\lambda(X_0 \cap \mathcal{C}) > 0$. We also have:

$$\lambda((X \setminus X_0) \cap \mathcal{C}) = \lambda(\mathcal{C} \setminus X_0) = \lambda(\mathcal{C}) - \lambda(X_0 \cap \mathcal{C}).$$

For $\lambda((X \setminus X_0) \cap \mathcal{C})$, it is important to remember that all n-cylinders have infinite measure (see [HLM24, Lem.5.8]). Thus we have $\lambda((X \setminus X_0) \cap \mathcal{C}) = +\infty$.

We can then consider $B_{\mathcal{C},0}$ a Borel subset of $(X \setminus X_0) \cap \mathcal{C}$ such that $\lambda(B_{\mathcal{C},0}) = \lambda(X_0 \cap \mathcal{C})$. Restricting the measure λ to $\mathcal{C}$ gives us a σ-finite space, and as H_n acts transitively on $\mathcal{C}$, the full group of the action of H_n on $\mathcal{C}$ is $\mathrm{Aut}(\mathcal{C}, \lambda_{\restriction\mathcal{C}})$. Applying Corollary 3.4.12 to $X_0 \cap \mathcal{C}$ and $B_{\mathcal{C},0}$ in that measurable space yields a measure-preserving bijection $T_{\mathcal{C}}$ in $[\mathcal{R}_{H_n \curvearrowright \mathcal{C}}] = \mathrm{Aut}(\mathcal{C}, \lambda_{\restriction\mathcal{C}})$ such that $T_{\mathcal{C}}(X_0 \cap \mathcal{C}) = B_{\mathcal{C},0}$.

As previously stated, for any n in $\mathbb{N}$ the set of all the n-cylinders is finite of cardinal 2^{n+1}, so we have

$$X_0 = \bigsqcup_{i=1}^{2^{n+1}} X_0 \cap \mathcal{C}_i,$$

where the cylinders $\mathcal{C}_i$ are all the possible n-cylinders. It is of note that $B_{\mathcal{C}_i,0} \cap B_{\mathcal{C}_j,0} = \emptyset$, for $i \neq j$, as the sets $B_{\mathcal{C},0}$ previously defined all live in different n-cylinders.

We can then construct the suitable sequence (T_n) of elements of $\mathrm{Aut}(X, \lambda)$ as follows: For all n in $\mathbb{N}$, we consider $(C_i)_{i=1}^{2^{n+1}}$ the different n-cylinders and $(B_{\mathcal{C}_i,0})_{i=1}^{2^{n+1}}$ the previously defined sets, and we define T_n by

$$T_n = \begin{cases} T_{\mathcal{C}_i} & \text{on } X_0 \cap \mathcal{C}_i \ , \ i \in \{1, \ldots, 2^{n+1}\} \\ T_{\mathcal{C}_i}^{-1} & \text{on } B_{\mathcal{C}_i,0} \ , \ i \in \{1, \ldots, 2^{n+1}\} \\ \mathrm{id}_X & \text{elsewhere.} \end{cases}$$

For all n in $\mathbb{N}$, T_n is a well-defined element of $\mathrm{Aut}(X, \lambda) = [\mathcal{R}_G]$ by construction. Working on n-cylinders at rank n ensures us that (T_n) converges orbitally in measure to the identity: Indeed, for any n we have $T_n \in [\mathcal{R}_{H_n \curvearrowright \mathcal{C}}]$, and so the associated cocycle c_{T_n} is in H_n. As the (H_n) form a neighbourhood basis of the identity e_G, in particular $c_{T_n} \to (x \mapsto e_G)$ in $\mathrm{L}^0(X, \lambda, G)$, which exactly means that (T_n) converges to id_X for τ_m^o.

Finally, for all n in $\mathbb{N}$:

$$\begin{aligned} \lambda(T_n(X_0) \Delta X_0) &= \lambda\left(T_n\left(\bigsqcup_{i=1}^{2^{n+1}} X_0 \cap \mathcal{C}_i\right) \Delta X_0\right) \\ &= \lambda\left(\left(\bigsqcup_{i=1}^{2^{n+1}} B_{\mathcal{C}_i,0}\right) \sqcup X_0\right) \\ &= \sum_{i=1}^{2^{n+1}} \lambda(B_{\mathcal{C}_i,0}) + \lambda(X_0) \\ &= \sum_{i=1}^{2^{n+1}} \lambda(X_0 \cap \mathcal{C}_i) + \lambda(X_0) = 2, \end{aligned}$$

which ensures us that (T_n) does not converge weakly to the identity. □

When the space acted upon is Polish and endowed with a non locally finite measure, Proposition 3.5.13 guarantees that τ_m does not refine τ_w. By uniqueness of the Polish topology on $\mathrm{Aut}(X, \lambda)$, it is impossible for $(\mathrm{Aut}(X, \lambda), \tau_m)$ to be Polish. It could happen however, that $(\mathrm{Aut}(X, \lambda), \tau_m^o)$ is Polish for a certain essentially transitive continuous action of G, and in that case τ_w would have to be equal to τ_m^o. Proposition 3.5.16 gives an example where that does not happen, showing that the hypotheses of Theorem 3.5.7 are minimal.

Remark 3.5.17. The topological properties of (X,λ) that we need to consider heavily depend on the weights (p_n). Indeed, suppose we chose p_n to be equal to $\frac{1}{2^{n+1}}$, for all n in $\mathbb{N}$. All n-cylinders where at least one coordinate is equal to 1 have finite measure, that is to say that any element of X which is not the sequence where all the coordinates are equal to 0 admits a neighbourhood of finite measure. Thus, (X,λ) is not locally finite, but the set of all the points which admit a neighbourhood of finite measure has full measure in X. This is the correct property to consider in our case, and the one we used in Proposition 3.3.13, and therefore in Theorem 3.5.7.

3.5.3 Induced actions on spaces of infinite measure

In this section we provide a family of examples of infinite measure-preserving actions of Polish groups, including non locally compact ones. In particular when the space is endowed with a topology on which the measure is locally finite, Theorem 3.5.7 applies. We start with a countable index subgroup acting on a finite measure space and induce the action on a countable product of the space.

Proposition 3.5.18. *Let G be a Polish group, with H a closed subgroup of countable index of G, such that H acts continuously and in a measure preserving manner on a Polish space X equipped with a locally finite σ-finite measure μ. Then G acts continuously in a measure-preserving manner on the space $Y = X \times {}^{G}\!/_{H}$, which we can endow with the locally finite σ-finite infinite measure $\mu \otimes \mathrm{counting}$. Moreover, if the H-action on X is essentially free, then so is the G-action on Y.*

Proof. Let T be a Borel fundamental domain for the action of H on G by right multiplication by the inverse, which exists since H is closed (see *e.g.* [Kec95, Thm. 12.17]). We will freely use the following identification: $T \simeq {}^{G}\!/_{H}$.

The group G naturally acts on T, and the action is given by

$$(g,t) \mapsto g * t := \text{ the only element of } T \cap gtH.$$

We can then define the cocycle of the G-action into H by setting:

$$c : (g,t) \in G \times T \longmapsto \text{the only } h \in H \text{ such that } (gt)h^{-1} \in T,$$

that is to say that $c(g,t)$ is the only h in H such that $(gt)h^{-1} = g * t$. We now verify that c satisfies the cocycle relation. We note $c(g_1, g_2 * t) = h_1$, that is to say that h_1 is the only element of H such that $(g_1(g_2 * t))h_1^{-1} \in T$. Similarly we define $c(g_2,t) = h_2$ and $c(g_1g_2,t) = h_{1,2}$, therefore h_2 and $h_{1,2}$ are the only elements of H such that $(g_2t)h_2^{-1}$ and $(g_1g_2t)h_{1,2}^{-1}$ are in T, respectively. We have

$$(g_1g_2t)h_{1,2}^{-1} = g_1g_2 * t = g_1 * (g_2 * t) = g_1(g_2 * t)h_1^{-1} = g_1((g_2t)h_2^{-1})h_1^{-1} = (g_1g_2t)(h_1h_2)^{-1}$$

which proves that c is a cocycle.

Finally, we define the action α as follows:

$$g \cdot_\alpha (x,t) = (c(g,t)\cdot x, g * t),$$

and the cocycle relation ensures us that it is a G-action on $X \times {}^{G}\!/_{H}$. We endow $Y = X \times {}^{G}\!/_{H}$ with the product topology of the Polish topology on X and the discrete topology on ${}^{G}\!/_{H}$, and with the product measure of μ and the counting measure on ${}^{G}\!/_{H}$.

In order for the G-action on Y to be continuous, it remains to show that for a fixed t in T, $g \mapsto c(g,t)$ is continuous. To this end we fix t in T and (g_n) converging to g in G. Recall that H being closed, it is also open, and therefore gtH contains an open neighbourhood of gt. By passing to a subsequence is necessary, we can thus assume that all the $g_n t$ are in gtH, which implies that $g_n * t = g * t$ for any n. We have

$$c(g_n,t) = g_n t (g_n * t)^{-1} = g_n t (g * t)^{-1} \underset{n\to\infty}{\longrightarrow} gt(g*t)^{-1} = c(g,t)$$

which means that α is continuous.

We now prove the essential freeness part of the statement. As H acts in an essentially free manner on X, there exists X_0 which is μ-conull in X and H-invariant, and such that $\forall x \in X_0 \colon h \cdot x = x \Rightarrow h = e_G$.

In particular, if we take g in G, x in X_0 and t in T such that

$$g \cdot_\alpha (x,t) = (c(g,t)\cdot x, g*t) = (x,t),$$

then thanks to the second coordinate we see that $g * t = t$, so $c(g,t)$ is the only h in H satisfying $(gt)h^{-1} = g * t = t$. But thanks to the first coordinate, we see that $c(g,t)$ is necessarily e_G, because x is in X_0. Therefore, $gt = t$, and thus $g = e_G$, which proves that α is essentially free. □

Remark 3.5.19. Notice that in the previous proposition Y is made of a countable number of copies of X, and as such Y with the product measure is locally finite when (X,μ) is locally finite. It is in particular true for the case of an action on a standard probability space. In particular one can construct such an example from any continuous G-action on a standard probability space. This is a particular case of the general construction known as *an induced action*, defined more generally for non-singular actions. It is of note that induction preserves ergodicity, we refer to [Tsa24, Sec. 3.2], which can be consulted for more details.

Example 3.5.20. Let $G = \mathfrak{S}_\infty$, which we recall is not locally compact. We start by considering the subgroup

$$H_0 \coloneqq \{\sigma \in \mathfrak{S}_\infty \colon \sigma(0) = 0\}$$

comprised of the permutations fixing the first coordinate when seen as acting on a product space. We see H_0 as acting naturally on the space of sequences $X = [0,1]^{\mathbb{N}}$ by permutation of the coordinates. It is clear that H_0 is a subgroup of G of countable index.

We set $Y = X \times {}^{G}\!/\!_{H_0}$, where ${}^{G}\!/\!_{H_0}$ denotes the set of cosets for the action by right multiplication by the inverse. When X is endowed with $\mu = (\mathrm{Leb}_{\restriction[0,1]})^{\otimes\mathbb{N}}$, we can endow Y with the product measure of μ and the counting measure on ${}^{G}\!/\!_{H_0} \simeq \mathbb{N}$, making it a locally finite σ-finite space with infinite measure. Proposition 3.5.18 applies, yielding a continuous infinite measure preserving action of $\mathfrak{S}_\infty$ on Y, on which the measure is locally finite.

Corollary 3.5.21. *The G-action α defined in* Example 3.5.20 *is essentially free. In particular, G embeds into $[\mathcal{R}_G]$.*

Proof. We only have to check that the H_0-action on X is essentially free. Each component of X being endowed with the Lebesgue measure on $[0,1]$, the coordinates of an element x of X are different, μ-almost surely. Therefore there exists a subset X_0 conull in X, which is the subset of sequences with distinct coordinates. As such, it is obviously H_0-invariant, and for every x in X_0, we have $h \cdot x = x \Rightarrow h = e_G$. By Proposition 3.5.18 again, the G-action is also essentially free on Y. The second part of the statement is given by Corollary 3.5.12. □

All continuous pmp actions of non-archimedean Roelcke precompact Polish groups fit our conditions. Tsankov actually proves that all boolean non-singular actions of groups of this family are isomorphic to countable disjoint copies of induced actions [Tsa24, Thm. 3.4].

For an action of a locally compact Polish group, it is once again possible to weaken the assumptions by using Theorem 3.2.11. We obtain the following, which gives examples in which Theorem 3.5.9 applies.

Corollary 3.5.22. *Let G be a locally compact Polish group, with H a closed subgroup of countable index of G, such that H acts continuously and in a measure preserving manner on a standard σ-finite space (X, μ). Then G acts continuously in a measure-preserving manner on the space $Y = X \times {}^{G}\!/_{H}$, which we can endow with a locally finite σ-finite infinite measure. Moreover, if the H-action on X is essentially free, then so is the G-action on Y.*

3.5.4 Orbit full groups are complete invariants for orbit equivalence

Let us define orbit equivalence for two equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on a standard σ-finite space (X, λ). One can consult [KM04] for more details on this topic, although it is in the case of countable group actions and probability spaces. In this whole section (X, λ) will be a standard σ-finite space, $\mathcal{R}$ and $\mathcal{R}'$ will be equivalence relations on (X, λ), and G and H will be two Polish groups, unless stated otherwise.

Definition 3.5.23. We say that $\mathcal{R}$ and $\mathcal{R}'$ are **orbit equivalent** if there exists a conull subset $X_0 \subseteq X$ and a Borel bijection $S : X_0 \to X_0$ which verify the following:

1. S preserves λ up to multiplication by a positive scalar;
2. for all $(x, y) \in X_0 \times X_0$, we have $(x, y) \in \mathcal{R} \iff (S(x), S(y)) \in \mathcal{R}'$.

We also say that S is an orbit equivalence between $\mathcal{R}$ and $\mathcal{R}'$.

Remark 3.5.24. Orbit equivalence is usually defined with the previous Borel bijection S being measure-preserving. This definition is a bit more general and is the one that should be considered for equivalence relations on σ-finite spaces.

An orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$ conjugates the corresponding full groups which are in particular isomorphic. Note that when that is the case, they are in fact homeomorphic by Theorem 1.1.24. The proof in the probability measure setting adapts verbatim to our context.

Lemma 3.5.25 ([CLM16, Lem. 3.25])**.** *Let G and H be two Polish groups acting in a Borel manner on (X, λ). Let $S \in \mathrm{Aut}(X, \lambda)$ be an orbit equivalence between the orbit full groups $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$. Then the conjugation by S is a group homeomorphism between $([\mathcal{R}_G], \tau_m^o)$ and $([\mathcal{R}_H], \tau_m^o)$.*

In [CLM16, Thm. 3.26], the authors prove that in the case of two equivalence relations on a standard probability space coming from Borel measure-preserving ergodic actions of Polish locally compact groups, the orbit full groups are not only invariants, but *complete invariants* of orbit equivalence. Their result relies heavily on Dye's reconstruction theorem [Dye63, Thm. 2], and to extend it to our σ-finite context, we will be needing a more general result from [Fre02], which applies to groups with *many involutions*. We start by noticing that full groups of Polish groups acting on standard σ-finite spaces have many involutions, which will be immediate thanks to Corollary 3.4.13.

Definition 3.5.26. A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\lambda)$ has **many involutions** if for every Borel subset $A \subseteq X$ of positive measure, there exists a non-trivial involution U in $\mathbb{G}$ such that the support of U is contained in A.

Proposition 3.5.27. *Any ergodic full group $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ has many involutions. In particular, if a Polish group G is acting in a Borel measure-preserving ergodic manner on (X,λ), then the full group $[\mathcal{R}_G]$ of the associated equivalence relation $\mathcal{R}_G$ has many involutions.*

Proof. Take any Borel subset $A \subseteq X$ of positive measure, and consider A_1 and A_2 two disjoint Borel subsets of A such that $0 < \lambda(A_1) = \lambda(A_2) < +\infty$. Corollary 3.4.13 applies (see Remark 3.5.4), and gives us an involution supported in A and mapping A_1 to A_2, which concludes the proof. □

Let us now state Fremlin's theorem. The theorem is very general, and the following formulation, which applies to our context is from [LM18, Thm. 3.18]. Recall that T is non-singular if $T_*\lambda \in [\lambda]$ and we denote by $\mathrm{Aut}(X,[\lambda])$ the group of non-singular bijections of (X,λ).

Theorem 3.5.28 ([Fre02, Thm. 384D])**.** *Let $\mathbb{G}$ and $\mathbb{H}$ be two subgroups of $\mathrm{Aut}(X,\lambda)$ with many involutions. Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some non-singular bijection. In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in $\mathrm{Aut}(X,[\lambda])$ such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

We will also need the following proposition, which is from [CLM16, Prop. 3.28]. Although stated in the case of a standard probability space, the proof adapts verbatim to our case, and is given once again for convenience.

Proposition 3.5.29. *Let G and H be two Polish locally compact groups acting in a Borel measure-preserving manner on (X,λ), and suppose that $[\mathcal{R}_G] \subseteq [\mathcal{R}_H]$. Then there exists a conull subset $X_0 \subseteq X$ such that*

$$\mathcal{R}_G \cap (X_0 \times X_0) \subseteq \mathcal{R}_H.$$

Proof. Let ν be the right Haar measure on G. As $[\mathcal{R}_G] \subseteq [\mathcal{R}_H]$, for all elements g in G, for λ-almost all x in X, we have $g \cdot x \in H \cdot x$. By Fubini's theorem, we have

$$0 = \int_G \int_X \mathbb{1}_{\{x \,|\, g\cdot x \notin H\cdot x\}} d\lambda d\nu = \int_X \int_G \mathbb{1}_{\{g \,|\, g\cdot x \notin H\cdot x\}} d\nu d\lambda,$$

which exactly means that the set

$$X_0 \coloneqq \{x \in X \mid \text{ for } \nu\text{-almost } g \in G \text{ we have } g \cdot x \in H \cdot x\}$$

is conull. We now fix a $x \in X_0$ and a $g_1 \in G$ such that $g_1 \cdot x \in X_0$, and let us show that $g_1 \cdot x \in H \cdot x$, which will exactly mean that $(x, g_1 \cdot x) \in \mathcal{R}_G \cap (X_0 \times X_0)$ is also in $\mathcal{R}_H$. By definition of X_0, the sets

$$\begin{cases} A \coloneqq \{g \in G \mid g \cdot x \in H \cdot x\} \\ B \coloneqq \{g \in G \mid g \cdot (g_1 \cdot x) \in H \cdot (g_1 \cdot x)\} \end{cases}$$

are conull in G, and therefore $Ag_1^{-1} \cap B$ is conull in G. Take now g in $Ag_1^{-1} \cap B$. Then, $gg_1 \cdot x \in (H \cdot x) \cap (H \cdot (g_1 \cdot x))$, which means that the H-orbits of x and $(g_1 \cdot x)$ intersect non trivially, so $g_1 \cdot x$ is in $H \cdot x$. □

We will finally be needing the following proposition, in which the main argument is from [Kec10, Sec. I.4], before combining Theorem 3.5.28 and Proposition 3.5.29.

Proposition 3.5.30. *If $\mathbb{G}$ is an ergodic subgroup of* $\mathrm{Aut}(X,\lambda)$*, any non-singular bijection S of (X,λ) that verifies $S\mathbb{G}S^{-1} \leqslant \mathrm{Aut}(X,\lambda)$ preserves λ up to multiplication by an element of $\mathbb{R}_+^*$.*

Proof. As S is non-singular, by the Radon-Nikodym theorem there exists a unique non-zero Borel function $f : X \to \mathbb{R}_+$ such that for any Borel subset A we have

$$S_*\lambda(A) = \int_A f d\lambda.$$

We will show that f is actually essentially constant. Let $U = STS^{-1}$ be an element of $S\mathbb{G}S^{-1}$. Start by noticing that U preserves $S_*\lambda$. Indeed, as T preserves λ, we have

$$U_*S_*\lambda = (STS^{-1})_*S_*\lambda = S_*T_*{S^{-1}}_*S_*\lambda = S_*T_*\lambda = S_*\lambda.$$

Now for any Borel subset A of X we have

$$U_*(S_*\lambda)(A) = \int_{U^{-1}A} f(x)d\lambda(x) = \int_A f(U^{-1}x)d\lambda(U^{-1}x) = \int_A f(U^{-1}x)d\lambda(x),$$

as $U = STS^{-1}$ preserves λ, because $U \in S\mathbb{G}S^{-1} \leqslant \mathrm{Aut}(X,\lambda)$. This means that the Radon-Nikodym derivative of $U_*(S_*\lambda)$ is $f \circ U^{-1}$. We previously proved that this pushforward measure $U_*(S_*\lambda)$ is equal to $S_*\lambda$, so by uniqueness $f = f \circ U^{-1}$, which means that f is constant, up to a null set, by Proposition 1.2.5. $\square$

We finally obtain the following.

Theorem 3.5.31. *Let G and H be two Polish locally compact groups acting in a Borel measure-preserving ergodic manner on a standard σ-finite space (X,λ). Let $\Psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ be a group isomorphism. Then there exists an orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$, such that Ψ is the conjugation by S,* i.e. *for all $T \in [\mathcal{R}_G]$ we have*

$$\Psi(T) = STS^{-1} \in [\mathcal{R}_H].$$

In particular, $([\mathcal{R}_G], \tau_m^o)$ and $([\mathcal{R}_H], \tau_m^o)$ are homeomorphic.

Proof. Proposition 3.5.27 ensures us that $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$ have many involutions. We apply Fremlin's theorem (Theorem 3.5.28) between $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$, then we apply Proposition 3.5.29 both ways to construct the conull subset X_0 of Definition 3.5.23, and Proposition 3.5.30 to verify that the non-singular bijection S preserves λ up to a multiplication by a positive scalar. Finally, Ψ is a homeomorphism by Lemma 3.5.25. $\square$

Remark 3.5.32. As noted by Emmanuel Roy, if we denote by $\mathrm{Aut}(X,\mathbb{R}_+^*\lambda)$ the group of Borel bijections of X that preserve λ up to multiplication by a positive scalar, by considering the natural map sending $T \in \mathrm{Aut}(X,\mathbb{R}_+^*\lambda)$ to this scalar,

$$1 \to \mathrm{Aut}(X,\lambda) \hookrightarrow \mathrm{Aut}(X,\mathbb{R}_+^*\lambda) \to \mathbb{R}_+^* \to 1$$

is a split exact short sequence. In particular $\mathrm{Aut}(X,\mathbb{R}_+^*\lambda)$ decomposes as a semi-direct product $\mathrm{Aut}(X,\lambda) \rtimes \mathbb{R}_+^*$. The conjugations by orbit equivalences we consider in this section are inner automorphisms of $\mathrm{Aut}(X,\mathbb{R}_+^*\lambda)$ that are not inner automorphisms of $\mathrm{Aut}(X,\lambda)$.

3.6 Polishability and uniqueness of the Polish topology on ergodic full groups

3.6.1 Refining the weak topology

The arguments of the beginning of this section have been used by Kallman in [Kal85] to prove that $\mathrm{Aut}(X,\lambda)$ (and $\mathrm{Aut}(X,\mu)$) carry a unique Polish topology, and later by Le Maître in [LM22] to prove that the normal subgroup $\mathrm{Aut}_f(X,\lambda)$ of finitely supported elements of $\mathrm{Aut}(X,\lambda)$ cannot carry a Polish topology. The main result of this section, Theorem 3.6.4, has been proved by Carderi and Le Maître in [CLM16, Sect. 4.2] for the case of a standard probability space, and we use the exact same arguments. The important observation was that Kallman's technique was generalizable to ergodic full groups thanks to the probability version of Corollary 3.4.13.

In this whole section, unless stated otherwise, $\mathbb{G}$ will be an ergodic full group, subgroup of $\mathrm{Aut}(X,\lambda)$, where as usual (X,λ) is a standard σ-finite space. We also fix a Hausdorff topology τ on $\mathbb{G}$.

Let us start by introducing the following notation: for any $\varepsilon > 0$, we set

$$F_\varepsilon^A := \{T \in \mathbb{G} \mid \lambda(T(A) \setminus A) \leqslant \varepsilon\}.$$

We will be needing the following lemmas, proved and used in [LM22, Sect. 2.2].

Lemma 3.6.1. *For any Borel subset A of X, the set $\mathbb{G}_{X\setminus A}$ is τ-closed.*

Proof. We prove that $\mathbb{G}_{X\setminus A} = \mathcal{C}(\mathbb{G}_A)$, where $\mathcal{C}(\mathbb{G}_A) = \{T \in \mathbb{G} \mid TU = UT \text{ for all } U \in \mathbb{G}_A\}$ is the centralizer of $\mathbb{G}_A$ in $\mathbb{G}$. First note that $\mathbb{G}_{X\setminus A}$ is obviously a group. We clearly have $\mathbb{G}_{X\setminus A} \leqslant \mathcal{C}(\mathbb{G}_A)$.

For the reverse inclusion, take an element T in $\mathbb{G}$ such that $T \notin \mathbb{G}_{X\setminus A}$ and such that there exists B a Borel subset of A, satisfying $T(B) \cap B = \emptyset$. If now U is non-trivial in $\mathbb{G}$, and $\mathrm{supp}\, U \subseteq B$, (which exists thanks to Corollary 3.4.13), U and T do not commute, which proves that $T \notin \mathcal{C}(\mathbb{G}_A)$.

We conclude that $\mathbb{G}_{X\setminus A}$ is closed as a centralizer in a topological group. ☐

Lemma 3.6.2. *For any Borel subsets A and B of X, the set $\mathbb{G}_{(A,B)}$ is τ-closed.*

Proof. We first establish that for all T in $\mathbb{G}$, we have the following equality:

$$\mathbb{G}_{T(A)} = T\mathbb{G}_A T^{-1}. \tag{3.1}$$

Indeed, if T is in $\mathbb{G}$ we have

$$\begin{aligned} T\mathbb{G}_A T^{-1} &= \left\{TST^{-1}, S \in \mathbb{G} \mid \mathrm{supp}\, S \subseteq A\right\} \\ &= \left\{U \in \mathbb{G} \mid \mathrm{supp}\,(T^{-1}UT) \subseteq A\right\}, \end{aligned}$$

and if U is in $\mathbb{G}_{T(A)}$, we have

$$\mathrm{supp}\, U \subseteq T(A) \Leftrightarrow T^{-1}\mathrm{supp}\, U \subseteq A \Leftrightarrow \mathrm{supp}\,(T^{-1}UT) \subseteq A.$$

Observe now that we have the following:

$$A \subseteq B \Longleftrightarrow \mathbb{G}_A \leqslant \mathbb{G}_B.$$

The direct implication is clear. For the reverse implication, consider a Borel subset A of X, which is not a subset of B. Corollary 3.4.13 ensures the existence of a T in $\mathbb{G}$ supported on $A \setminus B$, and therefore T is in $\mathbb{G}_A$ but not in $\mathbb{G}_B$.

Finally, equality (3.1) ensures us that

$$\mathbb{G}_{(A,B)} = \left\{T \in \mathbb{G} \mid \mathbb{G}_{T(A)} \leqslant \mathbb{G}_B\right\} = \left\{T \in \mathbb{G} \mid T\mathbb{G}_A T^{-1} \leqslant \mathbb{G}_B\right\}.$$

We conclude this proof by the same argument as the one used in the proof of Lemma 3.6.1. Indeed $\mathbb{G}_{(A,B)}$ is the set of all $T \in \mathbb{G}$ such that for all $U \in \mathbb{G}_A$, TUT^{-1} commutes with every element of $\mathbb{G}_{X\setminus B}$. This is a closed condition for our Hausdorff topology, which ensures us that $\mathbb{G}_{(A,B)}$ is τ-closed. □

Lemma 3.6.3. *Let $\varepsilon > 0$, and consider two Borel subsets $A \subseteq B \subseteq X$ such that $\lambda(B\setminus A) = \varepsilon$. We have $\mathbb{G}_{X\setminus A} \cdot \mathbb{G}_{(A,B)} = F_\varepsilon^A$.*

Proof. We observe that F_ε^A is left $\mathbb{G}_{X\setminus A}$-invariant. Indeed, let T be in F_ε^A, and U in $\mathbb{G}_{X\setminus A}$. We have $T(A) = (T(A) \cap A) \sqcup D$, where D is disjoint from A and has measure less than ε. Then,

$$\begin{aligned}U(T(A)) &= U((T(A) \cap A)) \sqcup U(D)\\ &= (T(A) \cap A) \sqcup U(D),\end{aligned}$$

as U is the identity on A, and sends D on a subset of the same measure, disjoint from A. Therefore, $\lambda(UT(A) \setminus A) = \lambda(U(D) \setminus A) = \lambda(U(D)) \leqslant \varepsilon$, which means that UT is in F_ε^A.

Having chosen A and B such that $\lambda(B \setminus A) = \varepsilon$ ensures us that $\mathbb{G}_{(A,B)}$ is a subset of F_ε^A, and so we have $\mathbb{G}_{X\setminus A} \cdot \mathbb{G}_{(A,B)} \subseteq F_\varepsilon^A$.

For the converse inclusion, we fix $T \in F_\varepsilon^A$. As $\lambda(T(A) \setminus A) \leqslant \varepsilon$ and $\lambda(B \setminus A) = \varepsilon$, we can find $U \in \mathbb{G}_{X\setminus A}$ such that $U(T(A)\setminus A) \subseteq (B\setminus A)$, thanks to Corollary 3.4.13. Therefore, UT sends A to a subset of B, which exactly means that UT is in $\mathbb{G}_{(A,B)}$, and then that T is in $\mathbb{G}_{X\setminus A} \cdot \mathbb{G}_{(A,B)}$, as $\mathbb{G}_{X\setminus A}$ is a group. □

Theorem 3.6.4. *Let (X, λ) be a σ-finite space, and let $\mathbb{G}$ be an ergodic full group on (X, λ).*

(1) Any Polish group topology on $\mathbb{G}$ refines the weak topology.

(2) The group $\mathbb{G}$ carries at most one Polish topology.

Proof. (1) Let τ be a Polish group topology on $\mathbb{G}$. Fix $\varepsilon > 0$, and $A \subseteq B$ two Borel subsets of Y such that $\lambda(B \setminus A) = \varepsilon$. The previous lemmas ensure us that F_ε^A is analytic, as the pointwise product of two closed sets. Therefore the identity map $(\mathbb{G}, \tau) \to (\mathrm{Aut}(X, \lambda), \tau_w)$, where τ_w is the weak topology, is Baire-measurable, hence continuous thanks to Proposition 1.1.24. So the Polish topology τ refines τ_w on $\mathbb{G}$.

(2) Let now τ and τ' be two Polish topologies on $\mathbb{G}$. By *(1)* both topologies refine the weak topology τ_w, so both of the following inclusion maps are Borel:

$$\begin{aligned}(\mathbb{G}, \tau) &\longrightarrow (\mathrm{Aut}(X, \lambda), \tau_w)\\ (\mathbb{G}, \tau') &\longrightarrow (\mathrm{Aut}(X, \lambda), \tau_w).\end{aligned}$$

Applying Theorem 1.1.7, we observe that any τ-Borel set is a τ_w-Borel set, and the same thing is true for τ'. The Borel σ-algebras induced by τ, τ' and τ_w are therefore the same. Theorem 1.1.24 applied both ways between $(\mathbb{G}, \tau)$ and $(\mathbb{G}, \tau')$ concludes the proof, as it ensures us that τ and τ' both refine each other. □

3.6.2 Hopf decomposition and further factorisation

In this section use the notions of *conservative and dissipative parts* of an infinite measure-preserving bijection, as well as the associated *Hopf decomposition* (see Section 1.2.4) to factorise any measure-preserving bijection T which has a support of infinite measure. We write it as a product of three bijections, two of which have disjoint supports with infinite measure, and the third having a support of finite measure. It is also possible to add a condition on the measure of the intersection of the supports with any fixed Borel set of positive measure. We use this factorisation in Section 3.6.3 to decrease the known Steinhaus exponent of an ergodic full group $\mathbb{G}$ with the uniform topology.

Remark 3.6.5. For any measure-preserving bijection T, we recall that we have the refined Hopf decomposition: $X = \mathfrak{D}(T) \sqcup \mathfrak{C}_f(T) \sqcup \mathfrak{C}_\infty(T)$. Moreover, as it is a partition of X into T-invariant parts, we have $T = T_{\mathfrak{D}} T_{\mathfrak{C}_f} T_{\mathfrak{C}_\infty}$, where $T_{\mathfrak{D}}$ is defined by T on the T-orbits in $\mathfrak{D}(T)$, and by id_X elsewhere, and the other constituents $T_{\mathfrak{C}_f}$ and $T_{\mathfrak{C}_\infty}$ are defined similarly. In particular, each factor is in the full group $[T]$ (in particular the support of each factor is included in $\operatorname{supp} T$), and those three bijections commute, as their supports consist of different T-orbits.

Remark 3.6.6. The T-action on $\operatorname{supp} T$ admits a Borel fundamental domain when T is either conservative periodic or dissipative. Indeed, if $\operatorname{supp} T \subseteq \mathfrak{D}(T)$ this is a direct consequence of the definition. If $\operatorname{supp} T \subseteq \mathfrak{C}_f(T)$, identify $(\operatorname{supp} T, \lambda_{\restriction \operatorname{supp} T})$ with $(\mathbb{R}, \mathrm{Leb})$ if it has infinite measure, and with an interval with the restriction of the Lebesgue measure if it has finite measure. Taking the smallest element in each orbit provides us with a suitable set.

We now factorise $T \in \mathrm{Aut}(X, \lambda)$ by treating separately the cases of T dissipative or conservative periodic (in which case we use a Borel fundamental domain), or T conservative aperiodic, which requires additional tools.

Lemma 3.6.7. *Let (X, λ) be a standard σ-finite space, and $T \in \mathrm{Aut}(X, \lambda)$ be such that the T-action on X has a Borel fundamental domain. Let $D \subseteq X$ be of positive measure (possibly infinite). We can write $T = T_1 T_2$, with T_1 and T_2 in $[T]$ satisfying the following:*

- $\operatorname{supp} T_1$ *and* $\operatorname{supp} T_2$ *are disjoint and have equal measure,*
- $\lambda(\operatorname{supp} T_1 \cap D) = \lambda(\operatorname{supp} T_2 \cap D)$.

Proof. Denote by A a Borel fundamental domain for the T-action on its support (see Remark 3.6.6). We denote by $T_{\mathrm{sat}}(C)$ the T-saturation of any subset C, and we have $T_{\mathrm{sat}}(A) = \operatorname{supp} T$. We consider (up to measure zero) the following set:

$$A_D \coloneqq \{x \in A : |\mathrm{orb}_\mathrm{T}(x) \cap D| \neq 0\},$$

and we will also denote by B the complement of A_D in A. There are two cases to consider, depending on the measure of A_D:

(1). If $\lambda(A_D) = +\infty$, then it suffices to cut A_D and B in half: write $A_D = A_D^1 \sqcup A_D^2$ with $\lambda(A_D^1) = \lambda(A_D^2) = +\infty$ and write $B = B^1 \sqcup B^2$ with $\lambda(B^1) = \lambda(B^2)$. We define (for $i \in \{1, 2\}$) T_i by T on $T_{\mathrm{sat}}(A_D^i \sqcup B^i)$, and by id_X elsewhere. We have

$$\lambda(\operatorname{supp} T_i \cap D) = \int_X \mathbb{1}_{A_D^i}(x) \times |\mathrm{orb}_\mathrm{T}(x) \cap D| \, d\lambda(x) \geqslant \lambda(A_D^i) = +\infty,$$

and $\operatorname{supp} T_1$ is disjoint from $\operatorname{supp} T_2$ since they are comprised of different T-orbits.

(2). If $\lambda(A_D) < +\infty$, cutting A_D in half and defining T_1 and T_2 respectively by T or id_X on each part is not enough, as there is no guarantee that the supports will be of equal measure, let alone their intersections with D.

We then cut both A_D and B in slices that depend on the cardinal of the orbits of their points. That is to say that for any n in $\mathbb{N}^*$ we set

$$A_{n,D} = \{x \in A_D : |\mathrm{orb}_T(x)| = n\} \text{ and } A_{\infty,D} = \{x \in A_D : |\mathrm{orb}_T(x)| = +\infty\}$$

as well as

$$B_n = \{x \in B : |\mathrm{orb}_T(x)| = n\} \text{ and } B_\infty = \{x \in B : |\mathrm{orb}_T(x)| = +\infty\}.$$

We have

$$\begin{cases} A_D &= \bigsqcup_{n\in\mathbb{N}^*\cup\{\infty\}} A_{n,D} \\ B &= \bigsqcup_{n\in\mathbb{N}^*\cup\{\infty\}} B_n. \end{cases}$$

We then furthermore cut the slices $A_{n,D}$ depending on how many times the orbit of their points intersect D: we define $A_{k,n,D} \coloneqq \{x \in A_{n,D} : |\mathrm{orb}_T(x) \cap D| = k\}$, for any $k \leqslant n$. We then have

$$A_D = \bigsqcup_{n\in\mathbb{N}^*\cup\{\infty\}} \bigsqcup_{k\leqslant n} A_{k,n,D}.$$

For any n we can then cut B_n in half: define B_n^1 and B_n^2 such that $B_n = B_n^1 \sqcup B_n^2$ and $\lambda(B_n^1) = \lambda(B_n^2)$, and do the same for $A_{k,n,D}$, for any $k \leqslant n$. We put the two families of pieces together: for $i \in \{1,2\}$ let A_D^i and B^i be defined by

$$\begin{cases} A_D^i = \bigsqcup_{n\in\mathbb{N}^*\cup\{\infty\}} \left(\bigsqcup_{k\leqslant n} A_{k,n,D}^i \right) \\ B^i = \bigsqcup_{n\in\mathbb{N}^*\cup\{\infty\}} B_n^i \end{cases}$$

and let (for $i \in \{1,2\}$) T_i be defined by T on $T_{\mathrm{sat}}(A_D^i \sqcup B^i)$, and by id_X elsewhere. By construction we have

$$\begin{aligned} \lambda(\operatorname{supp} T_1 \cap D) &= \sum_{n\in\mathbb{N}^*\cup\{\infty\}} \sum_{k\leqslant n} k\lambda(A_{k,n,D}^1) \\ &= \sum_{n\in\mathbb{N}^*\cup\{\infty\}} \sum_{k\leqslant n} k\lambda(A_{k,n,D}^2) = \lambda(\operatorname{supp} T_2 \cap D). \end{aligned}$$

We also have

$$\begin{aligned} \lambda(\operatorname{supp} T_1 \cap T_{\mathrm{sat}}(A_D)) &= \sum_{n\in\mathbb{N}^*\cup\{\infty\}} \sum_{k\leqslant n} n\lambda(A_{k,n,D}^1) \\ &= \sum_{n\in\mathbb{N}^*\cup\{\infty\}} \sum_{k\leqslant n} n\lambda(A_{k,n,D}^2) = \lambda(\operatorname{supp} T_2 \cap T_{\mathrm{sat}}(A_D)). \end{aligned}$$

And finally, we have

$$\begin{aligned} \lambda(\operatorname{supp} T_1 \cap T_{\mathrm{sat}}(B)) &= \sum_{n\in\mathbb{N}^*\cup\{\infty\}} n\lambda(B_n^1) \\ &= \sum_{n\in\mathbb{N}^*\cup\{\infty\}} n\lambda(B_n^2) = \lambda(\operatorname{supp} T_2 \cap T_{\mathrm{sat}}(B)). \end{aligned}$$

Those equalities hold whether the quantities considered are finite or not, and the supports of T_1 and T_2 are disjoint since they are comprised of different T-orbits. Finally $\lambda(\operatorname{supp} T_1) = \lambda(\operatorname{supp} T_2)$ thanks to the fact that $\operatorname{supp} T = T_{\text{sat}}(A) = T_{\text{sat}}(A_D) \sqcup T_{\text{sat}}(B)$. This concludes the proof, as T_1 and T_2 are in $[T]$ by construction. $\square$

For the next part, we will need to recall the definition of an induced bijection on a Borel subset.

Definition 3.6.8. Consider (X, λ) a standard σ-finite space. Let $T \in \operatorname{Aut}(X, \lambda)$, and $A \subseteq \mathfrak{C}(T)$ be a Borel subset with $\lambda(A) > 0$. Theorem 1.2.19 ensures us that for λ-almost every $x \in A$ there exists a (finite) **smallest return time** in A denoted by $n_A(x)$, *i.e.*

$$n_A(x) \coloneqq \min\{n \in \mathbb{N}^* \mid T^n(x) \in A\}.$$

We can then define the **induced bijection** T_A by $T_A(x) = T^{n_A(x)}(x)$ for all $x \in A$ and $T_A(x) = x$ for all $x \notin A$. Partitioning A in sets of the form $\{x \in A \mid n_A(x) = n\}$ easily yields that $T_A \in [T]$.

As stated before, the case of a T which is conservative aperiodic is the most complicated, as the T-action does not necessarily admit a Borel fundamental domain. We instead use the marker lemma to construct a Borel subset which intersects each orbit, and has controllable measure. The following is a classical infinite measure generalization of Lemma 1.2.14.

Lemma 3.6.9. *Let (X, λ) be a standard σ-finite space, and $T \in \operatorname{Aut}(X, \lambda)$ be conservative aperiodic on its support. For any $\varepsilon > 0$ there exists a Borel subset C of $\operatorname{supp} T$ such that C intersects λ-almost all the T-orbits, and $0 < \lambda(C) < \varepsilon$.*

Proof. Let $\varepsilon > 0$. Since λ is atomless and σ-finite we can write $\operatorname{supp} T$ as $\operatorname{supp} T = \bigsqcup_{n \in \mathbb{N}} X_n$, with $\lambda(X_n)$ finite for all $n \in \mathbb{N}$. By conservativity, for λ-almost all $x \in X_n$ the T-orbit of x intersects X_n infinitely many times, in other words after throwing away a null set, the restriction $\mathcal{R}_n$ of the equivalence relation $\mathcal{R}_T = \{(x, T^k x) \mid x \in X, k \in \mathbb{Z}\}$ to X_n is aperiodic. By the marker lemma [KM04, Lem. 6.7], for every n there is a decreasing sequence $(C_{n,k})_k$ of Borel subsets of X_n which intersects every $\mathcal{R}_n$-class and satisfies $\bigcap_k C_{n,k} = \emptyset$. In particular, since $\lambda(X_n)$ is finite, for some large enough k_n we have $\lambda(C_{n,k_n}) < \varepsilon 2^{-n-1}$. It follows that $C \coloneqq \bigsqcup_n C_{n,k_n}$ is as desired. $\square$

Lemma 3.6.10. *Let (X, λ) be a standard σ-finite space, and $T \in \operatorname{Aut}(X, \lambda)$ be conservative aperiodic on its support. Let $D \subseteq X$ be of positive measure (possibly infinite). For any $\varepsilon > 0$ we can write $T = T_1 T_2 T_\varepsilon$, with T_1, T_2 and T_ε in $[T]$ satisfying the following:*

- *$\operatorname{supp} T_1$ and $\operatorname{supp} T_2$ are disjoint and have equal measure,*
- *$\lambda(\operatorname{supp} T_1 \cap D) = \lambda(\operatorname{supp} T_2 \cap D)$,*
- *T_ε is aperiodic and $\lambda(\operatorname{supp} T_\varepsilon) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. We start by using Lemma 3.6.9 to define a Borel subset C of X intersecting λ-almost all the T-orbits, and such that $0 < \lambda(C) < \varepsilon$.

We then write $T = T T_C^{-1} T_C$. The support of T_C has measure less than ε, and $\operatorname{supp} T T_C^{-1}$ has infinite measure whenever $\operatorname{supp} T$ does.

Let us then prove that $T T_C^{-1}$ has finite orbits. The situation is summarized by Figure 3.3.

We consider one T-orbit, and identify it to $\mathbb{Z}$, equipped with its natural order. Let x_1 and x_2 be two consecutive elements of that orbit that are in C (Theorem 1.2.19 ensures us that they exist). The integer interval $\{x_1 + 1, \ldots, x_2\}$ is $T T_C^{-1}$-invariant. Indeed, $T T_C^{-1}(x_2) = T(x_1) = x_1 + 1$, and applying $T T_C^{-1}$ to an element between x_1 and x_2 merely

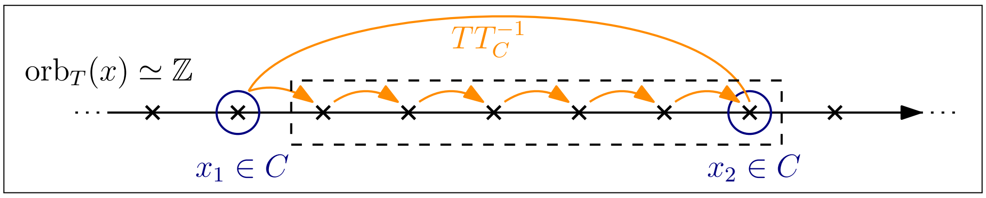

Figure 3.3: Visualisation of the periodicity of TT_C^{-1} on a T-orbit. On the example $n_C(x_1) = 6$.

moves it along the T-orbit, as T_C is trivial outside of C. This means that the TT_C^{-1}-orbit containing x_2 is finite, and contains $x_2 - x_1 = n_C(x_1)$ elements.

Thanks to Lemma 3.6.7, we can then write TT_C^{-1} as the product of two measure-preserving bijections T_1, T_2 with disjoint supports of equal measure and satisfying the first two parts of the statement. This concludes the proof,as $T_C = T_\varepsilon$ is suitable. □

Remark 3.6.11. The previous proof also yields the following, which is most likely well-known, but for which we have found no reference: for any $\varepsilon > 0$, for any conservative T in $\mathrm{Aut}(X, \lambda)$, there exists a periodic (hence conservative) T' in $[T]$ such that T' and T are ε-uniformly close.

We can now combine the previous lemmas to obtain the following factorisation.

Proposition 3.6.12. *Let (X, λ) be a standard σ-finite space, and consider T in $\mathrm{Aut}(X, \lambda)$. Let $D \subseteq X$ be of positive measure (possibly infinite). For any $\varepsilon > 0$ we can write $T = T_1T_2T_\varepsilon$, with T_1, T_2 and T_ε in $[T]$ satisfying the following:*

- $\mathrm{supp}\, T_1$ *and* $\mathrm{supp}\, T_2$ *are disjoint and of equal measure,*
- $\lambda(\mathrm{supp}\, T_1 \cap D) = \lambda(\mathrm{supp}\, T_2 \cap D)$*,*
- T_ε *is aperiodic and* $\lambda(\mathrm{supp}\, T_\varepsilon) < \varepsilon$*.*

Proof. We write $T = T_{\mathfrak{D}} T_{\mathfrak{C}_f} T_{\mathfrak{C}_\infty}$ as in Remark 3.6.5, with $T_{\mathfrak{D}}$, $T_{\mathfrak{C}_f}$ and $T_{\mathfrak{C}_\infty}$ commuting with each other.

We have $T_{\mathfrak{D}} = T_{\mathfrak{D}}^1 T_{\mathfrak{D}}^2$ and $T_{\mathfrak{C}_f} = T_{\mathfrak{C}_f}^1 T_{\mathfrak{C}_f}^2$ thanks to Lemma 3.6.7, where the supports of the two factors (for both factorisations) are disjoint, of equal measure, and meet D on sets of equal measure. Lemma 3.6.10 ensures us that $T_{\mathfrak{C}_\infty} = T_{\mathfrak{C}_\infty}^1 T_{\mathfrak{C}_\infty}^2 T_{\mathfrak{C}_\infty}^\varepsilon$, where the supports of $T_{\mathfrak{C}_\infty}^1$ and $T_{\mathfrak{C}_\infty}^2$ are disjoint and of equal measure, and meet D on sets of equal measure, and the support of $T_{\mathfrak{C}_\infty}^\varepsilon$ has measure less than ε.

We can finally define the following measure-preserving bijections:

$$\begin{cases} T_1 = T_{\mathfrak{D}}^1 T_{\mathfrak{C}_f}^1 T_{\mathfrak{C}_\infty}^1 \\ T_2 = T_{\mathfrak{D}}^2 T_{\mathfrak{C}_f}^2 T_{\mathfrak{C}_\infty}^2 \\ T_\varepsilon = T_{\mathfrak{C}_\infty}^\varepsilon \end{cases}$$

which concludes the proof in this case. Indeed by construction the supports of the factors of $T_{\mathfrak{D}}$ (respectively $T_{\mathfrak{C}_f}$, $T_{\mathfrak{C}_\infty}$) are included in the support of $T_{\mathfrak{D}}$ (respectively $T_{\mathfrak{C}_f}$, $T_{\mathfrak{C}_\infty}$) so there is no commutativity issue arising during the factorisation. □

3.6.3 Coarsening the uniform topology

The following question is very natural to ask: When is the uniform topology on a full group a Polish group topology? Carderi and Le Maître gave a complete answer in the probability case ([CLM16, Prop. 3.8, Thm. 4.7]), by using a technique of Kittrell and Tsankov on automatic continuity for ergodic full groups ([KT10, Thm. 3.1]). It works the same for infinite σ-finite measures. Let us start off by recalling the following.

Proposition 3.6.13 ([Dye59, Lem. 5.4]). *The restriction of the uniform metric to a full group of* $\mathrm{Aut}(X,\mu)$ *is complete, where* (X,μ) *is a standard probability space.*

This useful result has been used in [CLM16], and in order to generalize, we use Proposition 3.4.7 to answer the question of completeness, however separability proves to be the real obstacle in this case. Following [CLM16], we prove that a full group can only be τ_u-separable if and only if it is the full group of a countable equivalence relation. The following proof can be found in [LM14a, Prop. 1.25].

Proposition 3.6.14. *Let* $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ *be a full group. It is separable for the uniform topology if and only if it is the full group of a countable equivalence relation.*

Proof. Proposition 3.3.18 ensures us that proving the statement for any topology defined in Section 3.3.3 is enough, so we start by fixing a probability measure μ in $[\lambda]$.

($\Rightarrow$) First assume that Γ is a dense countable subgroup of $\mathbb{G}$. We can consider $[\Gamma]$ the full group generated by Γ, that is to say the saturation of Γ with regard to cutting and pasting. $[\Gamma]$ is closed, thanks to Proposition 3.4.7, and thus is equal to $\mathbb{G}$. Therefore $\mathbb{G}$ is the full group of the countable equivalence relation given by Γ.

($\Leftarrow$) Now assume that $\mathbb{G}$ is equal to $[\mathcal{R}]$, where is $\mathcal{R}$ is countable. For a Borel subset $A \subseteq \mathcal{R}$, we define

$$M_l(A) = \int_X |A_x| d\mu(x)$$

where $A_x = \{y \in X \mid (x,y) \in A\} \subseteq \mathrm{orb}_{\mathcal{R}}(x)$. Notice now that for any partial isomorphism ϕ in $[[\mathbb{G}]]$ we have $M_l(\mathrm{graph}(\phi)) = \mu(\mathrm{dom}(\phi))$. Thus, it follows from the theorem of Lusin-Novikov (see *e.g.* [Kec95, Thm. 18.10]) that M_l is σ-finite, and it is atomless because μ is atomless and $\mathcal{R}$ is countable. Therefore, if S, T are two elements of $\mathbb{G}$,

$$d_\mu(S,T) = \mu\left(\{x \in X \mid S(x) \neq T(x)\}\right) = \frac{1}{2} M_l(\mathrm{graph}(S)\Delta\mathrm{graph}(T)).$$

As $\hat{d}(A,B) \coloneqq M_l(A\Delta B)$ is a separable distance on $\mathrm{MAlg}_f(\mathcal{R}, M_l)$ by Proposition 3.2.2, we just proved that $\mathbb{G}$ is separable for the uniform topology, as a metric subspace of a separable metric space. ☐

We now prove, following Kittrell and Tsankov in [KT10], that should a full group $\mathbb{G}$ have a Polish topology, it would necessarily be weaker that the uniform topology. We need a few preliminary results, including a key proposition from [RS07], which gives us a condition called Steinhaus implying automatic continuity. It uses the notion of countable syndeticity, which we recall.

Definition 3.6.15. A subset V of a group G is **countably syndetic** if countably many translates of V cover G, *i.e.* if there exists $(g_n)_{n\in\mathbb{N}}$ a sequence of elements of G such that $\bigcup_n g_n V = G$.

Proposition 3.6.16 ([RS07, Prop. 2]). *Let G be a topological group. If there exists $n \in \mathbb{N}$ such that for any symmetric countably syndetic subset V of G, V^n contains a open neighbourhood of e_G (we say that G is n-**Steinhaus**), then any morphism $G \to H$, where H is a separable group, is continuous.*

*If there exists a integer $n \in \mathbb{N}^*$ such that G is n-Steinhaus, we say that G is **Steinhaus**. Therefore we have the following reformulation, more concise: any Steinhaus topological group has automatic continuity.*

We then have the following result, which follows from Lemma 3.4.21.

Lemma 3.6.17 ([Fre06, Lem. 494M]). *Let $\mathbb{G}$ be an ergodic full group, and $V \subseteq \mathbb{G}$ a symmetric subset. Let C be a Borel subset of X, and let U and U' respectively be an (A, B)-exchanging involution and an (A', B')-exchanging involution, with A, B, A' and B' Borel subsets of C satisfying the measure conditions of* Definition 3.4.15. *Suppose the following:*

1. *$\lambda(A) = \lambda(A')$ and $\lambda(C \setminus (A \sqcup B)) = \lambda(C \setminus (A' \sqcup B'))$;*
2. *$U \in V$;*
3. *for all T in $\mathbb{G}_C$, there exists a S in V, agreeing with T on C.*

Then U and U' are conjugate in $\mathbb{G}_C$ and U' is in V^3.

Proof. Lemma 3.4.21 directly yields a $T \in \mathbb{G}_C$ such that $UT = TU'$, so we only have to show that $U' \in V^3$. By the last hypothesis of the statement, there exists a $S \in V$ agreeing with T on C. We then have

$$\begin{cases} SU'(D) = TU'(D) = UT(D) = US(D) & \text{if } D \subseteq C, \\ SU'(D) = S(D) = US(D) & \text{if } D \subseteq X \setminus C. \end{cases}$$

So $U' = S^{-1}US$, which means that $U' \in V^3$, as $S^{-1} \in V$ by symmetry and $U \in V$. ☐

We can now prove the following. The details are from Fremlin ([Fre06, 494Y(i)], the proof is unpublished), based on the arguments of Kittrell and Tsankov. Proposition 3.6.12 is used in a later part of the proof to reduce the required exponent of the symmetric countably syndetic subset used.

Theorem 3.6.18. *Let $\mathbb{G}$ be an ergodic full group. Let V be a symmetric countably syndetic subset of $\mathbb{G}$. Then, V^{114} contains an open neighbourhood of e_G for the uniform topology. In particular, $(\mathbb{G}, \tau_u)$ is 114-Steinhaus.*

Proof. By syndeticity, we fix a sequence (ϕ_n) of elements of $\mathbb{G}$ such that $\bigcup_n \phi_n V = \mathbb{G}$. Do note at this point that $e_{\mathbb{G}} \in V^2$. Indeed, there exists n in $\mathbb{N}$ such that $e_{\mathbb{G}} \in \phi_n V$, so $\phi_n^{-1} \in V$. As V is symmetric, this ensures us that $e_{\mathbb{G}}$ is in V^2.

Claim 1. There exists a Borel subset $C' \subseteq X$ of infinite measure, such that for all $T \in \mathbb{G}_{C'}$, there exists $S \in V^2$ agreeing with T on C'.

Proof of Claim 1: Let $X = \bigsqcup_{n\in\mathbb{N}} X_n$, with $\lambda(X_n) = +\infty$ for any n. Suppose that for all $n \in \mathbb{N}$, there exists T_n in $\mathbb{G}_{X_n}$, such that there exists no S in V^2 agreeing with T_n on X_n. For any n in $\mathbb{N}$, do note that $V^2 = (\phi_n V)^{-1}(\phi_n V)$, and $T_n = e_{\mathbb{G}}^{-1} T_n$. Therefore, in order for T_n to disagree with all elements of V^2, either $e_{\mathbb{G}}$ or T_n has to disagree with every element of $\phi_n V$ on X_n. We can then define $T'_n \in \mathbb{G}_{X_n}$ as follows:

$$T'_{n\restriction X_n} = \begin{cases} e_{\mathbb{G}\restriction X_n} & \text{if } e_{\mathbb{G}} \text{ disagrees with every } S \text{ in } \phi_n V, \\ T_{n\restriction X_n} & \text{if } T_n \text{ disagrees with every } S \text{ in } \phi_n V. \end{cases}$$

If both $e_{\mathbb{G}}$ and T_n disagree with all S in $\phi_n V$, we can choose one or the other arbitrarily. Define now $T \in \mathbb{G}$ by

$$T_{\restriction X_n} = T'_{n\restriction X_n}.$$

By syndeticity, there exists $m \in \mathbb{N}$ such that $T \in \phi_m V$. This is impossible, as T agrees with T'_m on X_m by construction of T, but this contradicts the definition of T'_m, which cannot agree with an element of $\phi_m V$. Thus C' can be chosen to be one of the X_n. ◇

Claim 2. There exists an involution $U^* \in V^2$, with $C := \operatorname{supp} U^* \subsetneq C'$, and $\lambda(C) = \lambda(C' \setminus C) = +\infty$.

Proof of Claim 2: First consider $U \in \mathbb{G}$, an (A, B)-exchanging involution between two disjoint infinite measure Borel subsets A and B of C'. Its existence is ensured by Corollary 3.4.13.

Next we construct a family $(A_t)_{t\in[0,1]}$ of Borel subsets of A, with $A_0 = \emptyset$, $A_1 = A$, $A_t \subsetneq A_{t'}$ (up to null sets), and $\lambda(A_{t'} \setminus A_t) = +\infty$ whenever $t < t'$. To see how to construct such a sequence, we can construct it in $(\mathbb{R}, \mathrm{Leb}) \sim (A, \lambda_{|A})$. A_0 and A_1 are fixed, and for any $t \in]0, 1[$, we can consider

$$A_t = \bigcup_{n\in\mathbb{Z}} [n, n + t[\,.$$

This gives us an uncountable family of Borel subsets verifying the aforementioned conditions. As for all $t \in [0, 1]$, $U(A_t)$ is in B, which is disjoint from A, we can then define $U_t \in \mathbb{G}$, an $(A_t, U(A_t))$-exchanging involution supported in $A_t \sqcup U(A_t)$ and exchanging those two subsets. Corollary 3.4.13 once again ensures that U_t exists in $\mathbb{G}$. By syndeticity and cardinality, there exist $s < t$ in $[0, 1]$ and $n \in \mathbb{N}$ such that U_s and U_t are in $\phi_n V$. That means that both $\phi_n^{-1} U_s$ and $\phi_n^{-1} U_t$ are in V. We finally define

$$U^* = U_s^{-1} U_t = (\phi_n^{-1} U_s)^{-1} \phi_n^{-1} U_t \in V^2,$$

so we have $\operatorname{supp} U^* = (A_t \setminus A_s) \sqcup U(A_t \setminus A_s)$, with $\lambda(\operatorname{supp} U^*) = \lambda(C' \setminus \operatorname{supp} U^*) = +\infty$. $\diamondsuit$

Claim 3. If $U \in \mathbb{G}_C$ is an involution, then $U \in V^{12}$. In particular, $\mathbb{G}_C \subseteq V^{36}$.

Proof of Claim 3: *(i)* First assume that U is an involution in $\mathbb{G}_{C'}$, with $\lambda(\operatorname{supp} U) = \lambda(C' \setminus \operatorname{supp} U) = +\infty$. Proposition 3.4.17 ensures that there exists two disjoint Borel subsets A and B of C', both of infinite measure, such that U is an (A, B)-exchanging involution. The involution U^* is also an exchanging involution, so we define A^*, B^* such that U^* is an (A^*, B^*)-exchanging involution, with $\lambda(A) = \lambda(A^*) = \lambda(C' \setminus A) = \lambda(C' \setminus A^*) = +\infty$. Claim 2 states that U^* is in V^2 and Claim 1 ensures us that the third condition of Lemma 3.6.17 is verified, with V^2 playing the role of the symmetric set. Thus $U \in (V^2)^3 = V^6$.

(ii) Let now A and B be two Borel subsets of C, and let U be an (A, B)-exchanging involution in $\mathbb{G}_C$. If $\lambda(A) = \lambda(B) = +\infty$, fix $A_1 \subsetneq A$ and $A_2 = A \setminus A_1$, with $\lambda(A_1) = \lambda(A_2) = +\infty$. For $i \in \{1, 2\}$, define also $B_i = U(A_i)$ and U_i an (A_i, B_i)-exchanging involution, and do note that $U_i \in \mathbb{G}_{C'}$, as $\operatorname{supp} U_i = A_i \cup B_i \subseteq C \subsetneq C'$. Then $U = U_1 U_2$ can be written as the product of two involutions verifying the conditions of *(i)*, and as such is in V^{12}.

If $\lambda(A) = \lambda(B) < +\infty$, just consider A' and B' disjoint subsets of $C' \setminus C$, with $\lambda(A') = \lambda(B') = +\infty$. Fix $A_1 \subsetneq A$ and $A_2 = A \setminus A_1$ such that $0 < \lambda(A_1) = \lambda(A_2) < +\infty$. For $i \in \{1, 2\}$ define $B_i = U(A_i)$, and

$$U_i = \begin{cases} \text{an } (A_i, B_i)\text{-exchanging involution on } A \cup B, \\ \text{an } (A', B')\text{-exchanging involution on } A' \cup B', \\ e_{\mathbb{G}} \text{ elsewhere.} \end{cases}$$

The involutions U_i are supported in C' by construction, and verify the conditions of *(i)*, as $\operatorname{supp} U_i = A_i \sqcup B_i \sqcup A' \sqcup B' \subseteq C'$ satisfies $\lambda(\operatorname{supp} U_i) = \lambda(C' \setminus \operatorname{supp} U_i) = +\infty$, and $U = U_1 U_2$, which proves that $U \in V^{12}$.

Finally, Theorem 3.4.18 ensures us that $\mathbb{G}_C \subseteq V^{36}$. $\diamondsuit$

To conclude this proof, we now suppose that V^{114} does not contain an open neighbourhood of $e_{\mathbb{G}}$. We will show that it is possible to construct a sequence of measure-preserving bijections contradicting Claim 3.

Claim 4. For any two Borel subsets B and D of X such that $\lambda(B) < +\infty$ and $\lambda(D) = +\infty$ and any $\varepsilon > 0$, there exists a T in $\mathbb{G} \setminus V^{38}$ satisfying $\lambda(B \cap \operatorname{supp} T) \leqslant \varepsilon$ and $\lambda(D \setminus \operatorname{supp} T) = +\infty$.

Proof of Claim 4: Fix B, D and ε such as defined in the statement. Recalling the definition of neighbourhoods of the identity in $(\operatorname{Aut}(X, \lambda), \tau_u)$ from Remark 3.3.19, negating the fact that V^{114} contains an open neighbourhood of $e_{\mathbb{G}}$ exactly provides us with the following: there exists $T \in \mathbb{G} \setminus V^{114}$, such that $\lambda(B \cap \operatorname{supp} T) \leqslant \varepsilon$.

If $\lambda(D \cap \operatorname{supp} T) < +\infty$ then T is suitable, as $V^{38} \subseteq V^{114}$. If $\lambda(D \cap \operatorname{supp} T) = +\infty$, we use Proposition 3.6.12 to write T as a product of three measure-preserving bijections:

$$T = T^1_\infty T^2_\infty T_f.$$

T^1_∞ and T^2_∞ have disjoint supports of infinite measure, and T_f has a support of finite measure. Moreover, these three bijections have their supports included in $\operatorname{supp} T$, so they intersect B on sets of measure less than ε. For the second condition, T_f obviously intersects D on a set of finite measure, and T^1_∞ and T^2_∞ can be chosen such that the measures of the intersections of their supports with D are infinite. Thus, all three factors of T verify the conditions described in the statement of the claim, and at least one of the three is in $\mathbb{G} \setminus V^{38}$. $\diamondsuit$

We now define two sequences of Borel subsets of X, and two sequences of measure-preserving bijections. Fix $B_0 = \emptyset$ and $D_0 = C$, and for any n consider $T_n \in \mathbb{G} \setminus V^{38}$ given by Claim 4 with $(\phi_n^{-1}(B_n))$ and $(\phi_n^{-1}(D_n))$, as well as (T'_n) inductively defined as follows:

We have $\lambda(\phi_n^{-1}(B_n)) = \lambda(B_n) < +\infty$ and $\lambda(\phi_n^{-1}(D_n)) = \lambda(D_n) = +\infty$. We can choose T_n verifying $\lambda(\phi_n^{-1}(B_n) \cap \operatorname{supp} T_n) \leqslant 2^{-n}$ and $\lambda(\phi_n^{-1}(D_n) \setminus \operatorname{supp} T_n) = +\infty$. We then define $T'_n = \phi_n T_n \phi_n^{-1}$, such that we have the following

$$\begin{cases} \operatorname{supp} T'_n = \phi_n(\operatorname{supp} T'_n) \\ \lambda(B_n \cap \operatorname{supp} T'_n) \leqslant 2^{-n} \\ \lambda(D_n \setminus \operatorname{supp} T'_n) = +\infty. \end{cases} \tag{$*$}$$

We finally consider $D_{n+1} = D_n \setminus \operatorname{supp} T'_n$ (D_{n+1} has infinite measure) and B_{n+1} of finite measure such that $B_n \subseteq B_{n+1}$ and $\lambda(B_{n+1} \cap D_{n+1}) \geqslant n$.

Claim 5. Set $E = \bigcup_{n \in \mathbb{N}} \operatorname{supp} T'_n$. Then $\lambda(X \setminus E) = +\infty$.

Proof of Claim 5: We denote by E_n the support of T'_n, such that $E = \bigcup E_n$. Noticing that (B_n) is an increasing sequence and (D_n) is a decreasing sequence and using ($*$), we have

$$\begin{cases} \lambda(B_{n+1} \cap E_m) \leqslant \lambda(B_m \cap E_m) \leqslant 2^{-m} & (m \geqslant n+1) \\ \lambda\left(B_{n+1} \setminus \bigcup_{m \leqslant n} E_m\right) \geqslant \lambda(B_{n+1} \cap D_{n+1}) \geqslant n & (\forall\, n). \end{cases}$$

For any m we then have

$$\lambda(B_{n+1} \setminus E) \geqslant \lambda\left(B_{n+1} \setminus \bigcup_{m \leqslant n} E_m\right) - \sum_{m \geqslant n+1} \lambda(B_{n+1} \cap E_m) \geqslant n - 1$$

which concludes, as it is true for any n. $\diamondsuit$

The complement of E in X has infinite measure by Claim 4, so E can be sent to a subset of C: that is to say that there exists $T \in \mathbb{G}$ such that $T(E) \subseteq C$. By syndeticity again, there exists $n \in \mathbb{N}$ such that $T^{-1} \in \phi_n V$, *i.e.* $T\phi_n \in V$ by symmetry. Finally, $T\phi_n T_n \phi_n^{-1} T^{-1} = TT'_n T^{-1}$ has support $T(\operatorname{supp} T'_n) \subseteq T(E) \subseteq C$, which means that it is in V^{36} by Claim 3. This in turn means that $T_n \in V^{38}$, which contradicts the choice of T_n. $\square$

Combining Theorem 3.6.18 and Proposition 3.6.16, we obtain the following.

Theorem 3.6.19. *Let $\mathbb{G}$ be an ergodic full group. Any separable topology on $\mathbb{G}$ is weaker than the uniform topology.*

Proof. The topological group $(\mathbb{G}, \tau_u)$, where τ_u is the uniform topology, is Steinhaus thanks to Theorem 3.6.18, hence it has automatic continuity thanks to Proposition 3.6.16. Therefore the identity map $(\mathbb{G}, \tau_u) \to (\mathbb{G}, \tau)$ is continuous whenever τ is separable, which implies that τ_u refines τ. $\square$

Remark 3.6.20. Let G be a Polish group acting continuously and in a measure-preserving and ergodic manner on a Polish space X with an atomless σ-finite locally finite measure, and denote by $\mathbb{G}$ the orbit full group $[\mathcal{R}_G]$. Then for any Borel subset $A \subsetneq X$ of positive measure, the full group $\mathbb{G}_A$ is ergodic by Remark 3.4.16, and it is not *a priori* an orbit full group. As $\mathbb{G}_A$ is moreover closed in $\mathbb{G}$, it is a Polish group for the topology induced by τ_m^o. This yields examples of *a priori* non-orbit full groups where Theorem 3.6.4 and Theorem 3.6.19 apply.

3.6.4 Polishability and automatic continuity of the finitely supported elements of an ergodic full group

Recall that $\operatorname{Aut}_f(X, \lambda) = \{T \in \operatorname{Aut}(X, \lambda) \mid \lambda(\operatorname{supp} T) < \infty\}$, and that $\mathbb{G}_f = \mathbb{G} \cap \operatorname{Aut}_f(X, \lambda)$, for any $\mathbb{G} \leqslant \operatorname{Aut}(X, \lambda)$.

Definition 3.6.21. We endow $\operatorname{Aut}_f(X, \lambda)$ with $d_{u,f}$, which is defined by

$$d_{u,f} : S, T \mapsto \lambda\left(\{x \in X \mid S(x) \neq T(x)\}\right).$$

It corresponds to $d'_{u,X}$ if we allow subsets of infinite measure in Definition 3.3.15(3). It is a metric on $\operatorname{Aut}_f(X, \lambda)$ (and not on $\operatorname{Aut}(X, \lambda)$), and we call the **uniform finite topology** the topology induced by $d_{u,f}$, which we denote by $\tau_{u,f}$. It is immediate that $(\operatorname{Aut}_f(X, \lambda), \tau_{u,f})$ is a topological group and that $\tau_{u,f}$ refines τ_u.

In this section we consider the subgroup $\mathbb{G}_f$ of a fixed ergodic full group $\mathbb{G}$. As mentioned in Section 3.6.1, Le Maître proved in [LM22, Thm. 2.6] that $\operatorname{Aut}_f(X, \lambda)$ cannot carry a Polish group topology, and some of the arguments still hold if we replace $\operatorname{Aut}_f(X, \lambda)$ by $\mathbb{G}_f$.

Proving that $\operatorname{Aut}_f(X, \lambda)$ is not Polishable is done by displaying uncountably many measure-preserving bijections of the circle that are at a fixed distance of $3n$ of one another, contradicting the n-density of a countable subset of $\operatorname{Aut}_f(X, \lambda)$, which is easily established if we suppose that it is Polishable. As there is no reason for those bijections to be in $\mathbb{G}_f$, we instead prove that for any finitely supported T in $\mathbb{G}_f$, there exist in $\mathbb{G}_f$ an element whose support has measure $k\lambda(\operatorname{supp} T)$, for any k in $\mathbb{N}$. We then combine this property with the characterization of separability for the uniform topology obtained in Proposition 3.6.14 in order to contradict the n-density.

We start by defining what it means for a subgroup of a Polish group to be Polishable.

Definition 3.6.22. A subgroup H of a Polish group G is called **Polishable** if it admits a Polish group topology which refines the topology of G.

Lemma 3.6.23. *Let $\mathbb{G}$ be an ergodic full group, and $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X, \lambda)$. For any $k \geqslant 1$ we have an injective group homomorphism $\pi_k : T \mapsto T_k$ from $\mathbb{G}_f$ to $\mathbb{G}_f$, such that for any T in $\mathbb{G}_f$:*

$$\lambda(\operatorname{supp} T_k) = k\lambda(\operatorname{supp} T).$$

Moreover, for any two measure-preserving bijections S and T in $\mathbb{G}_f$ we have

$$d_{u,f}(\pi_k(S), \pi_k(T)) = k d_{u,f}(S, T).$$

Proof. For $k = 1$ it is evident, so let us fix $k \geqslant 2$, and $T \in \mathbb{G}_f$.

The proof is based on the phenomenon described in Remark 3.4.10, it is possible to decompose X into k parts of infinite measure, such that T acts on each part in the same way it acts on X. In this whole proof we identify $(X, \lambda) \sim (\mathbb{R}, \mathrm{Leb})$ and fix T an element of $\mathbb{G}_f$.

We start by dividing $\mathbb{R}$ into k parts which are copies of itself, and then we map $\mathbb{R}$ to one of its copies, k times. Consider for $i \in \{0, \dots, k-1\}$ the following partial isomorphism:

$$\begin{array}{cccll} \varphi_k^i & : & \displaystyle\bigsqcup_{n\in\mathbb{Z}} [n, n+1[& \longrightarrow & \displaystyle\bigsqcup_{n\in\mathbb{Z}} [kn+i, kn+1+i[\\ & & x \in [n, n+1[& \longmapsto & x + (k-1)n + i \in [kn+i, kn+1+i[. \end{array}$$

Each segment $[n, n+1[$ is sent by φ_k^i to the $(i+1)$th copy of itself. Each φ_k^i has $\mathbb{R}$ as its source, and has a range of infinite measure, which is not conull. Moreover, the ranges of the φ_k^i are disjoint, and

$$\bigsqcup_{i=0}^{k-1} (\mathrm{rng}(\varphi_k^i)) = \bigsqcup_{i=0}^{k-1} \left(\bigsqcup_{n\in\mathbb{Z}} [kn+i, kn+1+i[\right) = \mathbb{R}.$$

On Figure 3.4 we can see the division of $\mathbb{R}$ into $k = 3$ different subsets $\mathrm{rng}(\varphi_3^0)$, $\mathrm{rng}(\varphi_3^1)$ and $\mathrm{rng}(\varphi_3^2)$.

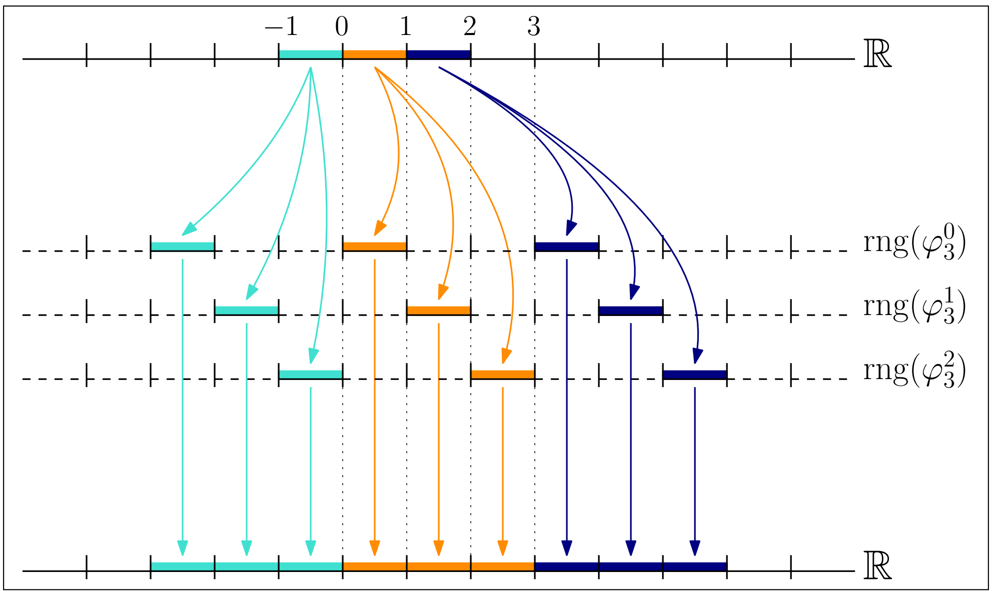

Figure 3.4: Visualisation of the $(\varphi_k^i)_{i\in\{0,\dots,k-1\}}$ for $k = 3$.

Proposition 3.4.11 gives us for each $i \in \{0, \dots, k-1\}$ an element ϕ_k^i in $[[\mathbb{G}]]$ such that $\phi_k^i(\mathbb{R}) = \mathrm{rng}(\varphi_k^i)$. We then define

$$T_k \coloneqq (\phi_k^0)T(\phi_k^0)^{-1}(\phi_k^1)T(\phi_k^1)^{-1}\dots(\phi_k^{k-1})T(\phi_k^{k-1})^{-1},$$

which acts as T on each one of the k copies of $\mathbb{R}$. We have

$$\mathrm{supp}(\phi_k^i)T(\phi_k^i)^{-1} = \bigsqcup_{n\in\mathbb{Z}} [kn+i, kn+1+i[$$

and therefore $\lambda(\mathrm{supp}\, T_k) = k\lambda(\mathrm{supp}\, T)$. We can see an example showing how T and T_k act on the real line on Figure 3.5.

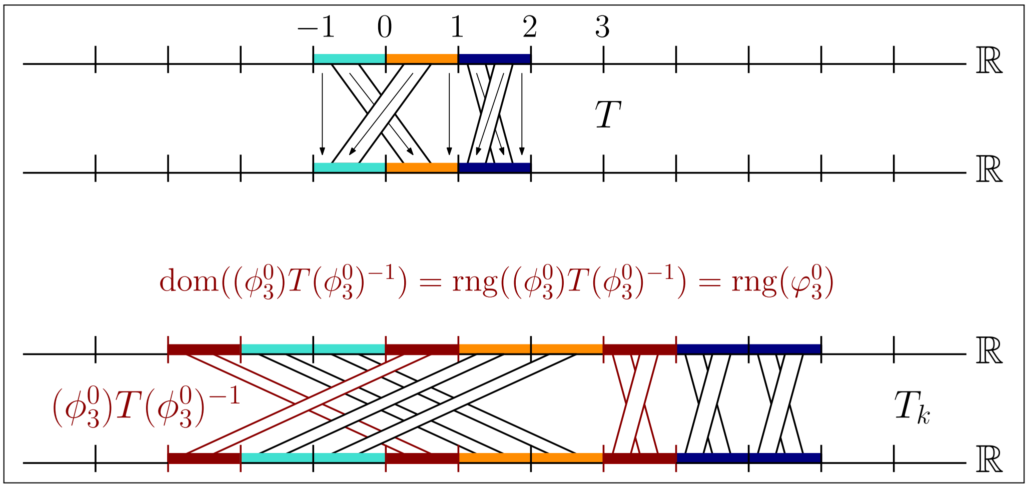

Figure 3.5: Visualisation of T_k for $k = 3$. The first copy of T is highlighted in red.

By construction of π_k, if S is another bijection in $\mathbb{G}_f$, we have $(S(x) \neq T(x)) \Leftrightarrow (S_k(x) \neq T_k(x))$, as we're just copying the way S and T act, k times. In particular, π_k is injective, and we have

$$d_{u,f}(S_k, T_k) = \lambda\left(\{x \in X \mid S_k(x) \neq T_k(x)\}\right) = k\lambda\left(\{x \in X \mid S(x) \neq T(x)\}\right) = kd_{u,f}(S,T).$$

Moreover, elements acting on different copies of $\mathbb{R}$ commute, and for λ-almost all $x \in \mathrm{rng}(\varphi_k^i)$ we have

$$\pi_k(TS)(x) = (\phi_k^i)TS(\phi_k^i)^{-1}(x) = (\phi_k^i)T(\phi_k^i)^{-1}(\phi_k^i)S(\phi_k^i)^{-1}(x) = \pi_k(T)\pi_k(S)(x),$$

so $\pi_k(TS) = \pi_k(T)\pi_k(S)$.

We conclude the proof by proving that the image T_k of T is indeed in $\mathbb{G}$, which is enough since its support has finite measure. It suffices to prove that each $(\phi_k^i)T(\phi_k^i)^{-1}$ is in $\mathbb{G}$. By definition $\phi_k^i \in [[\mathbb{G}]]$ means that it is obtained by cutting and pasting a sequence $(S_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{G}$ over two partitions (A_n) and (B_n) of $\mathbb{R}$ and $\mathrm{rng}(\varphi_k^i)$, respectively. Therefore $(\phi_k^i)T(\phi_k^i)^{-1}$ is obtained by cutting and pasting the sequence $(S_nTS_n^{-1})_{n\in\mathbb{N}}$ of elements of $\mathbb{G}$ over (B_n). As this holds for any $i \in \{0, \dots, k-1\}$, T_k is in $\mathbb{G}$, and thus it is in $\mathbb{G}_f$. □

By using this Lemma and some of the arguments used in [LM22], we then show that $\mathbb{G}_f$ cannot carry a Polish group topology. The proof relies on Proposition 3.6.14, in order to work with an uncountable set of measure-preserving bijections distant enough from each other.

Lemma 3.6.24 ([LM22, Lem. 2.4]). *For all $R > 0$, the set of all T in $\mathrm{Aut}(X,\lambda)$ such that $\lambda(\mathrm{supp}\, T) \leqslant R$ is closed in $(\mathrm{Aut}(X,\lambda), \tau_w)$.*

Proposition 3.6.25. *Let $\mathbb{G}$ be an ergodic full group, such that $\mathbb{G}$ is not the full group of a countable equivalence relation. The subgroup $\mathbb{G}_f \leqslant \mathrm{Aut}(X,\lambda)$ is not Polishable.*

Proof. Suppose that $\mathbb{G}_f$ is Polishable, that is to say that there exists a topology τ on $\mathbb{G}_f$ refining τ_w. For each n in $\mathbb{N}$, we consider the set

$$F_n \coloneqq \{T \in \mathbb{G}_f \mid \lambda(\operatorname{supp} T) \leqslant n\} = \{T \in \mathrm{Aut}_f(X,\lambda) \mid \lambda(\operatorname{supp} T) \leqslant n\} \cap \mathbb{G}.$$

By Lemma 3.6.24 the set $\{T \in \mathrm{Aut}_f(X,\lambda) \mid \lambda(\operatorname{supp} T) \leqslant n\}$ is closed, and Lemma 3.6.2 yields that $\mathbb{G} = \mathbb{G}_{(X,X)}$ is closed in $(\mathrm{Aut}(X,\lambda),\tau_w)$, so each F_n is closed. We have $\mathbb{G}_f = \bigcup_{n\in\mathbb{N}} F_n$, so the Baire Category Theorem ensures us that there exists n_0 such that F_{n_0} has nonempty interior. Since $(\mathbb{G}_f,\tau)$ is Polish, it is Lindelöf, and therefore $\mathbb{G}_f$ is covered by countably many F_{n_0}-translates. In other words, $\mathbb{G}_f$ contains a countable set which is n_0-dense for the uniform finite metric $d_{u,f}$. Indeed $d_{u,f}$ is invariant under group operations. We have then established that any element of $\mathbb{G}_f$ is at $d_{u,f}$-distance at most n_0 from an element of a countable set. We will show that this is not possible.

Recall that, in restriction to $\mathrm{Aut}_f(X,\lambda)$, the uniform finite metric $d_{u,f}$ induces a finer topology than the uniform topology as defined in Section 3.3.3.

We write $X = \bigcup_{n\in\mathbb{N}} X_n$, such that $X_n \subseteq X_{n+1}$ and $\lambda(X_n) < +\infty$ for all n in $\mathbb{N}$, and we consider $\mathbb{G}_{X_n} = \{T \in \mathbb{G} \mid \operatorname{supp} T \subseteq X_n\}$. We will now show that $\mathbb{G}_f = \bigcup_{n\in\mathbb{N}} \mathbb{G}_{X_n}$ is τ_u-dense in $\mathbb{G}$. Let $U \in \mathbb{G}$ be an involution. We define U_n by

$$U_n(x) = \begin{cases} U(x) & \text{if } x \in X_n \text{ and } U(x) \in X_n, \\ x & \text{else.} \end{cases}$$

For any n in $\mathbb{N}$, the involution U_n is in $\mathbb{G}_{X_n}$. Considering μ a probability measure in $[\lambda]$ and recall that d_μ induces τ_u. It is easy to see that $U_n \to_{\tau_u} U$, as

$$\begin{aligned} d_\mu(U_n,U) &= \mu\left(\{x \in X \mid U_n(x) \neq U(x)\}\right) \\ &\leqslant \mu\left(\{x \in X \mid x \notin X_n \text{ or } U(x) \notin X_n\}\right) \\ &\leqslant \mu(X \setminus X_n) + \mu(X \setminus U(X_n)), \end{aligned}$$

which tends to zero as n tends to infinity. This is sufficient, as Theorem 3.4.18 ensures us that any element T in $\mathbb{G}$ can be written as a product of three involutions, so we have proved the τ_u-density of $\mathbb{G}_f$.

Since $\mathbb{G}$ does not come from a countable equivalence relation, thanks to Proposition 3.6.14 we know that it is not separable for the uniform topology. Therefore, at least one of the $\mathbb{G}_{X_n}$ is not τ_u-separable. We then fix $\varepsilon' > 0$ and $n \in \mathbb{N}$ such that there are uncountably many elements $(T_t)_{t\in[0,1]}$ of $\mathbb{G}_{X_n}$ verifying $d_\mu(T_s,T_t) > \varepsilon$, for $s \neq t \in [0,1]$. This implies that $d_{u,f}(T_s,T_t) > \varepsilon$, for $s \neq t \in [0,1]$. We fix $s \neq t$ in $[0,1]$, and we choose k in $\mathbb{N}$ such that $k\varepsilon > n_0$. Lemma 3.6.23 gives us an injective application

$$\pi_k : T \in \mathbb{G}_f \longmapsto \pi_k(T) \in \mathbb{G}_f$$

such that $\lambda\left(\{x \in X \mid \pi_k(T_s)(x) \neq \pi_k(T_t)(x)\}\right) = k\lambda\left(\{x \in X \mid T_s(x) \neq T_t(x)\}\right)$. We then have

$$d_{u,f}(\pi_k(T_s),\pi_k(T_t)) = k d_{u,f}(T_s,T_t) > k\varepsilon > n_0.$$

Thus we have found uncountably many elements of $\mathbb{G}_f$ that are strictly more than n_0-distant from each other. By the pigeonhole principle, this contradicts the n_0-density of the countable subset of $\mathbb{G}_f$ that we constructed, which concludes the proof. $\square$

Remark 3.6.26. In the previous proof we in particular proved that for any ergodic full group $\mathbb{G}$, $\mathbb{G}_f$ is τ_u-dense in $\mathbb{G}$.

Combining this last result with the arguments of Section 3.6.1, we obtain the following Theorem. Indeed, by replacing $\mathbb{G}$ by $\mathbb{G}_f$ in Section 3.6.1, all the results hold, as it is still possible to send any Borel set of finite measure to another Borel set of the same measure with an element of $\mathbb{G}_f$, which is the only property that was used. Similarly to what we did in the proof of Theorem 3.6.4, we then consider τ a Polish topology on $\mathbb{G}_f$, and see that it refines the weak topology. But this is impossible according to Proposition 3.6.25. We have then proved the following.

Proposition 3.6.27. *Let $\mathbb{G}$ be an ergodic full group, such that $\mathbb{G}$ is not the full group of a countable equivalence relation. The group $\mathbb{G}_f$ cannot carry a Polish group topology.*

The rest of this section is dedicated to proving that an ergodic full group $\mathbb{G}$ arising from a countable equivalence relation provides a subgroup $\mathbb{G}_f$ which can be endowed with a Polish topology. In particular we obtain a characterization of such ergodic full groups.

Proposition 3.6.28. *Let $\mathbb{G}$ be a full group. Then $(\mathbb{G}_f, d_{u,f})$ is complete. Moreover, if $\mathbb{G}$ is the full group of a countable equivalence relation, then $(\mathbb{G}_f, \tau_{u,f})$ is separable.*

Proof. (1) We mimic the proof of Proposition 3.4.7, but it is easier here. We take (T_n) which is $d_{u,f}$-Cauchy in $\mathbb{G}_f$, and construct the limit T by Borel-Cantelli using the measure λ instead of μ. The important point is that imposing $d_{u,f}(T_n, T_{n+1}) < 2^{-n}$ (by passing to a subsequence if necessary) allows us to work in the space

$$\operatorname{supp} T_0 \cup \bigcup_{n \geqslant 0} \{x \in X \mid T_n(x) \neq T_{n+1}(x)\},$$

which has finite measure. Then by construction T is finitely supported and (T_n) converges to T for $d_{u,f}$.

(2) For separability we mimic the proof of Proposition 3.6.14($\Leftarrow$). The proof is very similar, but we keep λ instead of replacing it with an equivalent probability measure. The same arguments yield that $(\mathrm{MAlg}_f(\mathcal{R}, M_l), \widehat{d})$ is a separable metric space, where $\mathcal{R}$ is the countable equivalence relation on X associated with $\mathbb{G}$, M_l is defined by $M_l(A) = \int_X |\{y \in X \mid (x,y) \in A\}| \, d\lambda(x)$ for any Borel $A \subseteq \mathcal{R}$, and $\widehat{d}(A,B) = M_l(A \Delta B)$. Finally, for S, T in $\mathbb{G}_f$ we have

$$d_{u,f}(S,T) = \frac{1}{2} M_l(\mathrm{graph}(S) \Delta \mathrm{graph}(T)),$$

ensuring that $\mathbb{G}_f$ is separable, as a metric subspace of a separable metric space. $\square$

Putting together Proposition 3.6.27 and Proposition 3.6.28, we get the following characterization.

Theorem 3.6.29. *Let $\mathbb{G}$ be an ergodic full group on a standard σ-finite space. The following are equivalent:*

(i) *$\mathbb{G}$ is the full group of a countable equivalence relation;*

(ii) *$\mathbb{G}_f$ can be endowed with a Polish group topology.*

Moreover, when it exists, the Polish group topology on $\mathbb{G}_f$ refines the uniform topology, which itself refines the weak topology.

Having established automatic continuity for $(\mathbb{G}, \tau_u)$ in the previous section, it is very natural to ask if $(\mathbb{G}_f, \tau_{u,f})$ also has this property. We are very grateful to Emmanuel Roy for suggesting the following result, along with a complete proof.

Proposition 3.6.30. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be a full group. The group $(\mathbb{G}_f, \tau_{u,f})$ has automatic continuity.*

Proof. Fix a separable group H, and a group homomorphism $\psi : \mathbb{G}_f \to H$. We only have to show that $\psi(T_n) \to e_H$ for any sequence (T_n) that $\tau_{u,f}$-converges to id_X.

From any subsequence of (T_n), it is possible to extract a subsequence (T_k) such that

$$d_{u,f}(T_k, \mathrm{id}_X) = \lambda(\operatorname{supp} T_k) < 2^{-k}$$

for any k. As direct consequence, all the elements of this sub-subsequence are in $\mathbb{G}_A$, where $A = \bigcup_k \operatorname{supp} T_k$ has finite measure. Remark 3.4.16 and [CLM16, Thm. 4.7] (or Theorem 3.6.18) yield automatic continuity of $(\mathbb{G}_A, \tau_{u,f})$, which ensures that $\psi(T_k) \to e_H$. Convergence of a subsequence of any subsequence establishes sequential continuity of ψ, which is enough to conclude as the domain $\mathbb{G}_f$ of ψ is metrizable hence first countable. □

3.7 Algebraic and topological properties of ergodic full groups

3.7.1 Normal subgroups of an ergodic full group

In this section we use Theorem 3.4.18 to recover all the information of an ergodic full group from its involutions. In particular, by using Corollary 3.4.13 and the following well-known key lemma, we are able to retrieve many elements of an ergodic full group from the involutions contained in a normal subgroup. We recall that $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X, \lambda) = \{T \in \mathbb{G} \mid \lambda(\operatorname{supp} T) < +\infty\}$, and we say that a subgroup $N \leqslant \mathbb{G}$ is normalized by $\mathbb{G}_f$ if N satisfies $TNT^{-1} = N$ for any $T \in \mathbb{G}_f$. The following is classical but very useful.

Lemma 3.7.1. *Let (X, λ) be a standard σ-finite space, $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be an ergodic full group, and N be a non-trivial normal subgroup of $\mathbb{G}$. Let T be in N, and A be such that A is disjoint from $T(A)$. Then for any Borel subset $B \subseteq A$ and any involution V supported in B, the involution $U = [T, V]$ is in N and satisfies $\operatorname{supp} U = B \sqcup T(B)$. Moreover, if B has finite measure, the conclusion holds if we only assume that N is normalized by $\mathbb{G}_f$.*

Proof. Fix a Borel subset $B \subseteq A$. By Corollary 3.4.13, there exists an involution $V \in \mathbb{G}$ such that $\operatorname{supp} V = B$. Let us now prove that $U = [T, V] = TVT^{-1}V^{-1}$ is an involution in N with support $B \sqcup T(B)$. We have

$$(TVT^{-1})V = T(VT^{-1}V^{-1}).$$

From the left-hand side TVT^{-1} is an involution supported on $T(B)$ and V is an involution supported on B, so their product is an involution supported on $B \sqcup T(B)$. From the right-hand side, $T \in N$ and $VT^{-1}V^{-1} \in N$, so their product is in N. □

The previous Lemma implies in particular that non-trivial normal subgroups of $\mathrm{Aut}(X, \lambda)$ contain finitely supported elements. It is important to keep in mind also that conjugation by elements of $\mathrm{Aut}(X, \lambda)$ preserve the finiteness of the support. We then have the following.

Proposition 3.7.2. *Let (X, λ) be a standard σ-finite space and $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be an ergodic full group. Then any non-trivial subgroup $N \leqslant \mathbb{G}$ that is normalized by $\mathbb{G}_f$ contains $\mathbb{G}_f$. In particular $\mathbb{G}_f$ is simple.*

Proof. In this whole proof we fix a non-trivial $N \leqslant \mathbb{G}$ that is normalized by $\mathbb{G}_f$. To show that $\mathbb{G}_f \subseteq N$, we show that N contains all the involutions in $\mathbb{G}_f$.

(1) We start by proving that N contains an involution with a support of measure R, for any $R > 0$. Using Lemma 3.7.1 and Proposition 3.4.17, we consider an involution $U \in N$ and a Borel subset $A \subseteq X$ of positive and finite measure such that $\operatorname{supp} U = A \sqcup U(A)$.

Fix a positive real number $0 < t \leqslant 2\lambda(\operatorname{supp} U)$. Let $B \subseteq A$ be such that $\lambda(B) = \frac{t}{4}$, and set $C = B \sqcup U(B)$. Let now D be disjoint from $\operatorname{supp} U$, and such that $\lambda(D) = \lambda(C) = \frac{t}{2}$. Finally let $V \in \mathbb{G}_{C \sqcup D}$ be a (C, D)-exchanging involution, which exists by Corollary 3.4.13. We have

$$[U,V] = UVU^{-1}V^{-1} = \begin{cases} \mathrm{id}_X & \text{on } X \setminus (C \sqcup D), \\ U & \text{on } C, \\ VUV & \text{on } D, \end{cases}$$

so $\operatorname{supp}[U,V] = C \sqcup D$, which has measure $\lambda(\operatorname{supp}[U,V]) = t$, and $[U,V]$ is in N by Lemma 3.7.1. For any $R > 0$, starting with the newly obtained involution each time, iterating this construction enough times (which is a finite number of times as we can always double the measure the support of the previous involution) yields an involution $W \in N$ with $\lambda(\operatorname{supp} W) = R$.

(2) We now prove that any involution in $\mathbb{G}_f$ is in N. Let $U \in \mathbb{G}_f$ be an involution, and let $V \in N$ be an involution such that $\lambda(\operatorname{supp} V) = \lambda(\operatorname{supp} U)$, which exists by *(1)*. By Lemma 3.4.21 there exists $T \in \mathbb{G}_f$ such that $TVT^{-1} = U$, which proves that U is in N.

(3) Let T be in $\mathbb{G}_f$. By Theorem 3.4.18, we can write T as the product of three involutions in $\mathbb{G}_f$, which are all in N by *(2)*. Therefore T is in N, hence $\mathbb{G}_f \subseteq N$. □

We now only have take care of infinitely supported elements of N in order to obtain the following generalization of [Eig81] to any type II_∞ ergodic full group.

Theorem 3.7.3. *Let (X, λ) be a standard σ-finite space, and $\mathbb{G} \leqslant \operatorname{Aut}(X, \lambda)$ an ergodic full group. Let $N \trianglelefteq \mathbb{G}$ be normal and non-trivial. Then either $N = \mathbb{G}_f$ or $N = \mathbb{G}$.*

Proof. By Proposition 3.7.2, we have $\mathbb{G}_f \subseteq N$, so if $N \subseteq \mathbb{G}_f$, there is nothing to do. Assume then that there exists $T \in N$ such that $\lambda(\operatorname{supp} T) = +\infty$. We need to prove that N contains all the involutions in $\mathbb{G}$ with supports of infinite measure, then Theorem 3.4.18 will conclude the proof. In order to use Lemma 3.4.21, we must separately treat the cases of whether the measure of the complement of the support of an involution is infinite or finite. Let then U_0 and V_0 in $\mathbb{G}$ be such that

$$\begin{cases} \lambda(\operatorname{supp} U_0) = \lambda(X \setminus \operatorname{supp} U_0) = +\infty \\ \lambda(\operatorname{supp} V_0) = +\infty \text{ and } \lambda(X \setminus \operatorname{supp} V_0) < +\infty. \end{cases}$$

Our goal is to prove that U_0 and V_0 are in N, which will imply that their respective conjugacy classes are in N.

(1) Using Lemma 3.3.4 we consider a separator A for T (in particular $\lambda(A) = \lambda(X \setminus A) = +\infty$). Let now $U \in \mathbb{G}$ be an involution with $\operatorname{supp} U \subseteq A$ and $\lambda(\operatorname{supp} U) = \lambda(A \setminus \operatorname{supp} U) = +\infty$, which exists by Corollary 3.4.13. By Lemma 3.7.1, $[T,U]$ is an involution in N, and $\lambda(\operatorname{supp}[T,U]) = \lambda(X \setminus \operatorname{supp}[T,U]) = +\infty$. By Lemma 3.4.21, U_0 and $[T,U]$ are conjugated in $\mathbb{G}$, so U_0 is in N.

(2) As $\lambda(\operatorname{supp} V_0) = +\infty$, it is possible to write $\operatorname{supp} V_0 = B \sqcup C$, with $\lambda(B) = \lambda(C) = +\infty$. By *(1)* and Lemma 3.4.21, we can find in N two involutions V_1 and V_2 such that $\operatorname{supp} V_1 = B$ and $\operatorname{supp} V_2 = C$. The product V_1V_2 is in N and is an involution, its support satisfies $\lambda(\operatorname{supp} V_1V_2) = \lambda(\operatorname{supp} V_0) = +\infty$ and $\lambda(X \setminus \operatorname{supp} V_1V_2) = \lambda(X \setminus \operatorname{supp} V_0) < +\infty$, so Lemma 3.4.21 yields that V_0 and V_1V_2 are conjugated in $\mathbb{G}$. Therefore V_0 is in N. □

Remark 3.7.4. An ergodic full group $\mathbb{G}$ is never simple, as it will always admit $\mathbb{G}_f$ as a non-trivial normal subgroup. However from a topological point of view, as $\mathbb{G}_f$ is dense in $\mathbb{G}$ for τ_u (see Remark 3.6.26), we can say that $\mathbb{G}$ is topologically simple, *i.e.* it does not contain any non-trivial closed normal strict subgroup.

3.7.2 Contractibility of orbit full groups

The strategy used in [Kea70] and then in [CLM16] to prove that an orbit full group $(\mathbb{G}, \tau_m^o)$ is contractible cannot be directly used here as induced bijections are only defined for conservative bijections. We follow Danilenko's proof from [Dan95, Thm. 2.2] to circumvent this issue in the ergodic case.

Again we fix a standard σ-finite space (X, λ). The goal of this section is to prove the following proposition.

Proposition 3.7.5. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be an ergodic orbit full group, such that τ_m^o is Polish on $\mathbb{G}$. Then $(\mathbb{G}, \tau_m^o)$ is contractible.*

We fix such an ergodic orbit full group $\mathbb{G} = [\mathcal{R}_G]$, along with its Polish topology of orbital convergence in measure τ_m^o (recall that when G is locally compact Polish, $\mathbb{G}$ falls in this category by Theorem 3.5.9).

We denote by $\mathcal{P}$ the family of countable ordered partitions $P = (X_n)_{n\in\mathbb{N}}$ of X with $\lambda(X_n) = 1$ for any $n \in \mathbb{N}$. We endow it with the metric r defined by

$$r(P_1, P_2) = \sum_{n\in\mathbb{N}} \lambda(X_n^1 \Delta X_n^2)$$

for any $P_1 = (X_n^1)_{n\in\mathbb{N}}, P_2 = (X_n^2)_{n\in\mathbb{N}}$ in $\mathcal{P}$. Notice that the topology induced by r is the toplogy induced by the product topology on $\mathrm{MAlg}_f(X, \lambda)^{\mathbb{N}}$, so it is Polish by Proposition 3.2.2. Note also that this topology only depends on the measure class of λ, as on $\left\{A \in \mathrm{MAlg}_f(X, \lambda) \mid \lambda(A) \leqslant 1\right\}$ the topology from $\mathrm{MAlg}_f(X, \lambda)$ coincides with the one from $\mathrm{MAlg}(X, \lambda)$.

For any $P = (X_n)_{n\in\mathbb{N}} \in \mathcal{P}$, define

$$\mathbb{G}^P = \{T \in \mathbb{G} \mid \forall n \in \mathbb{N} : T(X_n) = X_n\}.$$

Until the end of this section we fix $P = (X_n)_{n\in\mathbb{N}}$ in $\mathcal{P}$. For any Borel subset A of positive measure we also recall that $\mathbb{G}_A = \{T \in \mathbb{G} \mid \mathrm{supp}\, T \subseteq A\}$, and naturally identify it with $\mathbb{G}_{A\restriction A} = \{T_{\restriction A} \in \mathrm{Aut}(A, \lambda_{\restriction A}) \mid T \in \mathbb{G}_A\}$.

Our immediate goal is to prove that any full group $(\mathbb{G}_A, \tau_m^o)$ is contractible (recall that $\mathbb{G}_A$ is closed in $\mathbb{G}$ by Lemma 3.6.1, hence Polish for τ_m^o). There is no reason for $\mathbb{G}_A$ to be an orbit full group, however in [CLM16, Cor. 4.3] the authors actually define a metric which is τ_m^o-compatible, and prove contractibility through it. We recall the important points.

Proposition 3.7.6 ([CLM16, Prop. 3.19, Cor. 3.20])**.** *Let $[\mathcal{R}_G]$ be an orbit full group, such that τ_m^o is Polish on $[\mathcal{R}_G]$. Fix a probability measure $\mu \in [\lambda]$, and let d_G be a right-invariant compatible bounded metric on G. Denote $G_x \coloneqq \mathrm{stab}_G(x)$. We identify the orbit $G \cdot x$ to the homogeneous space G/G_x, and endow it with the metric $d_x(gG_x, g'G_x) \coloneqq \inf_{h\in G_x} d_G(gh, g')$. Then the quotient metric $d_{[\mathcal{R}_G]}$ is defined by*

$$d_{[\mathcal{R}_G]}(T, S) = \int_X d_x(T(x), S(x))d\mu(x).$$

Moreover, for any sequence (T_n) and any T in $[\mathcal{R}_G]$, the following are equivalent:

(i) $T_n \to T$ for τ_m^o;

(ii) for any $\varepsilon > 0$, $\mu(\{x \in X \mid d_x(T_n(x), T(x)) > \varepsilon\}) \to 0$;

(iii) every subsequence of $(T_n)_{n\in\mathbb{N}}$ admits a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ such that for μ-almost all x in X, $T_{n_k}(x) \to T(x)$ in $G \cdot x$ for d_x.

Proof. We start by recalling that the quotient metric d_x as we defined it induces the quotient topology on G/G_x. Indeed, G_x is closed by [Kec95, Thm. 9.17], then [Gao09, Lem. 2.2.8] applies. Notice now that the right-invariance of d_G yields the right-invariance of $\tilde{d}_G$, that we recall is defined on $\mathrm{L}^0(X,\lambda,G)$ by $\tilde{d}_G(f,f') = \int_X d_G(f(x),f'(x))d\mu(x)$.

We define

$$K = \{f \in \mathrm{L}^0(X,\lambda,G) \mid \forall x \in X, f(x) \in G_x\} = \mathrm{Ker}\left(\Phi_{\restriction \widetilde{[\mathcal{R}_G]}}\right).$$

Recall the definition of the $*$-product from the proof of Theorem 3.5.7: for any $f \in \widetilde{[\mathcal{R}_G]}$ and $k \in K$ we have

$$(f * k)(x) = f(\Phi(k)(x))k(x) = f(x)k(x),$$

so the right K-cosets are the same in $\widetilde{[\mathcal{R}_G]}/K \cong [\mathcal{R}_G]$ for $*$ and for the pointwise product. By Proposition 3.3.14 and [Gao09, Lem. 2.2.8] again, the quotient metric induced by $\tilde{d}_G$ induces the quotient topology on $\mathrm{L}^0(X,\lambda,G)/K$, and by the previous argument, the quotient metric induced by $\tilde{d}_G$ on $\widetilde{[\mathcal{R}_G]}/K$ and the quotient metric induced by $\tilde{d}_G$ on $\mathrm{L}^0(X,\lambda,G)/K$ agree on $[\mathcal{R}_G]$. By definition, the latter is given by $\rho(f,f') = \inf_{k\in K} \tilde{d}_G(fk,f')$, therefore our goal is to show that

$$\rho(f,f') = \int_X d_x(f(x),f'(x))d\mu(x).$$

We fix $\varepsilon > 0$ and $f, f' \in \mathrm{L}^0(X,\lambda,G)$. For the first inequality we apply [Kec95, thm. 18.1] to

$$\left\{(x,g) \in X \times G \mid d_G(f(x)g, f'(x)) < d_x(f(x),f'(x)) + \varepsilon \text{ and } g \cdot x = x\right\},$$

yielding a function $k \in K$ satisfying $d_G(f(x)k(x), f'(x)) < d_x(f(x),f'(x)) + \varepsilon$ for any $x \in X$. In particular, we have $\rho(f,f') \leqslant \int_X d_x(f(x),f'(x))d\mu(x)$. The converse inequality is immediate, as for any $k \in K$ and any $x \in X$ we have $d_G(f(x)k(x),f'(x)) \geqslant d_x(f(x),f'(x))$ by definition of K and d_x.

The equivalence between (i), (ii) and (iii) is proved exactly as in [Moo76, Prop. 6] (Moore actually works with σ-finite spaces as well): $(i) \Rightarrow (ii)$ follows from Markov's inequality applied to $\mu(\{x \in X \mid d_x(T_n(x),T(x)) > \varepsilon\})$, $(ii) \Rightarrow (iii)$ follows from the general fact that convergence in (finite) measure implies convergence μ-a.e. along a subsequence. Finally $(iii) \Rightarrow (i)$ follows from the fact that (iii) gives convergence in measure of the sequence $(d_x(T_n(x),T(x)))$ to 0, then Lebesgue's dominated convergence theorem concludes the proof. □

Recall the definition of the induced bijection (see Definition 3.6.8). As we already mentioned, convergence in measure only depends on the measure class, so [CLM16, Prop. 4.2] generalizes to our context without trouble.

Proposition 3.7.7 ([CLM16, Prop. 4.2])**.** *Let $\mathbb{G}$ be an ergodic orbit full group on (X,λ), such that τ_m^o is Polish on $\mathbb{G}$. Let $A \subseteq X$ be of finite measure. The function which maps $(B,T) \in \mathrm{MAlg}(A,\lambda_{\restriction A}) \times \mathbb{G}_A$ to the induced bijection $T_B \in \mathbb{G}_A$ is τ_m^o-continuous.*

Proof. For any $A \subseteq X$ of finite measure, $T \in \mathbb{G}_A$, and $n \in \mathbb{N}^*$ we define

$$C_n(A,T) \coloneqq \{x \in A \mid T^n(x) \in A\}.$$

Set also $B_0(A,T) \coloneqq X \backslash A$ and $B_n(A,T) \coloneqq C_n(A,T) \backslash \bigcup_{m<n} C_m(A,T)$, so that $(B_n(A,T))_{n\in\mathbb{N}}$ is a partition of X and notice that for any $x \in B_n(A,T)$ we have $T_A(x) = T^n(x)$. The

important point is that A has finite measure so any element of $\mathbb{G}_A$ is conservative, thus the induced bijections are well-defined, and are contained in $\mathbb{G}_A$. Note also that

$$\Psi : (A, T) \in (\mathrm{MAlg}_f(X, \lambda) \times \mathbb{G}_A) \mapsto B_n(A, T) \in \mathrm{MAlg}(X, \lambda)$$

depends $(d_{X,\lambda}, \tau_w)$-continuously on (A, T). We now fix $\varepsilon > 0$ and $(A, T) \in \mathrm{MAlg}_f(X, \lambda) \times \mathbb{G}_A$. By virtue of $(B_n(A, T))_{n\in\mathbb{N}}$ being a partition of X, there exists $N > 0$ such that $\lambda(X \setminus \bigcup_{n<N} B_n(A, T)) < \varepsilon$. We now fix a probability measure $\mu \in [\lambda]$ and we let $\mathcal{U}$ be the set of couples $(A', T') \in \mathrm{MAlg}(A, \lambda_{\restriction A}) \times \mathbb{G}_A$ satisfying both following conditions:

(1) $\sum_{n=0}^{N} d_{[\mathcal{R}_G]}(T^n, T'^n) < \varepsilon$,

(2) $\sum_{n=0}^{N} \mu(B_n(A, T)\Delta B_n(A', T')) < \varepsilon$.

By τ_w-continuity of $T \mapsto T^n$, continuity of Ψ and Lemma 3.2.5 (since we want a condition on μ and not on λ, even though the Borel sets in play have finite λ-measure), $\mathcal{U}$ is open. We have

$$\begin{aligned}
d_{[\mathcal{R}_G]}(T_A, T'_{A'}) &= \int_X d_x(T_A(x), T'_{A'}(x))d\mu(x)\\
&\leqslant \sum_{n=0}^{N} \int_{B_n(A,T)\Delta B_n(A',T')} d_x(T_A(x), T'_{A'}(x))d\mu(x)\\
&+ \sum_{n=0}^{N} \int_{B_n(A,T)\bigcap B_n(A',T')} d_x(T_A(x), T'_{A'}(x))d\mu(x)\\
&+ \int_{X\setminus\bigcup_{n<N} B_n(A,T)} d_x(T_A(x), T'_{A'}(x))d\mu(x).
\end{aligned}$$

Without loss of generality it is possible to assume that d_x is bounded by 1, so by construction and (2) the first and third term are both less that ε. We finally have from (1):

$$\begin{aligned}
d_{[\mathcal{R}_G]}(T_A, T'_{A'}) &< \sum_{n=0}^{N} \int_{B_n(A,T)\bigcap B_n(A',T')} d_x(T_A(x), T'_{A'}(x))d\mu(x) + 2\varepsilon\\
&= \sum_{n=0}^{N} \int_{B_n(A,T)\bigcap B_n(A',T')} d_x(T^n(x), T'^n(x))d\mu(x) + 2\varepsilon\\
&< 3\varepsilon,
\end{aligned}$$

which concludes the proof. □

The previous proposition yields that the contraction path from [CLM16, Cor. 4.3] does not exit $\mathbb{G}_A$, so we have the following.

Proposition 3.7.8 ([CLM16, Cor. 4.3]). *For any Borel subset A of X of positive finite measure, the full group $(\mathbb{G}_A, \tau_m^o)$ is contractible.*

Proof. As A has finite measure, we may assume that it is $[0, 1]$, endowed with the Lebesgue measure. The homotopy between the identity map and the constant map $T \mapsto \mathrm{id}_A$ is given by $(s, T) \in [0, 1] \times \mathbb{G}_A \to T_{[s,1]} \in \mathbb{G}_A$. □

By definition of $\mathbb{G}^P$, the following application is a homeomorphism:

$$\begin{array}{ccc}
\mathbb{G}^P & \longrightarrow & \prod_{n\in\mathbb{N}} \mathbb{G}_{X_n}\\
T & \longmapsto & (T_{\restriction X_n})_{n\in\mathbb{N}}.
\end{array}$$

By Proposition 3.7.8 each $\mathbb{G}_{X_n}$ is contractible, and therefore $\mathbb{G}^P$ is contractible as a product. We then consider the following application:

$$\begin{array}{rcccl} \pi^P & : & \mathbb{G} & \longrightarrow & \mathcal{P} \\ & & T & \longmapsto & (T(X_n))_{n\in\mathbb{N}}. \end{array}$$

It is continuous for τ_w, hence for τ_m^o by Theorem 3.6.4, surjective by Corollary 3.4.13 and constant on the left $\mathbb{G}^P$-cosets. Danilenko proved the following.

Lemma 3.7.9 ([Dan95, Lem. 2.3])**.** *The Polish space $(\mathcal{P}, r)$ is contractible.*

We can now finish the proof of Proposition 3.7.5 just like in [Dan95]. For every partition $P' = (Y_n)_{n\in\mathbb{N}} \in \mathcal{P}$, by using Corollary 3.4.13 we define for any $n \in \mathbb{N}$, for any $x \in Y_n$:

$$\psi(P')(x) = U_n(x)$$

where U_n is the involution in $\mathbb{G}_{Y_n \Delta X_n}$ sending Y_n to X_n. By continuity of $(X_n, Y_n) \mapsto U_n$ (see Corollary 3.4.13), the injective map $\psi : P' \in \mathcal{P} \mapsto \psi(P') \in \mathbb{G}$ is τ_u-continuous, hence it is τ_m^o-continuous. Moreover, $\pi^P \circ \psi = \mathrm{id}_{\mathcal{P}}$, so π^P has a right inverse, hence it is a continuous projection map from $\mathbb{G}$ onto $\mathcal{P}$. From this we get two things. As π^P is surjective and constant on the left $\mathbb{G}^P$-cosets, we have the homeomorphism $\mathbb{G}/\mathbb{G}^P \cong \mathcal{P}$ (see *e.g.* [Mun00, Thm. 22.2]). Moreover, $\pi^P \circ \psi = \mathrm{id}_{\mathcal{P}}$ means that ψ is a global section for the fibre bundle $(\mathbb{G}, \pi^P, \mathbb{G}/\mathbb{G}^P, \mathbb{G}^P)$ (see [Ste51, § 7.4]), which implies (by [Ste51, § 8.3]) that we have the homeomorphism

$$\mathbb{G} \cong {}^{\mathbb{G}}\!/_{\mathbb{G}^P} \times \mathbb{G}^P \cong \mathcal{P} \times \mathbb{G}^P.$$

Thus $\mathbb{G}$ is contractible as a product of contractible spaces, which ends the proof of Proposition 3.7.5.

We conclude this section by the following remark: although the presence of non-conservative elements makes the proof more complicated for the contractibility of $\mathbb{G}$, there is no issue for $\mathbb{G}_f$. In particular, as [Kea70] shows continuity of $(B, T) \mapsto T_B$ for the uniform topology τ_u, which coincides with $\tau_{u,f}$ (see Definition 3.6.21) on $\mathbb{G}_f$, we get the following.

Proposition 3.7.10. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be the full group of a countable equivalence relation. Then $(\mathbb{G}_f, \tau_{u,f})$ is contractible.*

3.7.3 Density and genericity in orbit full groups

The goal of this section is to obtain density and category (in the sense of Baire) theorems for certain classes of measure-preserving bijections in full groups of $\mathrm{Aut}(X, \lambda)$.

Recall from Section 1.2.6 that in [Kre67, Thm. 8.1] the author proved that the conservative elements form a dense G_δ set for both τ_w and τ_u. For the rest of this section, APER and ERG will denote the set of aperiodic and ergodic elements of $\mathrm{Aut}(X, \lambda)$, respectively. Recall also the following results for the weak and uniform topologies, exhibiting different behaviours for the different topologies on $\mathrm{Aut}(X, \lambda)$.

Theorem 3.7.11 ([Sac71, Thm. 2.2 and Thm. 3.1])**.** ERG *is dense G_δ in $(\mathrm{Aut}(X, \lambda), \tau_w)$, but is nowhere dense in $(\mathrm{Aut}(X, \lambda), \tau_u)$.*

We aim to localize results of this flavour to full groups endowed with their relevant topologies. From Remark 3.6.11 and Remark 3.6.26, we get the following classical result. One can also consult [Fre06, Prop. 494C(c)].

Proposition 3.7.12. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be a full group. Then the set of periodic bijections in $\mathbb{G}_f$ is uniformly dense in $\mathbb{G}$.*

In particular the previous results yield τ-density when the considered full group is ergodic and has a separable topology τ, by Theorem 3.6.19. Results on aperiodic elements demand more work, but a lot has already been done for $\mathrm{Aut}(X,\lambda)$. We need a few adaptations to obtain result for our general Polish topologies on ergodic full groups. We have the following.

Theorem 3.7.13 ([CK79, Thm. 6]). *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ be an ergodic full group. Let T be in* ERG*, and S be in* APER*. For any Borel subset $E \subseteq X$ of finite measure, there exists $U \in \mathbb{G}_f$ such that $T = USU^{-1}$ a.e. on E.*

Proof. Choksi and Kakutani prove that there exists two families of pairwise disjoint Borel sets $(E_{k,i})$ and $(F_{k,i})$ (with $k \in \mathbb{N}$ and $i \in \{1,\dots,k+1\}$) satisfying the following:

- for any $k \in \mathbb{N}$, for all $i \in \{1,\dots,k\}$, $T(E_{k,i}) = E_{k,i+1}$ and $S(F_{k,i}) = F_{k,i+1}$;
- for all $k \in \mathbb{N}, i \in \{1,\dots,k+1\}$, $\lambda(E_{k,i}) = \lambda(F_{k,i})$;
- $\bigcup_{k\in\mathbb{N}} \bigcup_{i=1}^{k} E_{k,i} = E$.

Their conclusion is that the conjugate of S by the measure-preserving bijection $U \in \mathrm{Aut}(X,\lambda)$ exchanging the $E_{k,i}$ and the $F_{k,i}$ is equal to T a.e. on E, but thanks to Corollary 3.4.13, the conclusion holds even if we ask U to be in $\mathbb{G}_f$. $\square$

Their main result [CK79, Thm. 7] can also be extended to our context of ergodic full groups:

Theorem 3.7.14 ([CK79, Thm. 7]). *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ be an ergodic full group. If S is a fixed element in* $\mathrm{APER} \cap \mathbb{G}$*, the set of $\mathbb{G}_f$-conjugates of S, i.e. $\{USU^{-1} \mid U \in \mathbb{G}_f\}$ is τ_u-dense in* $\mathrm{APER} \cap \mathbb{G}$.

Proof. Fix $\varepsilon > 0$ and a metric compatible with τ_u on $\mathrm{Aut}(X,\lambda)$, as well as $S' \in \mathrm{APER} \cap \mathbb{G}$. By Proposition 1.2.40 (see also [CK79, Prop. 4]), there exists $T \in \mathrm{ERG}$ (not necessarily in $\mathbb{G}$) which is ε-close to S. By Theorem 3.7.13, T can be ε-approximated by a $\mathbb{G}_f$-conjugate of S, which means that this conjugate is 2ε-close to S'. $\square$

We also have the following generalization of a classical result (see *e.g.* [Kec10, Thm. 3.5]), which is of independent interest. It is most likely well-known, but we have found no reference, so we give a quick proof using a very powerful result from [Tse22].

Theorem 3.7.15. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ be an ergodic full group. Then* $\mathrm{ERG} \cap \mathbb{G}$ *is not empty.*

Proof. We let Γ be a τ_w-dense countable subgroup of $\mathbb{G}$. We view Γ as acting on (X,λ) in a non-singular manner, and use [Tse22, Thm. 1.3] to obtain an ergodic hyperfinite non-singular subgraph of the associated equivalence relation graph. This subgraph is generated by an ergodic non-singular T, which is then in $\mathbb{G}$ (in particular measure-preserving) by construction. $\square$

We also give a direct proof of the previous result.

Proof. Start by considering a subset $A \subseteq X$ of measure $\lambda(A) = 1$. By Remark 3.4.16, $\mathbb{G}_A$ is ergodic. By considering a τ_w-dense subgroup of $\mathbb{G}_A$ seen as a subgroup of $\mathrm{Aut}(A, \lambda_{\restriction A})$, which exists by [Kec10, Prop. 3.1], we can apply [Kec10, Thm. 3.5], which yields a bijection $T_A \in \mathbb{G}_A$ which is ergodic on A.

We identify A with the interval $[0,1]$, and denote by A_n the subinterval $[\frac{1}{n+1}, \frac{1}{n}]$, for any $n \geqslant 1$. We also write $X \setminus A = \bigsqcup_{k\geqslant 2} X_k$, with $\lambda(X_k) = \frac{1}{k}$ for any $k \geqslant 2$. We then iteratively define a sequence of elements $(\phi_n)_{n\geqslant 1}$ of $[[\mathbb{G}]]$ as follows. The first partial automorphism ϕ_1 is defined on $A_1 = [\frac{1}{2}, 1]$ by $\phi_1 = T_{A\restriction A_1}$. For clarity we also give the construction of ϕ_2 in detail. We first consider a subset B_2 of X_2 of measure $\lambda(B_2) = \frac{1}{2} - \frac{1}{3}$, and we consider U_2 the exchanging involution provided by Corollary 3.4.13 and exchanging A_2 and B_2. Then ϕ_2 is defined on $A_2 \sqcup B_2$, by

$$\phi_2 = \begin{cases} U_2 \text{ on } A_2, \\ T_A U_2^{-1} \text{ on } B_2. \end{cases}$$

For $n > 2$ and $2 \leqslant i \leqslant n$, we first define $n-1$ disjoint subsets

$$B_{n,i} \subseteq X_i \setminus \bigsqcup_{m<n} B_{m,i},$$

each of measure $\lambda(B_{n,i}) = \frac{1}{n} - \frac{1}{n+1}$, and set $B_n = \bigsqcup_{i=2}^{n} B_{n,i}$ (with these notations one might think of A_n as being $B_{n,1}$, and we also have $B_{2,2} = B_2$). By Corollary 3.4.13 again we can consider $n-1$ exchanging involutions $U_{n,1}, \ldots, U_{n,n-1}$ such that $U_{n,1}$ exchanges A_n and $B_{n,2}$, and for any $i \in \{3, \ldots, n-1\}$, $U_{n,i}$ exchanges $B_{n,i}$ and $B_{n,i+1}$. We can then define

$$\phi_n = \begin{cases} U_{n,1} \text{ on } A_n, \\ U_{n,i} \text{ on } B_{n,i}, \text{ for } 2 \leqslant i \leqslant n-1, \\ T_A U_{n,2}^{-1} \ldots U_{n,n}^{-1} \text{ on } B_{n,n}. \end{cases}$$

Notice that we in particular have $X_k = \bigsqcup_{l \geqslant k} B_{l,k}$. Figure 3.6 gives a visual representation of this whole construction.

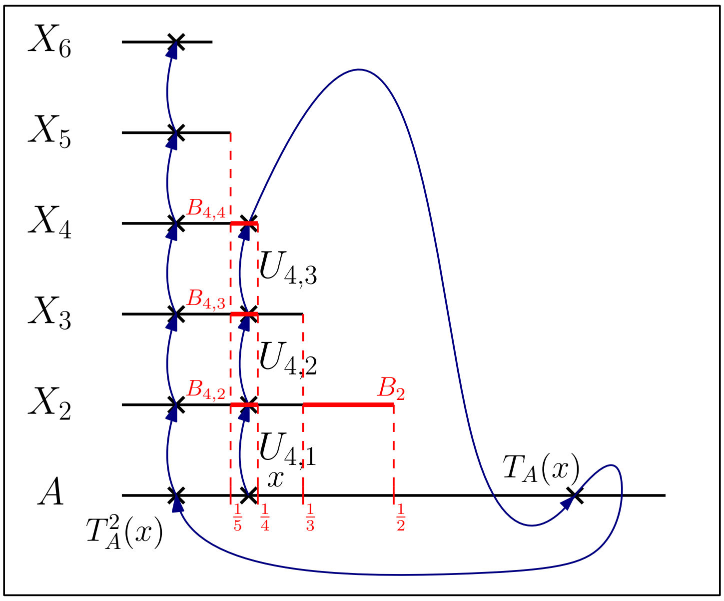

Figure 3.6: Visualisation of the proof of Theorem 3.7.15.

The proof is now almost over. Indeed, we only have to define T by cutting and pasting the sequence of partial automorphisms $(\phi_n)_{n \geqslant 2}$ along the sequence $(A_n \sqcup B_n)_{n \geqslant 2}$. By construction $(A_n \sqcup B_n)_{n \geqslant 2}$ is a partition of $A \sqcup (\bigsqcup_{k \geqslant 2} X_k) = X$, and each ϕ_n is in $[[\mathbb{G}]]$, so $T \in \mathbb{G}$. Finally, we notice that by construction, T_A is the bijection induced by T on A, and therefore ergodicity of T_A yields ergodicity of T (see *e.g.* [Bro76, Thm. 1.10]). $\square$

Recall that $\mathbb{G}_f$ is τ_u-dense (hence τ_m^o-dense when $(\mathbb{G}, \tau_m^o)$ is Polish by Theorem 3.6.19) in $\mathbb{G}$ (see Remark 3.6.26). We have the following infinite-measure analogue of [CLM16, Thm. 4.4]. Notice that we assume ergodicity.

Theorem 3.7.16. *Let $\mathbb{G} = [\mathcal{R}_G]$ be an ergodic orbit full group on (X, λ) associated with a Borel action of a Polish group (G, τ_G), such that τ_m^o is Polish on $\mathbb{G}$. The following are equivalent:*

(1) the set $\mathrm{APER} \cap \mathbb{G}$ is τ_m^o-dense in $\mathbb{G}$;

(2) the set $\mathrm{ERG} \cap \mathbb{G}$ is τ_m^o-dense in $\mathbb{G}$;

(3) the $\mathbb{G}_f$-conjugacy class of any element of $\mathrm{APER} \cap \mathbb{G}$ is τ_m^o-dense in $\mathbb{G}$;

(4) for all $A \in \mathrm{MAlg}(X, \lambda)$, there is a sequence (T_n) of elements of $\mathbb{G}$ such that $T_n \to \mathrm{id}_X$ for τ_m^o, and for all $n \in \mathbb{N}$, $\operatorname{supp} T_n = A$ and T_n is ergodic on its support;

(5) for all $A \in \mathrm{MAlg}_f(X, \lambda)$, there is a sequence (T_n) of elements of $\mathbb{G}_f$ such that $T_n \to \mathrm{id}_X$ for τ_m^o and such that $\operatorname{supp} T_n = A$ for all $n \in \mathbb{N}$;

(6) for any sequence (γ_n) in $\mathrm{Aut}(X, \lambda)$ such that any γ_n has almost no fixed points, there is a dense G_δ set (for τ_m^o) of elements S in $\mathbb{G}$ such that any product-word composed alternatingly of letters in $\{\gamma_n \mid n \in \mathbb{N}\}$ and $\left\{S^k \mid k \in \mathbb{Z} \setminus \{0\}\right\}$ has almost no fixed points;

(7) for any essentially free measure-preserving action $\Gamma \curvearrowright (X, \lambda)$ of a countable discrete group Γ, there is a dense G_δ set of elements of $\mathbb{G}$ inducing an essentially free action of $\Gamma * \mathbb{Z}$;

(8) for any $n \in \mathbb{N}$ there is a dense G_δ set (for the product of τ_m^o) of elements $(S_1, \ldots, S_n)$ in $\mathbb{G}^n$ inducing an essentially free action of $\mathbb{F}_n$.

These equivalent conditions moreover imply the following one:

(9) for any τ_G-neighbourhood V of e_G, the set $\bigcup_{g \in V} \operatorname{supp} g$ is conull;

and the converse implication (9) $\Longrightarrow$ (1-8) holds if the space X is endowed with a compatible Polish topology, with regard to which the measure is locally finite and the G-action is continuous.

Proof. We first notice that *(1)*, *(2)* and *(3)* are all equivalent. Indeed, by Theorem 3.7.15 there exists an element T in $\mathrm{ERG} \cap \mathbb{G}$ (in particular it is conservative aperiodic, see Proposition 1.2.23). This yields *(3)* $\Longrightarrow$ *(2)*. By Theorem 3.7.14, the $\mathbb{G}_f$ conjugacy class of the aforementioned ergodic element is τ_u-dense in $\mathrm{APER} \cap \mathbb{G}$. As τ_u refines τ_m^o, this yields *(1)* $\Longrightarrow$ *(3)*. Finally, *(2)* $\Longrightarrow$ *(1)* is immediate.

(2) $\Longrightarrow$ *(4)*: We choose a sequence (T_n) or ergodic bijections which converges to id_X. For any Borel subset A of positive measure the corresponding induced bijections are well defined. We fix such a subset A, by [CLM16, Prop. 4.2], the sequence $(T_{n,A})$ of induced bijections on A is as wanted, since any bijection is ergodic if and only its induced bijection on A is ergodic (see *e.g.* [Aar97, Prop. 1.5.2]).

(4) $\Longrightarrow$ *(5)* is immediate, *(6)* $\Longrightarrow$ *(7)* is straightforward as soon as we enumerate the elements of an acting countable group Γ and rewrite *(6)* as an essentially free action. Now *(7)* $\Longrightarrow$ *(8)* is a direct induction and *(1)* is a weaker reformulation of *(8)* for $n = 1$.

(5) $\Longrightarrow$ *(6)*: The proof closely follows the one from [Tör06, The Category Lemma]. We fix a partition (X_k) such that $X = \sqcup_{k \in \mathbb{N}} X_k$ and $\lambda(X_k) = 1$ for any $k \in \mathbb{N}$ for the rest of the proof. We will show that for any $(\varepsilon_i) \to 0$, $T \in \mathrm{Aut}(X, \lambda)$ has almost no fixed points if and only if

$$d'_{u, X_k}(T, \mathrm{id}_X) = \lambda(\{x \in X_k \mid T(x) \neq x\}) > 1 - \varepsilon_i$$

(see Definition 3.3.15 for the definition of pseudometrics inducing the uniform topology), for any $k, i \in \mathbb{N}$. We will need the following two preliminary lemmas, we prove the first one and refer to Törnquist's paper for the proof of the second one.

Lemma 3.7.17 ([Tör06],[CLM16]). *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \lambda)$ be an ergodic orbit full group satisfying condition (5) of* Theorem 3.7.16*. If $(A_i)_{i\in\mathcal{I}}$ is a finite family of disjoint Borel subsets of X of finite measure, then for any τ_m^o-neighbourhood $\mathcal{N}$ of id_X in $\mathbb{G}$ there is an element $T \in \mathcal{N}$ such that $\operatorname{supp} T = \sqcup_{i\in\mathcal{I}} A_i$ and $T(A_i) = A_i$ for any $i \in \mathcal{I}$.*

Proof. From condition *(5)*, for each $i \in \mathcal{I}$ we get a sequence (T_n^i) in $\mathbb{G}$ that τ_m^o-converges to id_X, and with $\operatorname{supp} T_n^i = A_i$ $(\forall n \in \mathbb{N})$. The sequence $(\prod_{i\in\mathcal{I}} T_n^i)$ still converges to id_X, and has $\sqcup_{i\in\mathcal{I}} A_i$ as its support, so for n large enough $T \coloneqq \prod_{i\in\mathcal{I}} T_n^i \in \mathcal{N}$, and is adequate. ♢

Lemma 3.7.18 ([Tör06, Lem. 2]). *Let ν be a finite Borel measure on X. Let $n \in \mathbb{N}^*$, and $T_1, \ldots T_n$ be Borel maps from X to itself such that*

$$\nu\left(\{x \in X \mid \forall i, j \in \{1, \ldots, n\} : i \neq j \Rightarrow T_i(x) \neq T_j(x)\}\right) > K > 0.$$

Then there exist finitely many disjoint non-null Borel subsets $E_1, \ldots, E_m$ such that the two following conditions are satisfied: $\nu(\bigsqcup_{l\leqslant m} E_l) > K$ and $T_i(E_l) \cap T_j(E_l) = \emptyset$ for any $l \leqslant m$ whenever $i \neq j$.

With these two lemmas, we can now prove the analogue of the Category Lemma from [Tör06], using only condition *(5)*. The proof adapts with a few modifications, and is given for convenience.

Proof of *(5)* $\Longrightarrow$ *(6)*: We consider $(\gamma_n)_{n\in\mathbb{N}}$ as in *(6)*, a a generator of the group $\mathbb{Z}$, and $\mathscr{A}_1 \cup \mathscr{A}_2 \coloneqq \{\gamma_n \mid n \in \mathbb{N}\} \cup \left\{a^k \mid k \in \mathbb{Z} \setminus \{0\}\right\}$ an alphabet. Then $\mathbb{F}_{alt}$ is the set of words formed with letters in $\mathscr{A}_1$ and $\mathscr{A}_2$ alternatingly. We will however write a^k as $a \ldots a$ (k times), and say that only a and a^{-1} are letters of W. For instance the word

$$W = aa\gamma_{i_8}a^{-1}a^{-1}\gamma_{i_5}aaa\gamma_{i_1}$$

has length $\mathrm{len}(W) = 10$, and notice that we count the letters from right to left, as we see them as functions (here the last letter of W is a). For a word $W \in \mathbb{F}_{alt}$, we consider the evaluation map $e_W : \mathrm{Aut}(X, \lambda) \to \mathrm{Aut}(X, \lambda)$ sending S to the product-word obtained by replacing a by S. By virtue of $\mathrm{Aut}(X, \lambda)$ being a topological group for τ_w, the map e_W is τ_w-continuous. We fix a non-trivial word W of length $\mathrm{len}(W) = n$, and our goal is to show that

$$\{S \in \mathbb{G} \mid \operatorname{supp} e_W(S) = X\}$$

is G_δ in $\mathbb{G}$ for τ_m^o. This is done by induction on $\mathrm{len}(W)$. We assume that the property holds for any word η such that $\mathrm{len}(\eta) < n$. We need to show that the set

$$G_{k,\varepsilon} \coloneqq \left\{S \in \mathbb{G} \mid d'_{u,X_k}(e_W(S), \mathrm{id}_X) > 1 - \varepsilon\right\}$$

is open and dense for τ_m^o, which yields the result for any choice of $(\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i \to 0$.

(i) $G_{k,\varepsilon}$ is open: this is a consequence of the following claim.

Claim. If $T \in \mathrm{Aut}(X, \lambda)$ satisfies $d'_{u,X_k}(T, \mathrm{id}_X) > K > 0$, then there is a τ_w-neighbourhood $\mathcal{N}$ of T such that $d'_{u,X_k}(S, \mathrm{id}_X) > K$ for any $S \in \mathcal{N}$.

Proof of Claim: We fix $\delta > 0$ such that $d'_{u,X_k}(T, \mathrm{id}_X) > K + \delta$. By Lemma 3.7.18 applied to T and id_X, there exists disjoint non-null Borel subsets $E_1, \ldots, E_m$ satisfying $\lambda(\bigsqcup_{l\leqslant m} E_l) > K + \delta$ (we can assume that the E_l are subsets of X_k), and $T(E_l) \cap E_l = \emptyset$ for any $l \leqslant m$. Consider the set

$$\mathcal{N} \coloneqq \left\{S \in \mathrm{Aut}(X, \lambda) \mid \forall\, l \leqslant m : \lambda(S(E_l)\Delta T(E_l)) < \frac{\delta}{m}\right\},$$

which is a τ_w-neighbourhood of T (see [Fre06, Prop. 494B]). For any $S \in \mathcal{N}$ we have $\lambda(S(E_l) \cap E_l) < \frac{\delta}{m}$, so

$$d'_{u,X_k}(S, \mathrm{id}_X) \geqslant \sum_{l \leqslant m} \lambda(\{x \in E_l \mid S(x) \neq x\}) \geqslant \sum_{l \leqslant m} (\lambda(E_l) - \lambda(\{x \in E_l \mid S(x) = x\}))$$
$$\geqslant \sum_{l \leqslant m} (\lambda(E_l) - \lambda(S(E_l) \cap E_l) \geqslant \sum_{l \leqslant m} \left(\lambda(E_l) - \frac{\delta}{m}\right) > K,$$

which means that $\mathcal{N}$ is the desired neighbourhood. $\diamondsuit$

The fact that $G_{k,\varepsilon}$ is τ_w-open is then immediate, since it is preimage by e_W of the open set $\left\{T \in \mathrm{Aut}(X, \lambda) \mid d'_{u,X_k}(T, \mathrm{id}_X) > 1 - \varepsilon\right\}$. It is then τ_m^o-open by Theorem 3.6.4.

(ii) $G_{k,\varepsilon}$ is dense: We denote by $W = W_n, \dots, W_1$ the words obtained from W by removing the leftmost letter when passing to W_i from W_{i+1}. They are uniquely determined, and satisfy $\mathrm{len}(W_i) = i$ and $W_{i+1} = \alpha W_i$ for a unique $\alpha \in \mathscr{A}_1 \cup \{a, a^{-1}\}$ (for all $i < n$). We fix $S \in \mathbb{G}$ and a τ_m^o-neighbourhood $\mathcal{N}$ of S. Our goal is to exhibit an element S' in $G_{k,\varepsilon} \cap \mathcal{N}$. By the inductive hypothesis we can assume

$$S \in \bigcap_{\{\eta \mid \mathrm{len}(\eta) < n\}} \bigcap_{k \in \mathbb{N}} \left\{S' \in \mathbb{G} \mid d'_{u,X_k}(e_\eta(S'), \mathrm{id}_X) = 1\right\}.$$

We denote by i_0 the largest $i < n$ such that $W_{i+1} = a^{\pm 1} W_i$, *i.e.* the rank of the letter at the right of the leftmost $a^{\pm 1}$ in W. We can moreover (up to a noncommittal swap) assume that it is a and not a^{-1}. In the example above where $\mathrm{len}(W) = 10$, $i_0 = 9$, and notice that i_0 is necessarily equal to $n - 1$ (if the last letter of W is a) or $n - 2$ (if the last letter of W is in $\mathscr{A}_1$).

For any $i < j < n$ we have $e_{W_i}(S) \neq e_{W_j}(S)$ λ-a.e. (in particular $\lambda_{\restriction X_k}$-a.e.), so by Lemma 3.7.18 we can find finitely many disjoint Borel subsets $E_1, \dots E_m$ of X_k such that

$$\begin{cases} \lambda\left(\bigsqcup_{l \leqslant m} E_l\right) > 1 - \varepsilon \\ e_{W_i}(S)(E_l) \cap e_{W_j}(S)(E_l) = \emptyset \text{ for any } i < j < n \text{ and any } l \leqslant m. \end{cases} \tag{$\star$}$$

We define the finite family $(A_i)_{i \in \mathcal{I}_1}$ as the family containing all the Borel sets of the form $e_{W_i}(S)(E_l)$ for $i < n$ and $l \leqslant m$. We also define

$$\mathrm{F}_{W,k}(S) \coloneqq X_k \setminus \operatorname{supp} e_W(S)$$

the set in X_k of all points fixed by the W-evaluation of S, up to measure 0.

If $\lambda(\mathrm{F}_{W,k}(S)) < \varepsilon$, that means that $d'_{u,X_k}(e_W(S), \mathrm{id}_X) > 1 - \varepsilon$ and there is nothing to do. By using the first line of ($\star$) we can therefore assume that $\lambda(\mathrm{F}_{W,k}(S) \cap E_l) > 0$ for a certain $l \leqslant m$, and up to a renaming we can assume that $\lambda(\mathrm{F}_{W,k}(S) \cap E_1) > 0$.

We then let $B = e_{W_{i_0}}(S)(\mathrm{F}_{W,k}(S) \cap E_1)$. By Lemma 3.7.17 applied to the family of disjoint subsets of B of the form $(A_i \cap B) \setminus \cup_{j \in \mathcal{I}_1 \setminus \{i\}} A_j$ we can find T in $S^{-1}\mathcal{N}$ such that $\operatorname{supp} T = B$, $T(A_i) = A_i$ for any $i \in \mathcal{I}_1$. We then define $S_1 = ST$, which satisfies

$$\begin{cases} S_1(A_i) = S(A_i) \text{ for all } i \in \mathcal{I}_1, \\ S_1 \in \mathcal{N}, \\ \text{a.e. on } E_1 \setminus \mathrm{F}_{W,k}(S), e_W(S_1) = e_W(S) \neq \mathrm{id}_X, \\ \text{a.e. on } \mathrm{F}_{W,k}(S) \cap E_1, e_W(S_1) \neq e_W(S) = \mathrm{id}_X. \end{cases}$$

Indeed, the first two line hold by construction of T, and the third and fourth lines hold because $\operatorname{supp} T = B$ and because of the second line of ($\star$): $e_{W_j}(S)(E_1)$ and $e_{W_j}(S_1)(E_1)$

do not meet B for $j < i_0$ by ($\star$) so they agree on E_1, then the last application of T makes them disagree (for the fourth line). We thus have $E_1 \subseteq \operatorname{supp} e_W(S_1)$.

We can then apply Lemma 3.7.18 again to $e_W(S_1)$ and id_X to obtain disjoint non-null Borel subsets $E'_1, \dots E'_p$ of $E_1 \subseteq X_k$ such that

$$\lambda\left(\bigsqcup_{q\leqslant p} E'_q \sqcup \bigsqcup_{1<l\leqslant m} E_l\right) > 1-\varepsilon$$

and $e_W(S_1)(E'_q) \cap E'_q = \emptyset$ for any $q \leqslant p$.

the proof is now almost over. We define $(A_i)_{i\in\mathcal{I}_2}$ to be the collection $(A_i)_{i\in\mathcal{I}_1}$ to which we adjoined all the $e_{W_i}(S_1)(E'_q)$, for $q \leqslant p$ and $i < n$.

If $\lambda(\mathrm{Fix}_{W,k}(S_1)) < \varepsilon$, we are done as $S_1 \in \mathcal{N}$, and if not, there exists $l \in 2, \dots, m$ such that $\lambda(\mathrm{Fix}_{W,k}(S) \cap E_l) > 0$, and once again we may assume that $\lambda(\mathrm{Fix}_{W,k}(S) \cap E_2) > 0$. We can apply the whole argument again to E_2 and $(A_i)_{i\in\mathcal{I}_2}$ replacing E_1 and $(A_i)_{i\in\mathcal{I}_1}$ respectively, and we obtain $S_2 \in \mathcal{N}$ satisfying $E_2 \subseteq \operatorname{supp} e_W(S_2)$. The main point is that the construction ensures that $S_1(A_i) = S_2(A_i)$ for all $i \in \mathcal{I}_2$, so we still have $e_W(S_2)(E'_q) \cap E'_q = \emptyset$ for all $q \leqslant p$. Therefore, in finitely many iterations of the argument we obtain S' satisfying the requirements: $S' \in \mathcal{N}$ and $d'_{u,X_k}(e_W(S'), \mathrm{id}_X) = \lambda(\{x \in X_k \mid e_W(S')(x) \neq x\}) = \lambda(X_k \setminus \mathrm{Fix}_{W,k}(S')) > 1-\varepsilon$.

The only thing left to do is to link *(9)* to the other equivalent conditions. The proof is almost the same as [CLM16, Thm. 4.4], as such it is detailed only for the sake of completeness.

Proof of *(1)* $\Longrightarrow$ *(9)*: This is done by contraposition. We choose a probability measure $\mu \in [\lambda]$. Assuming *(9)* is not satisfied, there exists a τ_G-neighbourhood V of e_G such that $\mu(\bigcup_{g\in V} \operatorname{supp} g) < 1-\delta$ for some $\delta > 0$. By definition of τ_m on $\widetilde{[\mathcal{R}_G]}$ (see Section 3.5.1 for the relevant notations) the set

$$\mathcal{U} := \left\{ f \in \widetilde{[\mathcal{R}_G]} \;\middle|\; \mu(\{x \in X : f(x) \notin V\}) > \frac{\delta}{2} \right\} \subseteq \mathrm{L}^0(X, \lambda, (G, \tau_G))$$

is a τ_m-neighbourhood of the identity in $\widetilde{[\mathcal{R}_G]}$. However for any $f \in \mathcal{U}$ we have $d_\mu(\Phi(f), \mathrm{id}_X) < 1$ by construction, so $\Phi(f)$ is not aperiodic, and therefore the projection of $\mathcal{U}$ on $[\mathcal{R}_G]$ is a τ_m^o-neighbourhood of id_X comprised of non-aperiodic elements, so *(1)* does not hold.

Proof of *(9)* $\Longrightarrow$ *(5)*: Recall that for this implication we have a Polish topology on X, that the G-action is continuous and that λ is locally finite with regard to the topology. For a τ_G-neighbourhood V of e_G in G, we say that $T \in \mathbb{G}$ is V*-uniformly small* if there exists $f \in \mathrm{L}^0(X, \lambda, (G, \tau_G))$ such that

$$\begin{cases} f(X) \subseteq V \\ \lambda\text{-a.e. on } X : T(x) = f(x)\cdot x \end{cases}$$

and the main point is that for any τ_m^o-neighbourhood $\mathcal{N}$ of id_X in $\mathbb{G}$, there exists a τ_G-neighbourhood V of e_G in G such that being V-uniformly small implies being in $\mathcal{N}$. We fix such sets $\mathcal{N}$ and V as before. Our goal is to prove that for any $A \in \mathrm{MAlg}_f(X, \lambda)$ there exists $T \neq \mathrm{id}_X$ which is V-uniformly small with $\operatorname{supp} T \subseteq A$. By a maximality argument (see Proposition 3.2.3), this will yield the existence of a $T \in \mathcal{N}$ with $\operatorname{supp} T = A$, concluding the proof.

By outer regularity (see Proposition 2.2.21, this is where the local finiteness is used), we fix an open set $B \subseteq X$ such that $A \subseteq B$ and $\lambda(B \setminus A) < \lambda(A)$, in particular B has finite measure. We also fix an open neighbourhood U of e_G in G such that $U = U^{-1}$ and $U^2 \subseteq V$.

Claim. There exists a countable family $(g_i)_{i\in\mathbb{N}}$ of elements of U and an a.e. partition $(A_i)_{i\in\mathbb{N}}$ of A such that for all $i\in\mathbb{N}$, $g_i(A_i)$ is a subset of B that is disjoint from A_i.

Proof of Claim: Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of open neighbourhoods of e_G in G, and let

$$S := \bigcap_{n\in\mathbb{N}} \bigcup_{g\in U_n} \operatorname{supp} g.$$

By hypothesis, S has full measure. Moreover since the action is continuous, for all $x\in S\cap B$ there is a $g\in U$ such that $g\cdot x\in B$ and $g\cdot x\neq x$. Again by continuity we can find an open neighborhood $W_x\subseteq B$ of x such that $g(W_x)$ and W_x are disjoint. We can now define the partition $(A_i)_{i\in\mathbb{N}}$ to be a countable open subcover of $(W_x)_{x\in S\cap B}$ which exists by Lindelöf's lemma. $\diamondsuit$

There are now two cases to consider.

• If there exists some $i\in\mathbb{N}$ such that $g_i(A_i)$ is not disjoint from A, we set $C_i := g_i^{-1}(g_i(A_i)\cap A)$, and define $T\in\mathbb{G}$ by

$$T(x)=\begin{cases} g_i\cdot x & \text{if } x\in C_i,\\ g_i^{-1}\cdot x & \text{if } x\in g_i(C_i),\\ x & \text{otherwise.}\end{cases}$$

• If for all $i\in\mathbb{N}$ we have that $g_i(A_i)$ is disjoint from A, then since $\lambda(B\setminus A)<\lambda(A)$ there exists $i\neq j$ in $\mathbb{N}$ such that $g_i(A_i)\cap g_j(A_j)$ is non-null. We then set $C_{i,j} := g_i^{-1}(g_i(A_i)\cap g_j(A_j))$ and define $T\in\mathbb{G}$ by

$$T(x)=\begin{cases} g_j^{-1}g_i\cdot x & \text{if } x\in C_{i,j},\\ g_i^{-1}g_j\cdot x & \text{if } x\in g_j^{-1}g_i(C_{i,j}),\\ x & \text{otherwise.}\end{cases}$$

In both cases, the element $T\in\mathbb{G}$ is V-uniformly small, so it is in $\mathcal{N}$, it is non-trivial, and it satisfies $\operatorname{supp} T\subseteq A$, which concludes the proof. $\square$

We finally obtain an analogue of [CLM16, Cor. 4.5] thanks to Corollary 2.4.4, Proposition 3.5.29 and Proposition 3.6.14. The proof is exactly the same, and as such, is not provided. The authors also provide a non-trivial example where *(i)* holds in [CLM16, Ex. 4.6], which generalizes to infinite measures.

Corollary 3.7.19. *Let G be a locally compact Polish group acting in a measure-preserving and ergodic manner on a standard σ-finite space (X,λ). Then exactly one of the following holds:*

(i) $\mathcal{R}_G$ is a countable measure-preserving equivalence relation;

(ii) The set $\mathrm{APER}\cap[\mathcal{R}_G]$ *is τ_m^o-dense in $[\mathcal{R}_G]$.*

Remark 3.7.20. Without using condition *(9)*, we can still deduce that *(i)* implies that *(ii)* does not hold. Indeed, if $\mathcal{R}_G$ is a countable measure-preserving equivalence relation, then $\tau_u=\tau_m^o$ by uniqueness of the Polish topology on $[\mathcal{R}_G]$ (Theorem 3.6.4) and Proposition 3.6.14, so it is not possible for *(4)* to hold, by definition of τ_u.

Given the ambiguous role that $\mathbb{G}_f$ plays, it is natural to ask whether we can find a statement on its elements that is equivalent to the statements in Theorem 3.7.16. It is not hard to see that the answer is positive if we can approach aperiodic elements of $\mathbb{G}$ by elements of $\mathbb{G}_f$ that are aperiodic on their supports. The following is most likely well-known.

Proposition 3.7.21. *Let T be a conservative aperiodic (resp. ergodic) element of* $\mathrm{Aut}(X,\lambda)$*. For any τ_u-neighbourhood N of T, there exists $T_f \in [T]_f \cap N$ such that $T_{f\restriction \operatorname{supp} T_f}$ is aperiodic (resp. ergodic).*

Proof. We start by describing T as a skyscraper as in step *(i)* of the proof of [CK79, Prop 2]: there exists $B \subseteq X$ with $\lambda(B) < 1$ and a sequence $(B_n)_{n\in\mathbb{N}}$ satisfying

$$\begin{cases} B_0 = B, \\ \lambda(B_{n+1}) \leqslant \lambda(B_n), \\ \lambda(B_n) \to_n 0, \\ X = \bigsqcup_{n\in\mathbb{N}} B_n, \end{cases}$$

as well as an aperiodic (resp. ergodic) measure-preserving bijection T_B of B and injective measure-preserving maps $\pi_n : B_{n+1} \to B_n$ satisfying for any $n \in \mathbb{N}$:

$$\begin{cases} T = \pi_n^{-1} & \text{on } \pi_n(B_{n+1}), \\ T = T_B \pi_1 \pi_2 \dots \pi_{n-1} & \text{on } B_n \setminus \pi_n(B_{n+1}). \end{cases}$$

Once this observation is made, the proof is over, since the sequence of induced bijections on $\bigsqcup_{0\leqslant i\leqslant n} B_i$ uniformly approaches T by construction, and each term of the sequence is in $[T]$, has a support of finite measure, and is aperiodic (resp. ergodic) on its support. $\square$

Remark 3.7.22. Thanks to Proposition 3.7.21 we can add the following equivalent conditions to Theorem 3.7.16:

(1') The set of elements of $\mathbb{G}_f$ aperiodic on their support is τ_m^o-dense in $\mathbb{G}$;

(2') The set of elements of $\mathbb{G}_f$ ergodic on their support is τ_m^o-dense in $\mathbb{G}$.

3.8 Spatial models for actions of ergodic full groups

In this short section we briefly discuss the possible spatial actions that an ergodic full group can admit. As usual we fix an ergodic full group $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$, and assume now that it is endowed with a topology that makes it a Polish group.

The first thing to notice is that our proof that the natural action of $\mathrm{Aut}(X,\lambda)$ is whirly (see Proposition 2.7.2) generalizes to $\mathbb{G}$, and even to $\mathbb{G}_f$, without any issue thanks to Corollary 3.4.13. Indeed, for any two Borel subsets A and B of positive measure, $\mathbb{G}_f$ can exchange two subsets of A and B respectively with an exchanging involution, and we can choose these subsets to be as small as needed. This guarantees that the involution is as $\tau_{u,f}$-close to id_X as needed. Since $\tau_{u,f}$ is finer than any Polish topology we can consider, from this fact along with Proposition 2.7.3 and Theorem 3.6.29 we thus have the following generalization of Proposition 2.7.4.

Proposition 3.8.1. *The natural action of $\mathbb{G}$ on (X,λ) cannot admit a spatial model. Moreover, if $\mathbb{G}$ is the full group of a countable equivalence relation, then the natural action of $\mathbb{G}_f$ on (X,λ) cannot admit a spatial model.*

The next question is whether these groups can admit other spatial actions preserving an infinite measure. By using Theorem 2.6.9 we can prove that the answer is negative in our usual locally finite setting.

Proposition 3.8.2. *Every continuous measure-preserving spatial action of $\mathbb{G}$ on a Polish space Y endowed with an atomless σ-finite locally finite measure η is trivial, i.e. the set of fixed points is conull. Moreover this also holds for $\mathbb{G}_f$ if $\mathbb{G}$ is the full group of a countable equivalence relation.*

Proof. Suppose by contradiction that there exists T in either $\mathbb{G}$ or $\mathbb{G}_f$ if the condition is respected, acting non-trivially, *i.e.* such that the measure of $\operatorname{supp} T \coloneqq \{y \in Y \mid T \cdot y \neq y\}$ is positive. By Ryzhikov's theorem (Theorem 3.4.18), we can write T as a product of three involutions in $\mathbb{G}$ (resp. $\mathbb{G}_f$). At least one of these three involutions is acting non-trivially, let us call it U, and denote by Y' a fundamental domain of the action of U on its support.

We next identify the full group $[U]$ generated by U and the group $\mathrm{L}^0(Y', \eta_{\restriction Y'}, \mathbb{Z}/2\mathbb{Z})$. Indeed, as any element of $[U]$ can only map points to themselves or to their image by U, one easily checks that elements of $[U]$ are involutions, with the product of two such involutions acting like U on the symmetric difference of the supports:

$$V_1 V_2 = \begin{cases} U \text{ on } \operatorname{supp} V_1 \Delta \operatorname{supp} V_2 \\ \mathrm{id}_X \text{ elsewhere,} \end{cases}$$

where V_1 and V_2 are any two elements of $[U]$. Thus it is straightforward that

$$\begin{array}{ccc} [U] & \longrightarrow & \mathrm{L}^0(Y', \eta_{\restriction Y'}, \mathbb{Z}/2\mathbb{Z}) \\ V & \longmapsto & \mathbb{1}_{Y' \cap \operatorname{supp} V} \end{array}$$

is a group isomorphism. It is moreover a topological group isomorphism, when $[U]$ is endowed with the uniform topology generated by d_ν (where ν is a probability measure in the class of $\eta_{\restriction Y'}$), and $\mathrm{L}^0(Y', \eta_{\restriction Y'}, \mathbb{Z}/2\mathbb{Z})$ is endowed with the topology of convergence in measure generated by $\tilde{d}$ (see Section 3.3 for the notations, and note also that both of these topologies only depend on the class of the measure). This is just a consequence of the definitions: for any $V_1, V_2 \in [U]$ we have

$$\begin{aligned} d_\nu(V_1, V_2) &= \nu(\{y \in Y \mid V_1(y) \neq V_2(y)\}) \\ &= \nu(\operatorname{supp} V_1 \Delta \operatorname{supp} V_2) \\ &= 2 \int_{Y'} d_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{1}_{\operatorname{supp} V_1}(y), \mathbb{1}_{\operatorname{supp} V_2}(y)) d\nu(y) \\ &= 2\tilde{d}(\mathbb{1}_{Y' \cap \operatorname{supp} V_1}, \mathbb{1}_{Y' \cap \operatorname{supp} V_2}). \end{aligned}$$

Do note that this identification is classical, and that this group is also isomorphic (as a topological group once again) to the additive group (when endowed with the symmetric difference) of the measure algebra $\mathrm{MAlg}(Y', \eta_{\restriction Y'})$ with the topology generated by the metric defined in Section 3.3.

Recall now that $\mathrm{L}^0(Y', \eta_{\restriction Y'}, \mathbb{Z}/2\mathbb{Z})$ is a Lévy group when endowed with this topology (see *e.g.* [Pes06, Thm. 4.2.2]). From Theorem 2.6.9 we deduce that any continuous spatial action of $([U], \tau_u)$ on (Y, η) is trivial. However by our hypothesis $[U]$ acts continuously spatially and non-trivially on (Y, η) when endowed with the topology inherited from $\mathbb{G}$. By Theorem 3.6.19, this topology is coarser than the uniform topology, so $([U], \tau_u)$ acts continuously spatially and non-trivially on (Y, η), a contradiction. □

Remark 3.8.3. The exact same argument works in the simpler case of spatial probability measure-preserving actions on a standard probability space by using the original Glasner Tsirelson and Weiss theorem (Theorem 2.6.8) instead of Theorem 2.6.9.

Appendix A

Fremlin's Reconstruction Theorem

Abstract

In this appendix we provide context and give a demonstration of a generalization of Dye's reconstruction theorem [Dye63, Thm. 2], proved by Fremlin [Fre02, Thm. 384D] following Eigen's arguments from [Eig82]. The theorem is stated for the very general setting of Dedekind complete boolean algebras and their automorphism groups, but we will specifically work with measure algebras of standard spaces.

Thoughts put into writing remain after we are gone, and they might spur someone into action far in the future. What else could you call that but a miracle?

Jolenta in *On the movements of the Earth*

Contents

Organization of the Appendix

This appendix is organized in the following way: Section A.1 contains both Dye and Fremlin's Reconstruction theorems, followed by the relevant definitions and some elementary facts. Section A.2 explains why Dye's setting satisfies the hypotheses of Fremlin's theorem, and in particular it establishes the link between the two different languages. The three subsections are specifically dedicated to order-completeness, the supports of Borel bijections, and to finding many involutions in our full groups, respectively. Finally, Section A.3 is dedicated to the proof of a version of the theorem specifically stated in the context of measure algebras, although the general proof is sensibly the same.

A.1 Statements and definitions

A.1.1 Dye's Reconstruction theorem

Let us start by recalling Dye's theorem (Theorem 1.3.13), in one of its most commonly found forms. For its most general form, see Theorem A.2.25.

Theorem A.1.1 ([Dye63, Thm. 2], see also [Kec10, Sec. I.4])**.** *Let $\mathbb{G}$ and $\mathbb{H}$ be two ergodic full groups on a standard probability space (X, μ). Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some measure-preserving bijection. In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in $\mathrm{Aut}(X, \mu)$ such that for all $T \in \mathbb{G}$, we have $\psi(T) = STS^{-1}$.*

A.1.2 Fremlin's theorem

We now state Fremlin's theorem in full generality, then we will give the corresponding definitions. Although the proof will be given specifically in the context of measure algebras, the level of generality described by Fremlin still provides some insight. The reader interested in having reformulations more adapted to the measurable context will find them in Section A.2.

Theorem A.1.2 ([Fre02, Thm. 384D])**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Dedekind complete boolean algebras, $\mathbb{G}$ and $\mathbb{H}$ two subgroups of $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{Aut}(\mathfrak{B})$ respectively, both having many involutions. If $\psi : \mathbb{G} \to \mathbb{H}$ is a group isomorphism, there exists a unique boolean isomorphism $S : \mathfrak{A} \to \mathfrak{B}$ such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

Definition A.1.3. A **boolean algebra** is a ring $(\mathfrak{A}, +, \times)$ such that $A^2 = A \times A = A$ for all A in $\mathfrak{A}$, and such that $\mathfrak{A}$ is unital with multiplicative identity $1_{\mathfrak{A}}$. A **boolean homomorphism** between two boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a unital ring homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$: for any A, B in $\mathfrak{A}$ a boolean homomorphism ϕ satisfies $\phi(A + B) = \phi(A) + \phi(B)$, $\phi(A \times B) = \phi(A) \times \phi(B)$ and $\phi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

We denote by $\mathrm{Aut}(\mathfrak{A})$ the group of bijective boolean homomorphisms from $\mathfrak{A}$ to itself.

Remark A.1.4. In particular, for any boolean algebra $\mathfrak{A}$, for all A in $\mathfrak{A}$ we have $A + A = 0$, in other words any element is equal to its additive inverse element. (Indeed we have $A + A = (A + A)^2 = (A + A) \times (A + A) = A^2 + A^2 + A^2 + A^2 = A + A + A + A$.) In particular, a boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ maps $0_{\mathfrak{A}}$ to $0_{\mathfrak{B}}$.

Remark A.1.5. The most natural example of a boolean algebra is the algebra of subsets of a set, equipped with the operations of symmetric difference and intersection (*i.e.* the law '$+$' is 'Δ' and the law '$\times$' is '$\cap$'), and with the (multiplicative) identity being the whole set. In this case the complement of a set A corresponds to $1_{\mathfrak{A}} \setminus A = 1_{\mathfrak{A}} \Delta A$, and the union of two sets A and B is $A \cup B = 1_{\mathfrak{A}} \Delta((1_{\mathfrak{A}} \Delta A) \cap (1_{\mathfrak{A}} \Delta B))$.

Notice that identifying subsets to their characteristic functions taking values in $\mathbb{Z}/2\mathbb{Z}$ gives a natural justification of the choice of notations: a boolean algebra is usually denoted by $(\mathfrak{A}, \Delta, \cap)$.

We will be needing the following equivalence about boolean algebra homomorphisms, the proof is straightforward and relies solely on the previously defined boolean properties.

Proposition A.1.6 ([Fre02, Prop. 312H])**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two boolean algebras, and $f : \mathfrak{A} \to \mathfrak{B}$ be a function. Then the following are equivalent:*

(1) f is a boolean homomorphism,

(2) $\forall A, B \in \mathfrak{A}$, we have $f(A \cap B) = f(A) \cap f(B)$ and $f(1_{\mathfrak{A}} \setminus A) = 1_{\mathfrak{B}} \setminus f(A)$,

(3) $\forall A, B \in \mathfrak{A}$, we have $f(A \cup B) = f(A) \cup f(B)$ and $f(1_{\mathfrak{A}} \setminus A) = 1_{\mathfrak{B}} \setminus f(A)$,

(4) $f(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$, and $\forall A, B \in \mathfrak{A}$ such that $A \cap B = 0_{\mathfrak{A}}$ we have $f(A \cup B) = f(A) \cup f(B)$ and $f(A) \cap f(B) = 0_{\mathfrak{B}}$.

Proof. *(1)* $\implies$ *(4)*: Immediate.

(4) $\implies$ *(3)*: Consider A, B in $\mathfrak{A}$. From *(4)* we get

$$f(A) = f(A \cap B) \cup f(A \setminus B),$$

and similarly for B, which yields

$$f(A \cup B) = f(A) \cup f(B \setminus A) = f(A \cap B) \cup f(A \setminus B) \cup f(B \setminus A) = f(A) \cup f(B).$$

In particular, for $B = 1_{\mathfrak{A}} \setminus A$ we get $f(A) \cup f(1_{\mathfrak{A}} \setminus A) = f(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$, and $f(A) \cap f(1_{\mathfrak{A}} \setminus A) = 0_{\mathfrak{B}}$, so by intersecting with $1_{\mathfrak{B}} \setminus f(A)$, we get $f(1_{\mathfrak{A}} \setminus A) = 1_{\mathfrak{B}} \setminus f(A)$, hence *(3)*.

(3) $\implies$ *(2)*: For any A, B in $\mathfrak{A}$ we have

$$f(A \cap B) = f(1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus A) \cup (1_{\mathfrak{A}} \setminus B))) = 1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus f(A)) \cup (1_{\mathfrak{B}} \setminus f(B))) = f(A) \cap f(B).$$

(2) $\implies$ *(1)*: For any A, B in $\mathfrak{A}$ we have

$$\begin{aligned} f(A \Delta B) &= f((1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus A) \cap (1_{\mathfrak{A}} \setminus B))) \cap (1_{\mathfrak{A}} \setminus (A \cap B))) \\ &= (1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus f(A)) \cap (1_{\mathfrak{B}} \setminus f(B))) \cap (1_{\mathfrak{B}} \setminus (f(A) \cap f(B)))) = f(A) \Delta f(B), \end{aligned}$$

and finally we also have $f(1_{\mathfrak{A}}) = f(1_{\mathfrak{A}} \setminus 0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus f(0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus 0_{\mathfrak{B}} = 1_{\mathfrak{B}}$ (because any element is its own Δ-inverse), so f is a boolean homomorphism, hence the implication. $\square$

A boolean algebra (or a field of sets) comes naturally with an order structure. We define *order-completeness*, or *Dedekind-completeness*.

Definition A.1.7. Seeing a boolean algebra $\mathfrak{A}$ as a field of sets, we can define a natural order $\subseteq$ by setting $A \subseteq B$ if and only if $A \cap B = A$. This makes $(\mathfrak{A}, \subseteq)$ a partially ordered set (poset), with least element $0_{\mathfrak{A}}$ and greatest element $1_{\mathfrak{A}}$. Setting $\sup\{A, B\} := A \cup B$ and $\inf\{A, B\} := A \cap B$ makes it a lattice.

Remark A.1.8. We have to take care when extending the notions of supremum and infimum to infinite families. For instance, in the measure algebra of a standard probability space (in particular it is non-atomic!) the uncountable union of all points is equal to the whole space. In the measure algebra, each point corresponds to 0, but the whole space is 1. In other words, an uncountable union does not correspond to a satisfying notion of supremum. We will see how to correctly define those notions in Section A.2.1.

Remark A.1.9 ([Fre02, Prop. 313La])**.** Any element of $\mathrm{Aut}(\mathfrak{A})$ is order-preserving.

Definition A.1.10. Any poset P is **Dedekind-complete** if any non-empty subset of P with an upper-bound admits a *least upper-bound* (a supremum).

Definition A.1.11. Let $\mathfrak{A}$ be a boolean algebra, and T an element of $\mathrm{Aut}(\mathfrak{A})$. We say that $A \in \mathfrak{A}$ **supports** T if $T(B) = B$ for any $B \subseteq 1_{\mathfrak{A}} \setminus A$. If $\operatorname{supp} T \coloneqq \inf \{A \in \mathfrak{A} \mid A \text{ supports } T\}$ is defined, we call it the **support of** T.

Proposition A.1.12 ([Fre02, Cor. 381F])**.** *If $\mathfrak{A}$ is Dedekind-complete, every element of $\mathrm{Aut}(\mathfrak{A})$ has a support.*

Definition A.1.13. A subgroup $\mathbb{G}$ of $\mathrm{Aut}(\mathfrak{A})$ has **many involutions** if for every non-zero $A \in \mathfrak{A}$ there exists an non-trivial involution V in $\mathbb{G}$ such that $\operatorname{supp} V \subseteq A$.

A.2 Why Fremlin's theorem generalizes Dye's theorem

The non-singular setting is the adequate one for stating Fremlin's theorem for measure algebras.

Definition A.2.1. We define on $\mathrm{Aut}(X, [\mu])$ the **weak topology** τ_w in the following way: (T_n) τ_w-converges to T if and only if for all Borel subsets of X one has $\mu(T_n(A) \Delta T(A)) \to 0$ and

$$\left\| \frac{d(T_{n*}\mu)}{d\mu} - \frac{d(T_*\mu)}{d\mu} \right\|_1 \to 0.$$

Proposition A.2.2 ([Ion65])**.** *The group $(\mathrm{Aut}(X, [\mu]), \tau_w)$ is a Polish group.*

The first important step is to establish the link between the conjugations obtained in the different statements of the theorem. As it is stated in Theorem A.1.2, the isomorphism obtained is an isomorphism of boolean algebras. As such, it preserves the neutral elements for both Δ and $\cap$. This is equivalent to saying (for measure algebras) that it is a non-singular bijection of the underlying measure spaces. In other words, for $\mathfrak{A} = \mathrm{MAlg}(X, \mu)$, we have

$$\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(X, [\mu]).$$

Indeed, the class of a non-singular bijection clearly yields an automorphism of the boolean algebra, and conversely a boolean isomorphism yields the class of non-singular bijection. If a genuine function acting on the points (up to null sets) is needed, we will need to use the separability of the measure algebra (see Proposition 1.2.25). This is the next proposition.

The notion of atom of a measure algebra is convenient, and as such it is recalled.

Definition A.2.3. An **atom** in a boolean algebra $\mathfrak{A}$ is a non-zero element $A \in \mathfrak{A}$ such that the only elements $\subseteq$-smaller than A are $0_{\mathfrak{A}}$ and A itself. Do note that for measure algebras it corresponds to the usual measure-theoretic notion of atoms (see [Fre02, Thm. 322B]).

Proposition A.2.4 ([LM14a, Thm. A.14, Cor. A.15])**.** *Let (X, μ) and (Y, ν) be two standard probability spaces.*

- *Any boolean homomorphism* $\Phi : \mathrm{MAlg}(Y,\nu) \to \mathrm{MAlg}(X,\mu)$ *yields a Borel non-singular application* $\varphi : (X,\mu) \to (Y,\nu)$ *that is unique up to null sets.*
- *If* $\Phi : \mathrm{MAlg}(X,\mu) \to \mathrm{MAlg}(Y,\nu)$ *is an isomorphism, there exist two conull Borel subsets* A *and* B *of* X *and* Y *respectively and a non-singular bijection* $\psi : A \to B$ *that is unique up to null sets.*

Proof. We start by fixing d_Y a compatible complete bounded metric on Y that induces its Borel structure. We consider the space $\mathrm{L}^0(X,\mu,Y)$ of measurable functions from X to Y, up to equality on μ-null sets, equipped with the metric defined by

$$d_\infty(f,g) := \operatorname*{ess\,sup}_{x\in X} d_Y(f(x),g(x)),$$

which is complete since d_Y is. Let now (A_n) be a sequence of representatives of a dense sequence in $\mathrm{MAlg}(X,\mu)$, which is separable by Proposition 1.2.25. Since Y is separable in particular it is Lindelöf so we can consider a sequence (y_k) such that $Y = \cup_k B(y_k, 2^{-n})$ for each $n \in \mathbb{N}$. By induction we then define for any $n \in \mathbb{N}$:

$$P_n := P_{n-1} \wedge (B(y_k, 2^{-n}))_{k\in\mathbb{N}} \wedge \{A_n\},$$

the algebra (not σ-algebra!) generated by P_{n-1}, $(B(y_k,2^{-n}))_{k\in\mathbb{N}}$, $\{A_n\}$. The sequence (P_n) is an increasing sequence of countable and atomic sub-algebras of $\mathcal{B}(Y)$, such that each atom of P_n has diameter less than 2^{-n} and $A_n \in P_n$ for any n.

We can now define the desired function as a limit in $\mathrm{L}^0(X,\mu,Y)$. For each $n \in \mathbb{N}$ and each atom $A \in P_n$, we choose $y_A \in A$ and define $\varphi_n \in \mathrm{L}^0(X,\mu,Y)$ by setting $\varphi_n(x) = y_A$ if and only if $x \in \Phi(A)$. By construction, (φ_n) is Cauchy for d_∞, we denote by φ its limit. By density of the classes of the A_n in $\mathrm{MAlg}(Y,\nu)$, φ lifts to Φ, and any other such lift also has to be the limit of the φ_n, so it is equal to φ up to a null set.

We now prove the second part of the statement. We apply the previous point to Φ and Φ^{-1}, yielding two Borel non-singular applications $\varphi' : Y \to X$ and $\varphi : X \to Y$. By uniqueness up to null sets and since $\Phi\Phi^{-1} = \mathrm{id}_{\mathrm{MAlg}(Y,\nu)}$ and $\Phi^{-1}\Phi = \mathrm{id}_{\mathrm{MAlg}(X,\mu)}$, we have two conull Borel subsets A and B of X and Y respectively, such that $(\varphi \circ \varphi')_{\restriction B} = \mathrm{id}_B$ and $(\varphi' \circ \varphi)_{\restriction A} = \mathrm{id}_A$. Setting $\psi = \varphi$ concludes the proof, as the uniqueness is a direct consequence of the first part. □

The version of the theorem that we will actually prove can be seen as a corollary of the most general version of Fremlin's theorem. The statement is the following:

Theorem A.2.5. *Let* (X,μ) *be a standard probability space and* $\mathbb{G}$ *and* $\mathbb{H}$ *be two subgroups of* $\mathrm{Aut}(X,[\mu])$ *with many involutions. Then any isomorphism between* $\mathbb{G}$ *and* $\mathbb{H}$ *is the conjugation by some non-singular bijection. In other words, for any group isomorphism* $\psi : \mathbb{G} \to \mathbb{H}$*, there exists* S *in* $\mathrm{Aut}(X,[\mu])$ *such that for all* $T \in \mathbb{G}$*, we have*

$$\psi(T) = STS^{-1}.$$

There are three things that we need to prove. Firstly, that measure algebras are Dedekind-complete, then that ergodic full groups have many involutions, and finally that for the case of a probability measure μ, the non-singular bijection S is in fact measure-preserving ([Fre02, Cor. 383K]).

The third verification has actually already been done: it is exactly Proposition 3.5.30. Indeed, taking λ to be a finite measure and naming k the element of $\mathbb{R}_+^*$ in the statement of Proposition 3.5.30, as $S_*\lambda = k \times \lambda$ gives the same measure to the whole space, it implies that $k = 1$, and thus S is actually measure-preserving.

A.2.1 Dedekind-completeness of measure algebras

We saw in the definition of boolean algebras that notions of infimum and supremum exist for finite families. As it is often the case with measure theory, it is easy to extend it to countable families (what is sometimes called *Dedekind σ-completeness*), but here the existence of infimum and supremum for arbitrary families is required.

One way of proving it also yields Dedekind-completeness for $\mathrm{MAlg}(X, \lambda)$ in the standard σ-finite context, it uses the notion of localizable measure space (see [Fre03, Def. 211G, Thm. 211Ld, Thm. 322Be]). The second way of doing things is through the separability and metric-completeness of $\mathrm{MAlg}(X, \mu)$ in the standard probability context, which we already established. This is enough for our needs, so we use this method.

Recall from Proposition 1.2.25 that $(\mathrm{MAlg}(X, \mu), d_\mu)$ is complete and separable whenever (X, μ) is a standard probability space (see also Proposition 3.2.2). We then have the following Proposition, ensuring the existence of supremums of arbitrary family of elements in a measure algebra. This is the probability version of Proposition 3.2.3, where a proof can be found. It crucially uses the completeness of the measure algebra.

Proposition A.2.6 ([LM14a, Lem. A.5, Prop. A.6])**.** *Consider (X, μ) a standard probability space, and $\mathrm{MAlg}(X, \mu)$ the associated measure algebra.*

(1) Any upwards directed family of elements of $\mathrm{MAlg}(X, \mu)$ admits a supremum, which is obtained as the limit of an increasing sequence of elements of the family.

(2) Any family of elements of $\mathrm{MAlg}(X, \mu)$ admits a supremum, which is obtained as the limit of an increasing sequence of finite reunions of elements of the family.

Remark A.2.7. If a family of elements of $\mathrm{MAlg}(X, \mu)$ is stable by countable union (in particular it is upwards directed), the supremum of the family is actually a maximum, which means that it is an element of the family.

Remark A.2.8. Of course, by taking complements, everything we said in this section remains true for infimum and minimum, with intersections rather than reunions.

A.2.2 Supports and separators

The goal of this section is to get to the existence of separators, but our first order of business is to link the notion of support defined in Section A.1 to the more 'usual' one used for bijections (see Definition 1.2.1). We chose to write this of section in the setting of general Borel bijections when possible, without any measure involved. Indeed, the proofs are quite elegant and not much more difficult. It is indeed also possible to express all of the following in the language of boolean algebras, and we refer the interested reader to [Fre02, 382]. The following is very useful. Like the authors, we adopt the convention that a partition can contain multiple times the empty set, and we thank Corentin Correia for simplifying the proof.

Lemma A.2.9 ([EG16, Lem. 5.1])**.** *Let T be a Borel bijection of a standard Borel space X. Then there exists a Borel partition $(A_k)_{k\in\mathbb{N}}$ of $\operatorname{supp} T$ such that for any k, A_k is disjoint from $T(A_k)$.*

Proof. Let $\mathcal{C}$ be a countable separating set, *i.e.* for any $x \neq y$ in X, there exists $C \in \mathcal{C}$ such that $x \in C$ but $y \notin C$ (on the real line, one can think of intervals with rational endpoints for example). Define now $\mathcal{B} := \{C \cap T^{-1}(X \setminus C) \mid C \in \mathcal{C}\}$.

Fix now x in $\operatorname{supp} T$. As $\mathcal{C}$ separates points, there exists $C \in \mathcal{C}$ such that $x \in C$ but $T(x) \notin C$, which means that $x \in C \cap T^{-1}(X \setminus C)$. Therefore, there exists $B \in \mathcal{B}$ such that $x \in B$, and $\mathcal{B}$ covers $\operatorname{supp} T$.

Moreover if $B \in \mathcal{B}$, there exists $C \in \mathcal{C}$ such that $B = C \cap T^{-1}(X \setminus C) \subseteq C$, so we have $T(B) = T(C) \cap (X \setminus C) \subseteq X \setminus C$. In particular B is disjoint from $T(B)$.

We conclude by defining the desired partition by setting $A_k := B_k \setminus \bigcup_{l<k} B_l$, where (B_n) is an enumeration of $\mathcal{B}$. $\square$

Lemma A.2.10. *Let T be a boolean automorphism of* $\operatorname{MAlg}(X,\mu)$*. We have that* $\operatorname{supp} T$ *(in the sense of Definition A.1.11) is equal to the class of* $\{x \in X \mid T(x) \neq x\}$ *in* $\operatorname{MAlg}(X,\mu)$*.*

Proof. In this proof we identify Borel subsets of X with their class in $\operatorname{MAlg}(X,\mu)$. We set $\mathcal{T} := \{A \in \operatorname{MAlg}(X,\mu) \mid T(B) = B \text{ for any } B \subseteq X \setminus A\}$. By definition, we have $\operatorname{supp} T = \inf \mathcal{T}$, and $\{x \in X \mid T(x) \neq x\} \in \mathcal{T}$, so $\operatorname{supp} T \subseteq \{x \in X \mid T(x) \neq x\}$.

For the converse inclusion, we first claim that $\{x \in X \mid T(x) \neq x\} \subseteq A$ for any $A \in \mathcal{T}$. Indeed, assume it is not the case (for a fixed $A \in \mathcal{T}$) and consider $\emptyset \neq C \subseteq \{x \in X \mid T(x) \neq x\} \setminus A$, such that, in particular, any subset of C is equal to its image by T. By Lemma A.2.9 we write $\{x \in X \mid T(x) \neq x\} = \bigsqcup A_k$ with A_k disjoint from $T(A_k)$ for any k, and define $B_k := C \cap A_k$. We must have that $T(B_k)$ and B_k are disjoint, but by assumption $T(B_k) = B_k$, since $B_k \subseteq C$. At least one of the B_k has to be nonempty, since C is nonempty, providing the contradiction. Notice now that $\mathcal{T}$ is stable by countable intersection, so $\operatorname{supp} T \in \mathcal{T}$ by Remark A.2.7. The proof is then over. $\square$

The previous lemma links the terminology used in the definition given at the beginning of the section, which corresponds to the "usual" notion of support, with the one from Section A.1. In other words, the ‘support’ of an automorphism of a measure algebra (seen as a boolean algebra) corresponds to the ‘support’ of that automorphism seen as a Borel bijection of the underlying space. We can now give the following lemmas, before getting to separators.

Lemma A.2.11. *Let T be a Borel bijection of a standard Borel space X and consider a Borel subset $A \subseteq X$. The following are equivalent:*

(i) $\operatorname{supp} T \subseteq A$;

(ii) $B \cap T(B) = \emptyset \implies B \subseteq A$;

(iii) $\emptyset \neq B \subseteq X \setminus A \implies B \cap T(B) \neq \emptyset$.

In particular, if A does not support T, there exists a non-empty Borel subset $B \subseteq X \setminus A$ such that $B \cap T(B) = \emptyset$.

Proof. *(i)* $\implies$ *(ii)*: Let B be such that $B \cap T(B) = \emptyset$. We have $(B \setminus A) \cap B \subseteq T(B \setminus A) \cap B \subseteq T(B) \cap B = \emptyset$, so $B \subseteq A$.

(ii) $\implies$ *(iii)* is immediate.

(iii) $\implies$ *(i)*: Assume $\operatorname{supp} T \nsubseteq A$. This means that there exists $\emptyset \neq C \subseteq \operatorname{supp} T \setminus A$ such that $T(C) \neq C$. As T is a bijection, $\emptyset \neq B := C \setminus T(C) \subseteq C \subseteq X \setminus A$ satisfies $T(B) \cap B \subseteq T(C) \setminus T(C) = \emptyset$, which concludes the proof. $\square$

The following measure-theoretic reformulation of Lemma A.2.11 is used widely, so we state it plainly.

Lemma A.2.12. *let T be an element of* $\mathrm{Aut}(X,[\lambda])$, *and let $D \subseteq X$ be non-trivial. The following are equivalent:*

(i) $\operatorname{supp} T \nsubseteq D$*;*

(ii) $T_{\restriction X\setminus D} \neq \mathrm{id}_{X\setminus D}$*;*

(iii) There exists $C \subseteq X \setminus D$ such that $C \neq \emptyset$ and $C \cap T(C) = \emptyset$.

The following is a rephrasing of Lemma 3.3.4 in the purely Borel setup, which guarantees the existence of separators. The proof is the same, so we omit it.

Lemma A.2.13. *Let T be a Borel bijection of a standard Borel space X. There exists a separator for T, i.e. a Borel subset A of* $\operatorname{supp} T$, *such that* $\operatorname{supp} T = A \sqcup \big(T(A) \cup T^{-1}(A)\big)$.

The notion of exchanging involution (see Definition 3.4.15) is fundamental as they play a crucial role in the proof of Fremlin's theorem. Indeed they encapsulate most of "the information of an ergodic full group", in a way. It naturally generalizes to the Borel setup.

With separators, it is easy to show that every non-singular bijection of a standard measure space is an exchanging involution:

Proposition A.2.14 ([Fre02, Cor. 382F])**.** *Let V be an involution in* $\mathrm{Aut}(X,[\lambda])$. *Then there exists two disjoint Borel subsets A and B such that $V = V_{A,B}$ is an (A,B)-exchanging involution.*

Proof. Let V be an involution. By Lemma A.2.13 there exists a Borel subset A such that $\operatorname{supp} V = A \sqcup (V(A) \cup V^{-1}(A))$. However V is an involution, therefore $A \sqcup (V(A) \cup V^{-1}(A)) = A \sqcup V(A)$. □

A.2.3 Ergodic full groups have many involutions

As stated before, having many involutions is a property for subgroups of $\mathrm{Aut}(X,[\mu])$, which is in particular satisfied by ergodic full groups. We give a direct proof of this fact, by actually proving something stronger: ergodic full groups have involutions with support *equal* to any Borel subset, not just *included* in them. In a second time, we recall the original statement of Dye in the context of type II groups, and after giving as few definitions as possible, we explain why this version of the theorem implies the first version of Dye's, while still being under the yoke of Fremlin's.

Strong version of having strongly many involutions

In this section, we actually prove something stronger than the fact that ergodic full groups have many involutions. The usual proof in the measure-preserving context is from [KLM15, Lem. 7.10] but stands only for countable groups. We already extended it by using Proposition A.2.2, and proved what we needed for a dense countable subgroup. Recall that we have the following.

Proposition A.2.15 ([Aar97, Prop. 1.6.7])**.** *Let* $\mathbb{G} \leqslant \mathrm{Aut}(X,[\mu])$ *be an ergodic subgroup, and let Γ be a countable τ_w-dense subgroup of $\mathbb{G}$. Then Γ is ergodic.*

If there exists a finite or infinite σ-finite measure λ in $[\mu]$ that is preserved by $\mathbb{G}$, we have done all the work already. From Proposition 3.4.11 and Corollary 3.4.13 we get the following.

Corollary A.2.16. *Let* $\mathbb{G} \leqslant \mathrm{Aut}(X,[\mu])$ *be an ergodic full group. If there exists a finite or infinite σ-finite measure λ in $[\mu]$ that is preserved by $\mathbb{G}$, then for any Borel subsets A and B of X such that $\lambda(A \setminus B) = \lambda(B \setminus A)$, there exists an involution V in $\mathbb{G}$ such that $V(A) = B$, and such that V is the identity on $X \setminus (A \Delta B)$. In particular, $\mathbb{G}$ has many involutions.*

The proof in the type III context (when no σ-finite measure is preserved) is a bit different, and can be found for instance in [KLM15]. Once again it stands for countable groups (or equivalently countable Borel equivalence relations, see e.g. [KLM15, Prop. 8.1]), and we extend it without complications thanks to Proposition A.2.15. We briefly recall the necessary background, starting with a key result of Hajian and Ito regarding weakly invariant sets.

Definition A.2.17 ([HI69]). Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be any group. A Borel subset $W \subseteq X$ is said to be **weakly wandering under** $\mathbb{G}$ if there exists an infinite countable subset $I \subseteq \mathbb{G}$ such that

$$\mu(T(W) \cap T'(W)) = 0$$

for any $T \neq T'$ in I.

Theorem A.2.18 ([HI69, Thm. 1]). *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be a group. The following are equivalent:*

(1) there exists a finite measure in $[\mu]$ that is invariant under $\mathbb{G}$;

(2) there does not exist any non-null Borel set that is weakly wandering under $\mathbb{G}$.

Notice that the direct implication is immediate. The converse is very far from immediate.

The following theorem is a classical result of descriptive set theory. We do not recall the proof.

Theorem A.2.19 (The Borel Schröder-Bernstein Theorem, see [Kec95, Thm. 15.7]). *Let X and Y be standard Borel spaces, and $f : X \to Y$, $g : Y \to X$ be two Borel injections. Then there exist Borel subsets $A \subseteq X$ and $B \subseteq Y$ such that $f(A) = Y \setminus B$ and $g(B) = X \setminus A$. In particular X and Y are isomorphic as Borel spaces.*

The following two proofs originally use the language of countable Borel equivalence relations, we rephrase them purely in terms of actions of (countable) groups.

Lemma A.2.20 ([KLM15, Lem. 12]). *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be an ergodic full group. If there does not exist any finite or σ-finite measure in $[\mu]$ that is preserved by $\mathbb{G}$, then for any non-null Borel subset $B \subseteq X$, the group $\mathbb{G}_B := \{g \in \mathbb{G} \mid \operatorname{supp} g \subseteq B\}$ also does not admit any finite or σ-finite measure in $[\mu_{\restriction B}]$ which is preserved by $\mathbb{G}_B$.*

Proof. We start with an argument similar to the one used in Proposition 3.4.11 (recall if necessary that $\mathrm{Aut}(X, [\mu])$ is Polish thanks to Proposition A.2.2). As such, we let $\Gamma = (\gamma_n)$ be an ergodic countable subgroup of $\mathbb{G}$, which is τ_w-dense.

We define (A_n) inductively by setting

$$\begin{cases} A_0 = (\gamma_0^{-1} B) \cap (X \setminus B) \\ A_{n+1} = (\gamma_{n+1}^{-1} B) \bigcap \left((X \setminus B) \setminus \bigsqcup_{m \leqslant n} A_m \right). \end{cases}$$

By ergodicity, the Γ-saturation of $\sqcup A_n$ is conull (it can't be null as that would mean that B is Γ-invariant), and therefore (A_n) is a partition of $X \setminus B$ such that $\gamma_n A_n \subseteq B$ for any $n \in \mathbb{N}$.

By contraposition, assume that $\nu_B \in [\mu_{\restriction B}]$ is a σ-finite measure on B that is preserved by $\mathbb{G}_B$. We extend ν to the whole space by setting

$$\nu(A) := \nu_B(A \cap B) + \sum_{n \in \mathbb{N}} \nu_B(\gamma_n(A_n \cap A))$$

for any Borel subset $A \subseteq X$. By construction, ν is a σ-finite measure in $[\mu]$.

It then remains to show that it is $\mathbb{G}$-invariant. To this end, we first show that

$$\mathbb{G} = [\mathbb{G}_B \cup (\gamma_{n\restriction A_n})]_D$$

where $(\gamma_{n\restriction A_n})$ is a sequence of partial isomorphisms and $[\cdot]_D$ denotes the closure under the operation of cutting and pasting. Indeed, let g be in $\mathbb{G}$ and fix n and m (not necessarily distinct):

$$\begin{cases} \text{if } C_1 := B \cap g^{-1}(B), \text{ notice that } g_{\restriction C_1} \in [[\mathbb{G}_B]], \\ \text{if } C_2^n := B \cap g^{-1}(A_n), \text{ notice that } (\gamma_n g)_{\restriction C_2^n} \in [[\mathbb{G}_B]], \\ \text{if } C_3^n := A_n \cap g^{-1}(B), \text{ notice that } (g\gamma_n^{-1})_{\restriction \gamma_n C_3^n} \in [[\mathbb{G}_B]] \\ \text{if } C_4^{n,m} := A_n \cap g^{-1}(A_m), \text{ notice that } (\gamma_m g \gamma_n^{-1})_{\restriction \gamma_n C_4^{n,m}} \in [[\mathbb{G}_B]]. \end{cases}$$

Note that, while they are not necessarily distinct, the countable collection of those sets covers X. It then remains to notice that ν_B (a measure on B) is preserved by $[[\mathbb{G}_B]]$ while $\nu_B(\gamma_n(A_n \cap \cdot))$ (a measure on A_n) is preserved by $[[(\gamma_{n\restriction A_n})]]$. The proof is now over, as $B \sqcup \bigsqcup_{n\in\mathbb{N}} A_n$ is a partition of X, and ν is thus (piecewise)-preserved by $\mathbb{G}$, a contradiction. □

Proposition A.2.21 ([KLM15, Prop. 11])**.** *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,[\mu])$ be an ergodic full group. If there does not exist any finite or σ-finite measure in $[\mu]$ that is preserved by $\mathbb{G}$, then for any two non-null Borel subsets A and B of X there exists $\varphi \in [[\mathbb{G}]]$ such that* $\mathrm{dom}(\phi) = A$ *and* $\mathrm{rng}(\phi) = B$ *(up to null sets).*

Proof. Consider the following notation. For any two non-null Borel subsets $A, B \subseteq X$ we will write $A \prec_{\mathbb{G}} B$ if there exists $\varphi \in [[\mathbb{G}]]$ such that $\mathrm{dom}(\phi) = A$ and $\mathrm{rng}(\phi) \subseteq B$.

We start by using Theorem A.2.19 to reduce the problem: if φ_1 witnesses $A \prec_{\mathbb{G}} B$ and φ_2 witnesses $B \prec_{\mathbb{G}} A$, then there exist two Borel subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\varphi_1(A') = B \setminus B'$ and $\varphi_2(B') = A \setminus A'$. By cutting and pasting we can then find $\varphi \in [[\mathbb{G}]]$ with $\mathrm{dom}(\phi) = A$ and $\mathrm{rng}(\phi) = B$. Therefore, by symmetry it is enough to prove that $A \prec_{\mathbb{G}} B$ for any two non-null Borel subsets.

We now consider the group $\mathbb{G}_B = \{T \in \mathbb{G} \mid \mathrm{supp}\, T \subseteq B\}$, which does not admit any preserved finite or σ-finite measure in $[\mu_{\restriction B}]$ by Lemma A.2.20. By Theorem A.2.18, we find a non-null subset $W \subseteq B$ and a sequence $(T_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{G}_B$ such that the sets $T_k(W)$ and $T_{k'}(W)$ are disjoint (and non-null) whenever $k \neq k'$.

Again, consider $\Gamma = (\gamma_n)$ a countable ergodic subgroup that is τ_w-dense in $\mathbb{G}$. By ergodicity of $\mathbb{G}$, and therefore of Γ by Proposition A.2.15, we can use once again the argument used in the proof of Lemma A.2.20. This yields a partition (A_k) of A, such that $A_k \prec_{[\Gamma]} W$ (and therefore $A_k \prec_{\mathbb{G}} W$) for any k, denote by ψ_k the corresponding partial isomorphism. But now recall that W is sent to $T_k(W)$ by T_k, for any k. This means that for any k, one has $A_k \prec_{\mathbb{G}} T_k(W)$, witnessed by $\phi_k := T_{k\restriction \mathrm{rng}(\psi_k)}\psi_k$, a partial isomorphism satisfying

$$\begin{cases} \mathrm{dom}(\phi_k) = A_k \\ \mathrm{rng}(\phi_k) = T_k(\mathrm{rng}(\psi_k)) \subseteq T_k(W). \end{cases}$$

By disjointness of the (A_k) in A and of the $g_k(W)$ in a non-null subset of W, by cutting and pasting we get $A \prec_{\mathbb{G}} B$. □

With Proposition A.2.21 it is immediate to establish that type III ergodic full groups of $\mathrm{Aut}(X,[\mu])$ have "strongly" many involutions.

Corollary A.2.22. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,[\mu])$ be an ergodic full group. If there does not exist any finite or σ-finite measure in $[\mu]$ that is preserved by $\mathbb{G}$, then for any two non-null Borel subsets A and B such that $\mu(A \setminus B) > 0$ and $\mu(B \setminus A) > 0$, there exists an involution V in $\mathbb{G}$ such that $V(A) = B$, and such that V is the identity on $X \setminus (A\Delta B)$. In particular, $\mathbb{G}$ has many involutions.*

Proof. Proposition A.2.21 yields a partial isomorphism ϕ such that $\mathrm{dom}(\phi) = A \setminus B$ and $\mathrm{rng}(\phi) = B \setminus A$, up to null sets. We define

$$V := \begin{cases} \phi & \text{on } A \setminus B \\ \phi^{-1} & \text{on } B \setminus A \\ \mathrm{id}_X & \text{on } X \setminus (A \Delta B) \end{cases}$$

which is in $\mathbb{G}$ and is suitable. $\square$

Without any mention of ergodicity, Fremlin actually proved that for a full group, the weak and strong versions of having many involutions are equivalent. His statement is once again in the general language of boolean algebras, but we give the non-singular version, without the proof.

Theorem A.2.23 ([Fre02, Thm. 382Q]). *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be a full group with many involutions. Then for any subset $A \subseteq X$ of positive measure, there exists an involution $U \in \mathbb{G}$ with $\mathrm{supp}\, U = A$.*

Dye's original statement

The statement of Theorem A.1.1 is given in its simplest form for people interested in ergodic theory, and more specifically in ergodic full groups. However, in [Dye63], the author works with *type II full groups*, which is a condition equivalent (for full groups!) to having many involutions. We give this general statement, with no mention of ergodicity (in particular, even working with full groups of measure-preserving bijections, the conjugation obtained is only non-singular, since Proposition 3.5.30 crucially uses ergodicity).

The adequate language to talk about type II full groups is the language of relative atoms in the measure algebras. We chose to remain concise on this subject matter, and to give only the necessary proofs for our needs with the reconstruction theorem(s). In particular, we do not define type I. We refer the interested reader to [LM14a, Sec. 1.4].

Definition A.2.24. Let N be a closed subalgebra of $M := \mathrm{MAlg}(X, \mu)$. We say that:

- $\emptyset \neq A \in M$ is **an atom relatively to** N if for all $B \subseteq A$ there exists $C \in N$ such that $B = A \cap C$,
- N is of **type II** if M does not have any atoms relatively to N;
- a full group $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ is of **type II** if the algebra $M_{\mathbb{G}}$ of $\mathbb{G}$-invariant elements of M is of type II.

Theorem A.2.25 ([Dye63, Thm. 2]). *Let $\mathbb{G}$ and $\mathbb{H}$ be two full groups of type* II *on a standard probability space (X, μ). Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some non-singular bijection. In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in $\mathrm{Aut}(X, [\mu])$ such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

For an ergodic (full) group $\mathbb{G}$, the algebra $M_{\mathbb{G}}$ of $\mathbb{G}$-invariant elements of $\mathrm{MAlg}(X, \mu)$ is equal to $\{\emptyset, X\}$, so an atom relatively to $M_{\mathbb{G}}$ is just an atom. In particular, $M_{\mathbb{G}}$ is of type

II, and thus $\mathbb{G}$ is also of type II. It remains to show that being of type II implies having many involutions. It happens to be an equivalence. The following proof uses some notions of countable Borel equivalence relations, and we refer to [LM14a] or to [KLM15] for more on this subject.

Proposition A.2.26. *Let (X,μ) be a standard probability space and $\mathbb{G} \leqslant \mathrm{Aut}(X,[\mu])$ be a full group. The following are equivalent:*

(1) $\mathbb{G}$ is of type II*;*

(2) $\mathbb{G}$ has many involutions;

(3) for any weakly-dense countable subgroup $\Gamma \leqslant \mathbb{G}$, for any Borel subset $A \subseteq X$, $\mathcal{R}_{\Gamma\restriction A}$ does not admit a Borel fundamental domain.

Proof. The first thing is once again that, $\mathrm{Aut}(X,[\mu])$ being Polish (Proposition A.2.2), we can consider a countable τ_w-dense subgroup $\Gamma \leqslant \mathbb{G}$. We fix such a subgroup Γ, and immediately have $M_{\mathbb{G}} \subseteq M_\Gamma$. The converse inclusion directly follows from density, and thus $M_{\mathbb{G}} = M_\Gamma$.

We will not prove the implication *(3)* $\implies$ *(1)*. It is done in [LM14a, Prop. 1.44] and stated in the probability-measure-preserving context, but generalizes to the non-singular one.

(1) $\implies$ *(3)*: By contraposition, assume that there exists a non-trivial Borel subset $A \subseteq X$ such that $\mathcal{R}_{\Gamma\restriction A}$ admits a Borel fundamental domain D. Let C be a non-trivial Borel subset of D, we have $\Gamma\text{-sat}(C) \in M_\Gamma$, where $\Gamma\text{-sat}(C)$ is the Γ-saturation of C, and $\Gamma\text{-sat}(C)$ exists by Proposition A.2.6 (for more on this see [LM14a, Prop. 1.8]). Therefore, by virtue of D being a fundamental domain we have $C = \Gamma\text{-sat}(C) \cap D$, so D is an atom relatively to M_Γ. Since $M_{\mathbb{G}} = M_\Gamma$, $\mathbb{G}$ is not of type II.

(3) $\implies$ *(2)*: Consider a non-trivial $A \subseteq X$. We know that the restriction of $\mathcal{R}_\Gamma$ to any Borel subset does not admit a Borel fundamental domain, therefore the set $\{x \in A \mid \Gamma \cdot x \cap A \text{ is finite}\}$ is null. Let $\Gamma = (\gamma_n)$ be an enumeration of Γ. Denote by N the integer defined by

$$N := \min\{n \in \mathbb{N} \mid \mu(\{x \in A \mid \gamma_n(x) \neq x \text{ and } \gamma_n(x) \in A\}) > 0\},$$

which exists by the previous argument. This means that there exists a Borel subset $B \subseteq \operatorname{supp}\gamma_N \cap A \cap \gamma_N^{-1}(A)$ satisfying $\mu(B) > 0$ and $\mu(B \cap \gamma_N(B)) = 0$. We define a $(B, \gamma_N(B))$- exchanging involution V by

$$\begin{cases} V(C) = \gamma_N(C) \text{ for any } C \subseteq B \\ V(C) = \gamma_N^{-1}(C) \text{ for any } C \subseteq \gamma_N(B) \\ V(C) = C \text{ for any } C \subseteq X \setminus (B \cup \gamma_N(B)). \end{cases}$$

As $\mathbb{G}$ is a full group, the involution V is in $\mathbb{G}$, its support is contained in A, and since A is arbitrary $\mathbb{G}$ has many involutions.

(2) $\implies$ *(3)*: By contraposition again, assume that there exists a non-trivial Borel subset $A \subseteq X$ such that $\mathcal{R}_{\Gamma\restriction A}$ admits a Borel fundamental domain D. Assume that there exists a non-trivial involution $V \in \mathbb{G}$ with $\operatorname{supp} V = B \sqcup V(B) \subseteq D$ by Proposition A.2.14. We have $B = \Gamma\text{-sat}(B) \cap D$, $\Gamma\text{-sat}(B) \in M_\Gamma = M_{\mathbb{G}}$, and $V(D) = D$, so

$$V(B) = V(\Gamma\text{-sat}(B)) \cap V(D) = \Gamma\text{-sat}(B) \cap D = B,$$

a contradiction. ∎

We then have the following chain of implications with the different versions of the reconstruction theorem, Fremlin's version being the "strongest", and the ergodic statement of Dye's theorem being the "weakest".

$$\boxed{\text{Theorem A.1.1} \Leftarrow \text{Theorem A.2.25} \Leftarrow \text{Theorem A.2.5} \Leftarrow \text{Theorem A.1.2}}$$

A.3 A proof of Fremlin's reconstruction theorem

Before jumping to the proof of Theorem A.2.5, we give the following lemma, which is very useful when working with involutions.

Lemma A.3.1 ([Fre02, Lem. 384.A]). *Let $\mathbb{G}$ be a subgroup of the group of Borel bijections of X with many involutions. For any non-trivial Borel subset B of X, there exists $T \in \mathbb{G}$ of order exactly 4 and such that $\operatorname{supp} T \subseteq B$.*

Proof. The proof amounts to following the construction illustrated in Figure A.1.

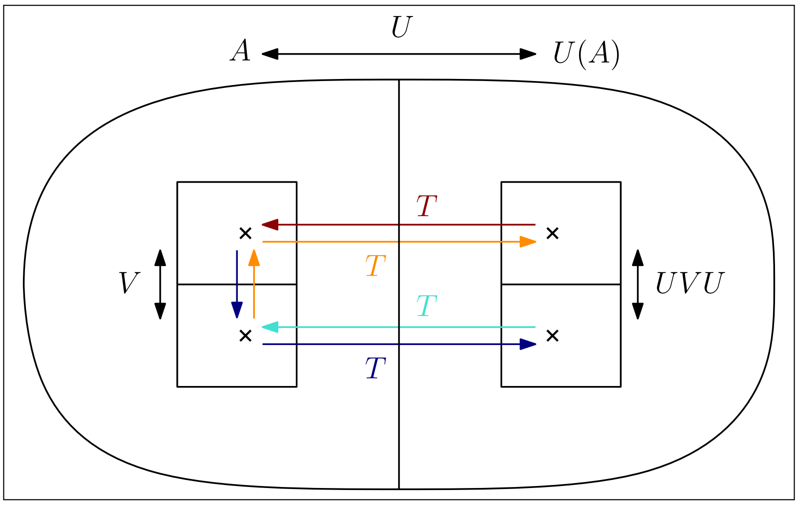

Figure A.1: Construction of the bijection of order 4.

As $\mathbb{G}$ has many involutions, there exists an involution $U \in \mathbb{G}$ such that $\operatorname{supp} U \subseteq B$. By Proposition A.2.14, there exists A such that $\operatorname{supp} U = A \sqcup U(A)$. Let V be an involution in $\mathbb{G}$ such that $\operatorname{supp} V \subseteq A$. Then, $UVU = UVU^{-1}$ in an involution with support equal to $U(\operatorname{supp} V)$, therefore it commutes with V. This means that $UVUV = VUVU$ is an involution. Consequently, $T = UV$ has order 4 (by looking at the supports we see that UV and $UVUV$ are not trivial, and so $UVUVUV$ isn't either). $\square$

We now fix all the necessary ingredients: (X, μ) is a standard probability space, $\mathbb{G}$ and $\mathbb{H}$ are two subgroups of $\operatorname{Aut}(X, [\mu])$ with many involutions, and $\psi : \mathbb{G} \to \mathbb{H}$ is a group isomorphism.

As previously stated, the proof is given in a measured context. As such, any equality between sets is to be understood as an equality in the measure algebra, that is to say an equality up to a null set. In particular, for two Borel subsets A and B of X, by $A \cap B = \emptyset$ we mean that $\mu(A \cap B) = 0$.

A.3.1 Linking measure-theoretic properties to algebraic properties

The goal of this section is to describe the support of a fixed involution purely in group-theoretic terms. Fix a non-trivial involution $V \in \mathbb{G}$. By Proposition A.2.14, there exists two disjoint Borel subsets A and B such that $V = V_{A,B}$, which means that $\operatorname{supp} V = A \sqcup B =$

$A \sqcup V(A)$. The involution V and these Borel subsets are fixed for the whole section. In the drawings, we shall imagine V to be the axial symmetry across the vertical separation of the support. We will be using the following notations:

$$\begin{array}{rcl}
\mathcal{C}_V & \coloneqq & C(V) = \{T \in \mathbb{G} \mid TV = VT\} \\
\mathcal{D}_V & \coloneqq & \{\text{involutions in } \mathcal{C}_V \text{ commuting with all their } \mathcal{C}_V\text{-conjugates}\} \\
 & = & \left\{T \in \mathcal{C}_V \mid T \text{ involution such that } \forall S \in \mathcal{C}_V : STS^{-1}T = TSTS^{-1}\right\} \\
\mathcal{E}_V & \coloneqq & C(\mathcal{D}_V) = \{T \in \mathbb{G} \mid \forall S \in \mathcal{D}_V : ST = TS\} \\
\mathcal{F}_V & \coloneqq & Sq(\mathcal{E}_V) = \left\{T^2 \mid T \in \mathcal{E}_V\right\} \\
\mathcal{G}_V & \coloneqq & C(\mathcal{F}_V) = \{T \in \mathbb{G} \mid \forall S \in \mathcal{F}_V : ST = TS\}\,.
\end{array}$$

Here C designates the centralizer in $\mathbb{G}$, and Sq designates the set of squared elements.

These sets already appear in the proof of Eigen's analogous theorem (see [Eig82]), however the distinction between the measure-theoretic properties and the group-theoretic properties was done by Fremlin.

We begin the study of the various "centralizer/squared"-sets we just defined. Our goal is Lemma A.3.8, which will be the heart of the construction of the desired conjugation.

Lemma A.3.2 (Property of $\mathcal{C}_V$)**.** *For any T in $\mathcal{C}_V$, we have $T(\operatorname{supp} V) = \operatorname{supp} V$.*

Proof. Recall that in general we have $T(\operatorname{supp} V) = \operatorname{supp}(TVT^{-1})$, and $TVT^{-1} = V$ since T is in $\mathcal{C}_V$. □

Definition A.3.3. For any Borel subset C of positive measure satisfying $V(C) = C$, we define the **induced exchanging-involution** V_C by

$$\begin{cases} V_C(D) = V(D) \text{ for any } D \subseteq C \\ V_C(D) = D \text{ for any } D \subseteq X \setminus C. \end{cases}$$

Notice that V_C is the $(C \cap A, C \cap B)$-exchanging involution associated with V.

Remark A.3.4. If C and D have positive measure and are such that $V(C) = C$ and $V(D) = D$, then $V_C V_D = V_{C\Delta D} = V_D V_C$, as illustrated by Figure A.2.

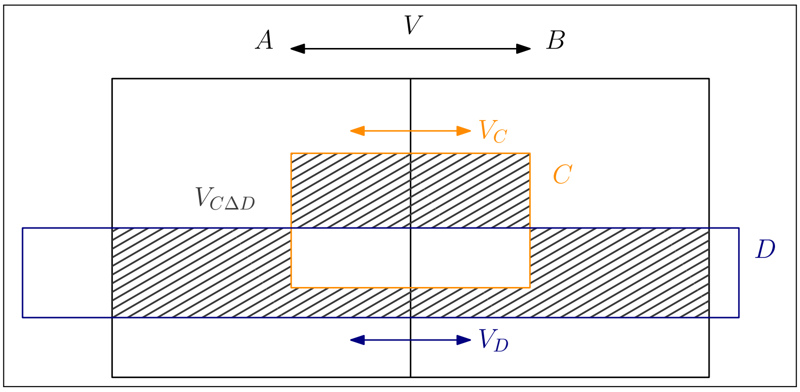

Figure A.2: Commutativity of induced exchanging-involutions.

Lemma A.3.5 (Properties of $\mathcal{D}_V$)**.** *We have the following:*

(i) For any T in $\mathcal{D}_V$, $\operatorname{supp} T \subseteq \operatorname{supp} V = A \sqcup B$;

(ii) For any Borel subset C of positive measure satisfying $V(C) = C$, V_C is in $\mathcal{D}_V$.

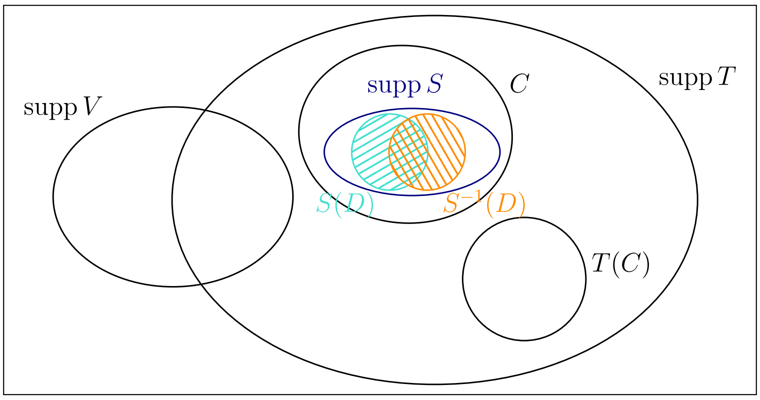

Figure A.3: Visualisation of the sets in play.

Proof. (i) By contraposition, assume that $T \in \mathcal{C}_V$ is such that $\operatorname{supp} T \nsubseteq \operatorname{supp} V$. By Lemma A.2.12, there exists $C \subseteq X \setminus \operatorname{supp} V$ of positive measure such that $C \cap T(C) = \emptyset$. By Lemma A.3.1, there exists S of order exactly 4 such that $\operatorname{supp} S \subseteq C$.

The bijections S and V commute, as their supports are disjoint, so $S \in \mathcal{C}_V$. Moreover, $S \neq S^{-1}$ so there exists a non-trivial $D \subseteq C$ such that $S(D) \neq S^{-1}(D)$. Figure A.3 illustrates the different sets coming into play in this proof.

We have

$$C \cap T(D) = C \cap TS^{-1}(D) = \emptyset$$

therefore S and S^{-1} are trivial on $T(D)$ and on $TS^{-1}(D)$, and since T is an involution we have

$$STS^{-1}T(D) = ST^2(D) = S(D) \neq S^{-1}(D) = T^2S^{-1}(D) = TSTS^{-1}(D).$$

As T does not commute with its S-conjugate (with $S \in \mathcal{C}_V$), we have $T \notin \mathcal{D}_V$.

(ii) For any $S \in \operatorname{Aut}(X, [\lambda])$, it is easy to check that $SV_CS^{-1} = (SVS^{-1})_{S(C\cap A),S(C\cap B)}$. In particular, for $S = V$, we obtain $VV_CV^{-1} = V_C$, which means that V_C is in $\mathcal{C}_V$.

Now for any $S \in \mathcal{C}_V$ (in particular $VS(C) = S(C)$), we have

$$SV_CS^{-1} = (SVS^{-1})_{S(C\cap A),S(C\cap B)} = V_{S(C\cap A),S(C\cap B)} = V_{S(C)}.$$

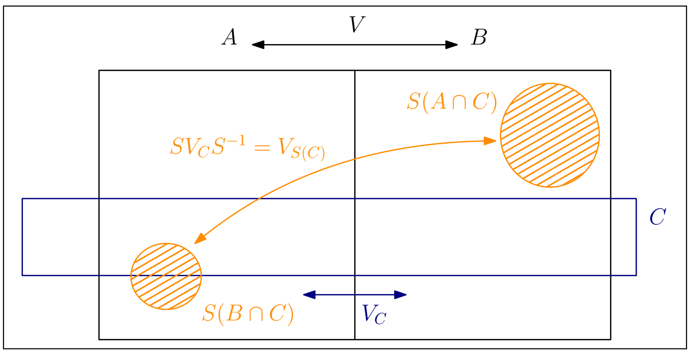

Figure A.4: An example of a conjugate of an induced exchanging-involution.

Figure A.3 illustrates the previous equality. By Remark A.3.4, $V_{S(C)}V_C = V_CV_{S(C)}$, and as S is arbitrary in $\mathcal{C}_V$, we proved that $V_C \in \mathcal{D}_V$. $\square$

Lemma A.3.6 (Properties of $\mathcal{E}_V$). *We have the following:*

(i) $\mathcal{E}_V \subseteq \mathcal{C}_V$;

(ii) *Let* T *be in* $\mathcal{E}_V$. *For any* $C \subseteq \operatorname{supp} V$ *we have* $T(C) \subseteq C \cup V(C)$;

(iii) *Let* T *be in* $\mathcal{E}_V$. *For any* $C \subseteq \operatorname{supp} V$ *we have* $T^2(C) = C$;

(iv) *If* $T \in \mathbb{G}$ *is such that* $\operatorname{supp} V \cap \operatorname{supp} T = \emptyset$, *then* $T \in \mathcal{E}_V$.

Proof. *(i)* The involution V is in $\mathcal{D}_V$, so the result follows.

(ii) Assume the contrary: let C be of positive measure, and such that $T(C)$ is not contained in $C_0 \coloneqq C \cup V(C)$ (note that $V(C_0) = C_0$). In particular if $C_1 \coloneqq T(C_0) \setminus C_0$, then $C_1 \supseteq T(C) \setminus C_0 \neq \emptyset$.

By *(i)* and Lemma A.3.2, $C_1 \subseteq T(\operatorname{supp} V) = A \sqcup B$. We also have $VT = TV$ so in particular $VT(C_0) = TV(C_0) = T(C_0)$, and therefore $V(C_1) = VT(C_0) \setminus V(C_0) = T(C_0) \setminus C_0 = C_1$. Define then $D \coloneqq C_1 \cap B$.

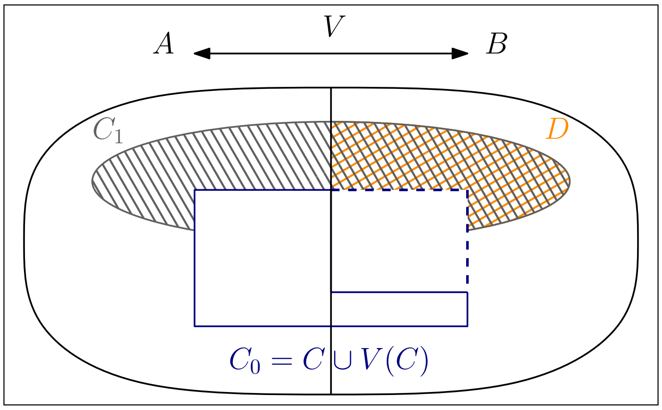

Figure A.5: Visualisation of C_0, C_1 and D.

Figure A.5 illustrates the sets in play, the ellipsoid representing the T-image C_1 of C_0, from which C_0 has been removed. Now we notice the following two facts:

- $C_1 \cap T(D) = \emptyset$. Indeed, $C_1 \subseteq T(C_0)$ so $C_1 \cap T(D) \subseteq T(C_0 \cap D) \subseteq T(C_0 \cap C_1) = \emptyset$.
- $C_1 = V(D) \sqcup D$. Indeed, $V(D) = V(C_1) \cap V(B) = C_1 \cap A$, so $C_1 = C_1 \cap (A \sqcup B) = (C_1 \cap A) \sqcup (C_1 \cap B) = V(D) \sqcup D$.

Therefore, we have

$$V_{C_1} T(D) = T(D) \neq TV(D) = TV_{C_1}(D).$$

Indeed, the first equality comes from the first point, the last equality is a direct consequence of the definitions, and the non-equality comes from the second point. Since T is in $\mathcal{E}_V = C(\mathcal{D}_V)$ and $V_{C_1} \in \mathcal{D}_V$ by Lemma A.3.5, this is a contradiction, as they do not commute.

(iii) By Lemma A.2.12, $\operatorname{supp} T = \sup\{D \subseteq X \mid D \cap T(D) = \emptyset\}$ and so we have $C \cap \operatorname{supp} T = \sup\{D \subseteq C \mid D \cap T(D) = \emptyset\}$. Fix $D \subseteq C \cap \operatorname{supp} T$ such that $T(D) \cap D = \emptyset$. Since $D \subseteq C \subseteq \operatorname{supp} V$, by *(ii)* we have $T(D) \subseteq (D) \cup V(D)$, so in fact $T(D) \subseteq V(D)$ and thus by taking the supremum, $T(C \cap \operatorname{supp} T) \subseteq V(C \cap \operatorname{supp} T)$. By *(i)* we moreover know that $TV = VT$, so we have

$$T^2(C \cap \operatorname{supp} T) \subseteq TV(C \cap \operatorname{supp} T) = VT(C \cap \operatorname{supp} T) \subseteq V^2(C \cap \operatorname{supp} T) = C \cap \operatorname{supp} T.$$

Of course, the previous inclusion holds on $C \setminus \operatorname{supp} T$, so $T^2(C) \subseteq C$, and this holds for any $C \subseteq \operatorname{supp} V$. But by *(i)* and Lemma A.3.2, we also have $\operatorname{supp} V = T(\operatorname{supp} V) = T^2(\operatorname{supp} V)$, so $\operatorname{supp} V \setminus T^2(C) = T^2(\operatorname{supp} V \setminus C) \subseteq \operatorname{supp} V \setminus C$. We have proved both inclusions, so $T^2(C) = C$.

(iv) This results from Lemma A.3.5, as bijections with disjoint supports commute. ☐

Lemma A.3.7 (Properties of $\mathcal{F}_V$). *If S is in $\mathcal{F}_V$, then $\operatorname{supp} V \cap \operatorname{supp} S = \emptyset$. Moreover, for any $\emptyset \neq C \subseteq X \setminus \operatorname{supp} V$, there exists a non-trivial involution S in $\mathcal{F}_V$, with $\operatorname{supp} S \subseteq C$.*

Proof. The first part of the statement is immediate from item *(iii)* of Lemma A.3.6. For the second part, we know from Lemma A.3.1 that there exists $T \in \mathbb{G}$ of order exactly 4 with $\operatorname{supp} T \subseteq C$, and from item *(iv)* of Lemma A.3.6 we get that $T \in \mathcal{E}_V$, so $T^2 \in \mathcal{F}_V$ and is a non-trivial involution. ☐

Lemma A.3.8 (Properties of $\mathcal{G}_V$). *The set $\mathcal{G}_V$ is comprised of all the elements in $\mathbb{G}$ that are supported by* $\operatorname{supp} V$*:*

$$\mathcal{G}_V = \{T \in \mathbb{G} \mid \operatorname{supp} T \subseteq \operatorname{supp} V\}.$$

Proof. ($\supseteq$) Let $T \in \mathbb{G}$ be such that $\operatorname{supp} T \subseteq \operatorname{supp} V$. For any $S \in \mathcal{F}_V$ we know from Lemma A.3.7 that $\operatorname{supp} V \cap \operatorname{supp} S = \emptyset$, so T and S commute, which means that $T \in \mathcal{G}_V$.

($\subseteq$) Let $T \in \mathbb{G}$ be such that $\operatorname{supp} T \nsubseteq \operatorname{supp} V$. By Lemma A.2.12 consider $C \subseteq X \setminus \operatorname{supp} V$ such that $C \cap T(C) = \emptyset$. By Lemma A.3.7 we know that there exists an involution $S \in \mathcal{F}_V$ with $\operatorname{supp} S \subseteq C$. Therefore, for any $D \subseteq \operatorname{supp} S$ such that $S(D) \neq D$ we have

$$TS(D) \neq T(D) = ST(D),$$

which means that T does not commute with S, hence $T \notin \mathcal{G}_V$. ☐

A.3.2 Constructing the conjugation

Now armed with all the necessary properties of the previously defined sets, we can go back to the isomorphism $\psi : \mathbb{G} \to \mathbb{H}$ in order to construct the associated conjugation. It is immediate that $\psi(V)$ is an involution, and it is straightforward to check that the following hold:

$$\begin{aligned} \psi(\mathcal{C}_V) &= \mathcal{C}_{\psi(V)}, \\ \psi(\mathcal{D}_V) &= \mathcal{D}_{\psi(V)}, \\ \psi(\mathcal{E}_V) &= \mathcal{E}_{\psi(V)}, \\ \psi(\mathcal{F}_V) &= \mathcal{F}_{\psi(V)}, \\ \psi(\mathcal{G}_V) &= \mathcal{G}_{\psi(V)}, \end{aligned}$$

and in particular from Lemma A.3.8, for any $T \in \mathbb{G}$ we see that

$$\operatorname{supp} T \subseteq \operatorname{supp} V \iff \operatorname{supp} \psi(T) \subseteq \operatorname{supp} \psi(V). \tag{$\star$}$$

We now define $S : \mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(X, \mu)$ and $S^* : \mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(X, \mu)$ to be such that for any $C \in \mathrm{MAlg}(X, \mu)$ we have

$$\begin{cases} S(C) := \sup \{\operatorname{supp} \psi(V) \mid V \in \mathbb{G} \text{ involution and } \operatorname{supp} V \subseteq C\}, \\ S^*(C) := \sup \left\{\operatorname{supp} \psi^{-1}(V) \mid V \in \mathbb{H} \text{ involution and } \operatorname{supp} V \subseteq C\right\}. \end{cases}$$

In particular S and S^* are ($\subseteq$)-order-preserving. We have to check that S and S^* are well-defined non-singular bijections of X (equivalently boolean automorphisms of $\mathrm{MAlg}(X, \mu)$), that $S^* = S^{-1}$, that S is the desired conjugation, and that it is unique in that regard.

By symmetry (as ψ is an isomorphism) it is enough to do the necessary checks on S.

Lemma A.3.9. *For any Borel subset $A \subseteq X$ and any involution $V \in \mathbb{G}$, if $\operatorname{supp} \psi(V) \subseteq S(A)$, then $\operatorname{supp} V \subseteq A$.*

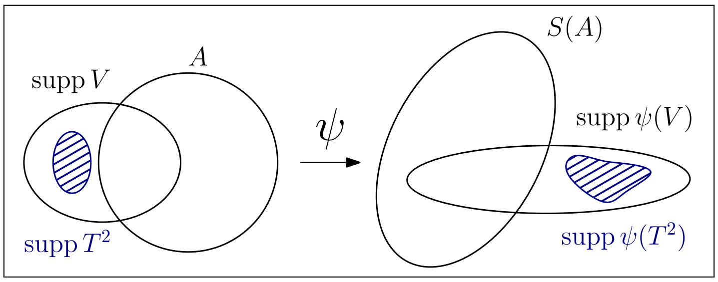

Figure A.6: Visualisation of the proof of Lemma A.3.9.

Proof. We do a proof by contraposition. By Lemma A.3.1 there exists an element $T \in \mathbb{G}$ of order exactly 4, with $\operatorname{supp} T \subseteq \operatorname{supp} V \setminus A$. Therefore, T^2 is a non-trivial involution with $\operatorname{supp} T^2 \subseteq \operatorname{supp} V$, so from ($\star$) we get that $\operatorname{supp} \psi(T^2) \subseteq \operatorname{supp} \psi(V)$. Again the sets in play are illustrated by Figure A.6.

Now if $V' \in \mathbb{G}$ is an involution with $\operatorname{supp} V' \subseteq A$, then $T \in \mathcal{E}_{V'}$ by disjointness of the supports and Lemma A.3.6, and therefore $T^2 \in \mathcal{F}_{V'}$. This yields that $\psi(T^2) \in \mathcal{F}_{\psi(V')}$, and by Lemma A.3.7 $\operatorname{supp} \psi(T^2) \cap \operatorname{supp} \psi(V') = \emptyset$. As V' is arbitrary, $\operatorname{supp} \psi(T^2) \cap S(A) = \emptyset$, so

$$\emptyset \neq \operatorname{supp} \psi(T^2) \subseteq \operatorname{supp} \psi(V) \setminus S(A),$$

which concludes the proof. □

Proposition A.3.10. *For any $A \in \mathrm{MAlg}(X, \mu)$ we have $S^*S(A) = A$, and similarly, $SS^*(A) = A$.*

Proof. We assume that A is non-trivial.

($\supseteq$) Assume the contrary, consider $\emptyset \neq B \subseteq A \setminus S^*S(A)$, and an involution $V \in \mathbb{G}$ such that $\operatorname{supp} V \subseteq B$. We know that $\psi(V)$ is an involution in $\mathbb{H}$ and that $\operatorname{supp} \psi(V) \subseteq S(A)$, so

$$0 \neq \operatorname{supp} V = B \cap \operatorname{supp} \psi^{-1}\psi(V) \subseteq B \cap S^*S(A),$$

which is the contradiction.

($\subseteq$) Let now V be an involution in $\mathbb{H}$ such that $\operatorname{supp} V \subseteq S(A)$. Then $\psi^{-1}(V)$ is an involution in $\mathbb{G}$ with $\operatorname{supp} \psi\psi^{-1}(V) = \operatorname{supp} V \subseteq S(A)$. By Lemma A.3.9, this means that $\operatorname{supp} \psi^{-1}(V) \subseteq A$. Again, as V is arbitrary, this means that $S^*S(A) \subseteq A$. □

Proposition A.3.11. *The functions S and $S^* = S^{-1}$ are boolean automorphisms of* $\mathrm{MAlg}(X, \mu)$.

Proof. We start by noticing that since S and S^* are order-preserving, they send X to itself, and $\emptyset$ to itself. Indeed, $S(X) \subseteq X$, and $S^*(X) \subseteq X$, so by applying S, we get $X \subseteq S(X)$, thus $S(X) = X$. Similarly, $\emptyset \subseteq S(\emptyset)$, and $\emptyset \subseteq S^*(\emptyset)$, so by applying S, we get $S(\emptyset) \subseteq \emptyset$, thus $S(\emptyset) = \emptyset$.

We now prove that S is a boolean automorphism by using characterization *(4)* of Proposition A.1.6. To that end we fix A, B disjoint in $\mathrm{MAlg}(X, \mu)$, then we aim to establish that $S(A) \cap S(B) = \emptyset$ and $S(A \cup B) = S(A) \cup S(B)$.

We start with $S(A) \cap S(B)$. Let $C \subseteq S(A) \cap S(B)$, and let $V \in \mathbb{H}$ be an involution with $\operatorname{supp} V \subseteq C$. We have

$$\operatorname{supp} \psi\psi^{-1}(V) = \operatorname{supp} V \subseteq C \subseteq S(A) \cap S(B),$$

so by Lemma A.3.9 applied to $S(A)$ and $S(B)$ separately, $\operatorname{supp} \psi^{-1}(V) \subseteq A \cap B = \emptyset$. This means that $\psi^{-1}(V)$ is the identity, so V is the identity, so $C = \emptyset$. Therefore $S(A) \cap S(B) = \emptyset$.

We finally have to take care of $S(A) \cup S(B)$. Since S is order-preserving we have $S(A) \cup S(B) \subseteq S(A \cup B)$. Fix then $C \subseteq S(A \cup B) \setminus (S(A) \cup S(B))$ and $D \subseteq A \cup B$ such that $S(D) = C$. We also have $S(D \cap A) \subseteq S(D) \cap S(A) = C \cap S(A) = \emptyset$, and similarly for B. By applying S^*, which sends $\emptyset$ to itself, this means that $D \cap A = D \cap B = \emptyset$. This concludes the proof, as $C = S(D) = S((D \cap A) \cup (D \cap B)) = S(\emptyset) = \emptyset$, yielding that $S(A \cup B) \subseteq S(A) \cup S(B)$. $\square$

We are now almost done. We only need to verify that ψ is the conjugation by S, and that it is the unique such automorphism. We prove the following final lemma, before finishing the proof with Proposition A.3.13.

Lemma A.3.12. *For any involutions $V \in \mathbb{G}$ and $V' \in \mathbb{H}$ we have $S(\operatorname{supp} V) = \operatorname{supp} \psi(V)$ and $S^{-1}(\operatorname{supp} V') = \operatorname{supp} \psi^{-1}(V')$.*

Proof. By definition of S we have $\operatorname{supp} \psi(V) \subseteq S(\operatorname{supp} V)$. For the converse inclusion, by Proposition A.3.10 and the definition of S^{-1} we have

$$\operatorname{supp} \psi(V) = SS^{-1}(\operatorname{supp} \psi(V)) \supseteq S(\operatorname{supp} \psi^{-1}\psi(V)) = S(\operatorname{supp} V),$$

which concludes the proof. $\square$

Proposition A.3.13. *For any $T \in \mathbb{G}$, we have $\psi(T) = STS^{-1}$. Moreover, S is the only non-singular bijection of (X, μ) satisfying this.*

Proof. Assume the contrary: for a non-trivial $T \in \mathbb{G}$ the map $F := \psi(T)^{-1}STS^{-1}$ is not id_X. By Lemma A.2.12 this means that there exists $\emptyset \neq A \subseteq X$ such that $F(A) \cap A = \emptyset$. By applying $\psi(T)$ we get

$$STS^{-1}(A) \cap \psi(T)(A) = \emptyset. \tag{$*$}$$

Let now $V \in \mathbb{H}$ be a non-trivial involution with $B := \operatorname{supp} V \subseteq A$. By Lemma A.3.12, $\operatorname{supp} \psi^{-1}(V) = S^{-1}(B)$, and so by the general fact about the support of a conjugate we get that

$$\operatorname{supp}(T\psi^{-1}(V)T^{-1}) = T(\operatorname{supp} \psi^{-1}(V)) = TS^{-1}(B),$$

which yields (from Lemma A.3.12 again):

$$\operatorname{supp}(\psi(T)V\psi(T)^{-1}) = \operatorname{supp}(\psi(T\psi^{-1}(V)T^{-1})) = S(\operatorname{supp}(T\psi^{-1}(V)T^{-1})) = STS^{-1}(B).$$

On the other hand, $\operatorname{supp}(\psi(T)V\psi(T)^{-1}) = \psi(T)(B)$. From the previous two equalities we get that $STS^{-1}(B) = \psi(T)(B)$, contradicting ($*$).

We now turn to the uniqueness, and assume that S_1, S_2 are two distinct non-singular bijections. We have $S_2^{-1}S_1 \neq \mathrm{id}_X$, so by Lemma A.2.12 again there exists $\emptyset \neq A \subseteq X$ such that $(S_2^{-1}S_1)(A) \cap A = \emptyset$. We then consider an involution $V \in \mathbb{G}$ with $\operatorname{supp} V \subseteq A$. We have $\operatorname{supp}((S_2^{-1}S_1)V(S_2^{-1}S_1)^{-1}) = (S_2^{-1}S_1)(\operatorname{supp} V) \subseteq (S_2^{-1}S_1)(A)$, so $(S_2^{-1}S_1)V(S_2^{-1}S_1)^{-1}$ cannot be equal to V. Therefore

$$S_2^{-1}S_1V \neq VS_2^{-1}S_1.$$

Multiplying by S_2 on the left and by S_1^{-1} on the right, we get that the conjugates of V by S_1 and S_2 are different. $\square$

Appendix B

Examples of ergodic infinite measure-preserving bijections

Abstract

This very informal appendix aims to present different ∞-mp ergodic bijections on standard σ-finite spaces. Constructing examples of such bijections is 'a non-trivial task' according to Halmos, but since Kakutani's skyscraper construction in 1943, which to our knowledge is the first significant and non-trivial one, many other examples have been studied. We go into a bit more details and provide some proofs in the first sections about Kakutani skyscrapers, as they represent the most relevant examples for our work, and remain more concise for the other sections, with some exceptions.

Malgré moi l'infini me tourmente.

Alfred de Musset

Contents

B.1 Kakutani Skyscrapers

The Kakutani skyscraper construction if one of the first (if not the first) meaningful example of ∞-mp ergodic bijection (see [Kak43]), and it is in a way the reverse of the induced bijection construction.

We start by describing a very concrete example before giving the general construction.

B.1.1 Basic skyscraper example

We use probability measure-preserving ergodic bijections to construct a Kakutani skyscraper ∞-mp ergodic bijection. Let $X_1 = [0, 1[$ be equipped with the Lebesgue measure μ, and $T = r_\theta$, the irrational rotation of angle θ in $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1[$. Define $X_n = [0, \frac{1}{n}[$ for any $n > 1$, and

$$X := \bigsqcup_{n \in \mathbb{N}^*} X_n \times \{n\}\,.$$

We have $X \subseteq X_1 \times \mathbb{N}$, and it is equipped with the product measure of μ with the counting measure. For (x, n) in $X_n \times \{n\}$ we then define S by

$$S(x, n) = \begin{cases} (x, n+1) & \text{if } x \in X_{n+1} \\ (T(x), 1) & \text{else.} \end{cases}$$

Since $\mu(X_n) = \frac{1}{n}$, X is an infinite σ-finite standard space, and for any x in X_1 we have $\mathrm{orb}_T(x) \times \{1\} = (X \times \{1\}) \cap \mathrm{orb}_S(x, 1)$, therefore S is ergodic because T is, hence S is totally conservative and aperiodic. Figure B.1 shows part of the positive S-orbit of an element $(x, 1)$ of the base.

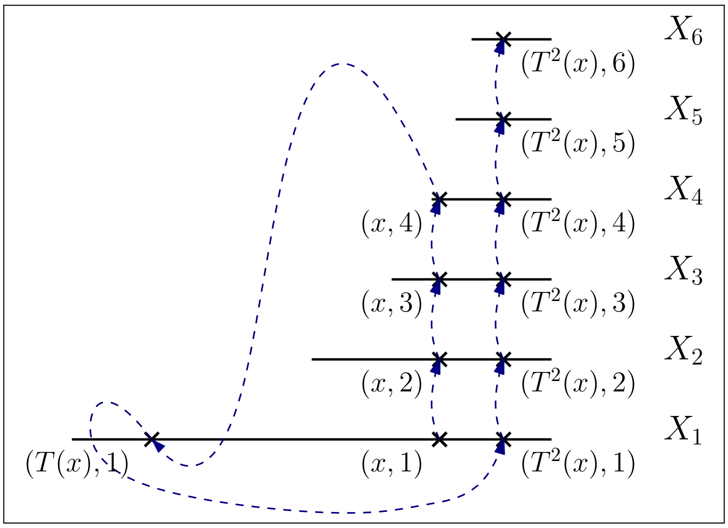

Figure B.1: Positive S-orbit on the first floors of X.

Remark B.1.1. Here we chose the irrational rotation r_θ as the ergodic bijection acting on the base X_1 of the skyscraper, but of course any other choice would yield a satisfying ∞-mp ergodic bijection. For instance in [EHIP14, 4.1.2] the authors detail the case where the adding machine was chosen instead. We give more details about this example later.

B.1.2 General skyscraper construction

In this section we give the general construction, which has varied applications. As stated earlier, the Kakutani skyscraper is in a way an reverse construction to the induced bijection construction defined in Definition 1.2.11. We give two equivalent definitions commonly found in the literature, as we believe they provide a different intuition and nicely complement each other.

Our first definition explicitly describes the building blocks of the skyscraper through the height function. This definition builds the skyscraper vertically while the second definition does it horizontally.

Definition B.1.2 (First skyscraper definition, [EHIP14, 4.1.1])**.** The base of the skyscraper is defined in the following way: we let $(A, \mathcal{B}_A, \lambda_A)$ be a standard σ-finite measure space of finite or infinite total mass, and we let T_A be a λ_A-preserving conservative self-bijection of A.

We let $(B_k)_{k\in N}$ be a sequence of mutually disjoint Borel subsets of A satisfying

$$\begin{cases} A = \bigsqcup_{k\in\mathbb{N}} A_k \\ \lambda(A_k) \to 0. \end{cases}$$

We define this time the height function simply as an increasing sequence of nonnegative integers $(\varphi_k)_{k\in\mathbb{N}} = \{0 \leqslant \varphi_0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \dots\}$, and for each $k \in \mathbb{N}$ we consider $\varphi_k + 1$ isomorphic (as measured spaces) copies of A_k (the first copy being A_k itself), that we denote by $A_k = A_k^0, A_k^1, A_k^2, \dots, A_k^{\varphi_k}$. We denote by R_k all the isomorphisms of the form

$$R_k : A_k^i \to A_k^{i+1}, \text{ for } i \text{ in } \{0, \dots, \varphi_k\} \text{ and } k \geqslant 0.$$

The measure space we consider is the following:

$$X \coloneqq \bigsqcup_{k\in\mathbb{N}} \bigsqcup_{i=0}^{\varphi_k} A_k^i$$

and we endow it with the natural σ-algebra of all measurable subsets of X (as induced by $\mathcal{B}_A$), and the measure λ naturally defined on $(X, \mathcal{B})$ by summing the measure of the components. We can finally define the bijection T by setting

$$T(x) \coloneqq \begin{cases} R(x) & \text{if } x \in A_k^i \text{ for } i \text{ in } \{0, \dots, \varphi_k\} \text{ and } k \geqslant 0 \\ T_A(R^{-\varphi_k}(x)) & \text{if } x \in A_k^{\varphi_k} \text{ for } k \geqslant 0. \end{cases}$$

The $\bigsqcup_{i=0}^{\varphi_k} A_k^i$ correspond to 'vertical slices', and for any $n \in \mathbb{N}^*$ the 'horizontal' n-th floor is

$$X_n \coloneqq \bigsqcup_{k\in D_n} A_k^n = \bigsqcup_{k=l_n}^{+\infty} A_k^n$$

where $D_n \coloneqq \{k \in \mathbb{N} \mid \varphi_k + 1 \geqslant n\}$ and $l_n \coloneqq \min D_n$. We can visualise the construction on Figure B.2.

This first definition is 'external' in a sense, we define a base and construct the skyscraper above it, with no ambient space to speak of. The point of view chosen for the second definition is 'internal', in that we build the skyscraper from Borel subsets of a σ-finite space (X, λ). They are indeed equivalent.

Definition B.1.3 (Second skyscraper definition, [CK79, §3.])**.** We define the base of the skyscraper as a Borel subset $A \subseteq X$ of finite or infinite measure, and we let T_A be a λ-preserving conservative self-bijection of A. We also let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Borel subsets satisfying

$$\begin{cases} A_0 = A, \\ \lambda(A_{n+1}) \leqslant \lambda(A_n), \\ \lambda(A_n) \to_n 0, \\ X = \bigsqcup_{n\in\mathbb{N}} A_n, \end{cases}$$

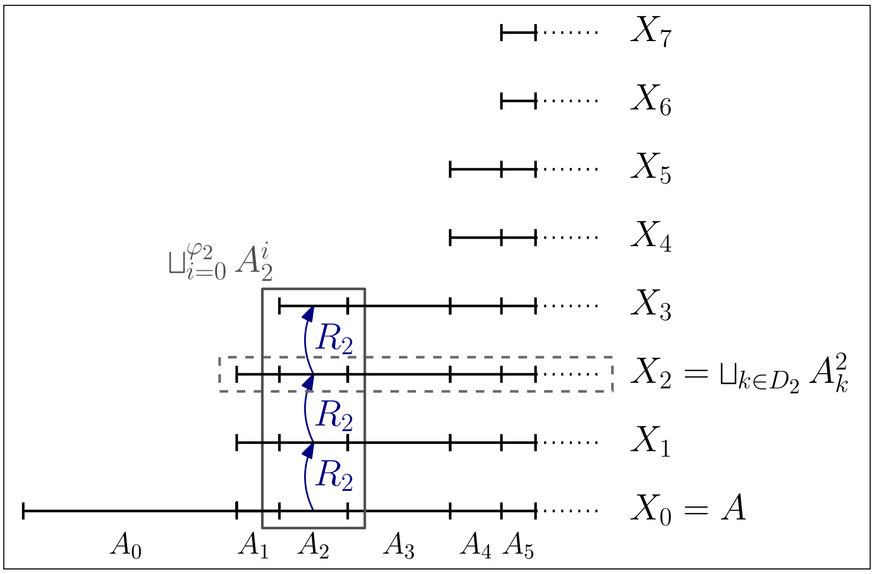

Figure B.2: Vertical slices of a skyscraper.

We also consider injective measure-preserving maps $\pi_n : A_{n+1} \to A_n$ and define T by setting for any $n \in \mathbb{N}$:

$$\begin{aligned} T &= \pi_n^{-1} && \text{on } \pi_n(A_{n+1}), \\ T &= T_A \pi_0 \pi_1 \dots \pi_{n-1} && \text{on } A_n \setminus \pi_n(A_{n+1}). \end{aligned}$$

With this definition, the 'horizontal' n-th floor X_n is simply A_n.

On Figure B.3 we can see the Borel subsets (A_n) coming from $\mathbb{R}_+$, which provides a nice way to visualise this construction. The maps π_n then just mean 'going down one floor', and in purple we represented the image by T of a point $x \in A_3 \setminus \pi_3(A_4)$.

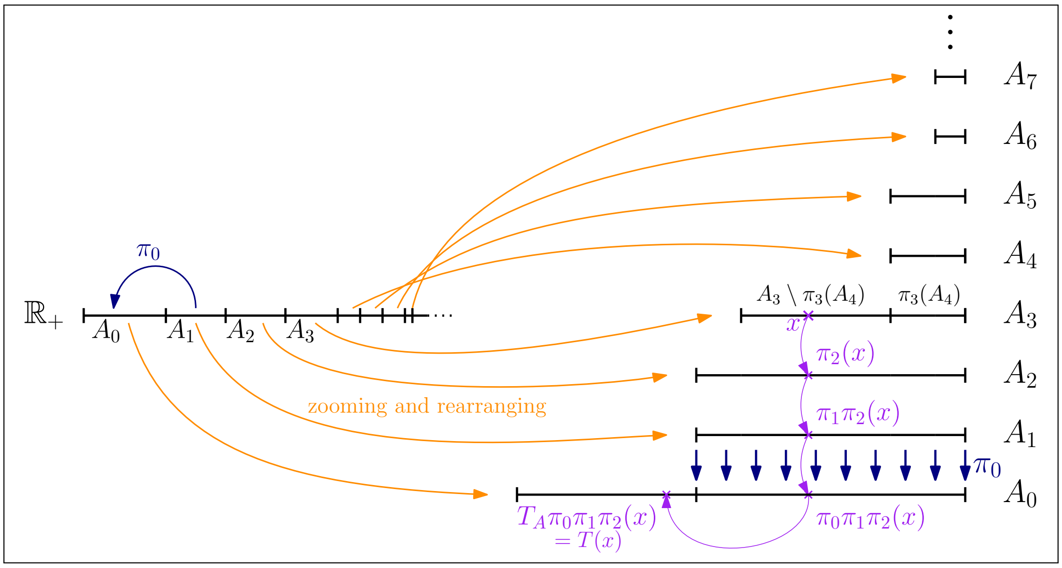

Figure B.3: Internal construction of a skyscraper from $(X, \lambda) = (\mathbb{R}_+, \mathrm{Leb}_{\restriction \mathbb{R}_+})$.

The skyscraper construction satisfies some nice properties, some of which are given in the following proposition. In particular the last one guarantees that we can construct many ∞-mp ergodic bijection this way. We provide the proofs of the next two results, as those properties are quite indispensable in our work. We prove it using the notations of Definition B.1.3.

Proposition B.1.4 ([CK79, §3]). *Let T be a skyscraper of base (A, T_A). The following hold:*

1. *$T \in \mathrm{Aut}(X, \lambda)$;*
2. *T is conservative;*
3. *T_A is the induced bijection of T on A in the sense of Definition 3.6.8;*
4. *T is ergodic if and only if T_A is ergodic on A.*

Proof. *1.* It is clear by construction.

For *2.* and *3.* the only thing that we have to prove is that for λ-a.e. $x \in X$, there exists a positive power $n \in \mathbb{N}^*$ of T sending x to the base A. For *2.* the conservativity of T_A then concludes. This holds by convergence of $(\lambda(A_n))$ to 0: since the π_n are measure preserving, for any x, $x \in A_k$ for some $k \in \mathbb{N}$ and there exists a large enough integer n such that $\pi_{n-1}^{-1} \ldots \pi_{k+1}^{-1}\pi_k^{-1}(x) \in A_n \setminus \pi_n(A_{n+1})$. By definition of T, the integer $n-k+1$ is suitable. For *3.*, we notice that if $k = 0$, then $n - k + 1$ is exactly the smallest return time in A of x.

4. We now move on to ergodicity. We first notice that for any T-invariant set B in X, the set $B \cap A$ is T_A-invariant. Indeed, using the fact that the family (A_n) is a Kakutani-Rokhlin partition we write

$$B \cap A = \bigsqcup_{n \in \mathbb{N}} B \cap \{x \in A \mid n_A(x) = n\} = \bigsqcup_{n \in \mathbb{N}} T^n(B) \cap \{x \in A \mid n_A(x) = n\} = T_A(B \cap A).$$

Now assume that T_A is ergodic. By the previous argument, $B \cap A$ is either null or conull. We have $\lambda(B \cap T^n(A)) = \lambda(T^n(B \cap A)) = \lambda(B \cap A)$, so it is either null or conull in A. As taking the towers above B yields a partition of the T-saturation of B, this means that B is either null or conull in X.

The converse implication can be proved by very similar arguments: if B is a T_A-invariant subset of A, then the towers above B form a T-invariant subset of X, and ergodicity of T concludes. □

Those skyscraper constructions are our first examples, but they are in a way 'universal', as by Rokhlin's lemma any aperiodic conservative element of $\mathrm{Aut}(X, \lambda)$ can be represented by a skyscraper. We have the following.

Proposition B.1.5 ([CK79, Prop. 2(i)]). *Let $T \in \mathrm{Aut}(X, \lambda)$ be conservative aperiodic. For any $\varepsilon > 0$ there exists a Borel subset $C \subseteq X$ such that $\lambda(C) < \varepsilon$ and such that T is a skyscraper over (C, T_C).*

Proof. The case of a finitely supported T is easier and taken care of by a straightforward application of Rokhlin's lemma, so let's assume that $\operatorname{supp} T$ has infinite measure. Fix $\varepsilon > 0$ and let $\operatorname{supp} T = \bigsqcup_{n \in \mathbb{N}} X_n$, with $\lambda(X_n) = \varepsilon$ for all n in $\mathbb{N}$. Do note at this point that for λ-almost all $x \in X_n$ we have $\mathrm{orb}_{T_{X_n}}(x) = \mathrm{orb}_T(x) \cap X_n$, therefore we can work with the T_{X_n} orbits, which are infinite by Theorem 1.2.19. In each X_n, we apply Rokhlin's (aperiodic) lemma (see Theorem 1.2.13) to find a subset $A_n \subseteq X_n$ such that A_n, $T_{X_n}(A_n)$, $\ldots$, $T_{X_n}^{2^{n+2}}(A_n)$ are pairwise disjoint, and such that B_n the complement of the tower in X_n satisfies

$$\lambda(B_n) = \lambda\left(X_n \setminus \bigsqcup_{k=0}^{2^{n+2}} T_{X_n}^k(A_n)\right) < 2^{-(n+2)}\varepsilon.$$

Define now $C_n = A_n \sqcup B_n$ and notice that C_n intersects all the T_{X_n}-orbits on X_n, so $C = \bigsqcup_n C_n$ intersects all the T-orbits. By construction $\lambda(A_n) < 2^{-(n+2)}\varepsilon$ and we then have

$$0 < \lambda(C) = \sum_{n\in\mathbb{N}} \lambda(C_n) = \sum_{n\in\mathbb{N}} (\lambda(A_n) + \lambda(B_n)) < \varepsilon,$$

which concludes the proof, as T is clearly a skyscraper over C. □

Remark B.1.6. The skyscraper construction is 'universal' in the sense of Proposition B.1.5 for ∞-mp ergodic bijections, but finer dynamical properties are encoded by the parameters of the floors. For example [EHIP14, 4.1.2] and [EHIP14, 4.3.1] exhibit very different behaviours.

Example B.1.7. Using the notations of Definition B.1.2, we quickly present the historical example of Hajian and Kakutani from [HK70], which is also the one exposed in [EHIP14, 4.1.2]. The base space X_0 is just the unit interval endowed with the usual Lebesgue measure, and T_A is the dyadic odometer. We then set $A_k = \left[\frac{1}{2^k}, \frac{1}{2^{k+1}}\right[$ and define the height function φ by $\varphi_0 = 0$ and

$$\varphi_k = 2 + 2^3 + \ldots + 2^{2k-1}$$

for any $k > 0$. This was the first example of an ∞-mp ergodic bijection for which an exhaustive weakly wandering sequence was exhibited (we recall that $(n_i)_{i\in\mathbb{N}}$ is an exhaustive weakly wandering sequence for T if there exists W of positive measure such that $X = \bigsqcup_{i\in\mathbb{N}} T^{n_i}(W)$).

B.2 Maharam extensions

Our second general class of examples of ∞-mp ergodic bijections is the class of Maharam extensions of non-singular ergodic bijections. The main idea behind the Maharam extension is to compensate the 'loss of measure-preservation' by considering infinitely many copies of the original space (equivalently, adding a new coordinate), and by moving 'vertically' along those copies in a proportional manner to the 'horizontal' movement in the base space. The main point is that the Radon-Nikodym derivative quantifies exactly 'how much of the measure-preservation is lost'.

We present two versions of this construction, a *continuous* Maharam extension with uncountably many copies (*i.e.* $X = Y \times \mathbb{R}$), and a *discrete* Maharam extension with countably many copies (*i.e.* $X = Y \times \mathbb{Z}$), which can only be defined in some cases.

B.2.1 Maharam $\mathbb{R}$-extension

This section presents a classical variation of the historical construction from [Mah64] (see [Aar97, 3.4] for a more modern exposition). We start with the general definition then give a specific example also from [Aar97] (the first example of an ergodic Maharam $\mathbb{R}$-extension is from [Kri76]).

We let $(Y, \mathcal{B}(Y), \mu)$ be a standard probability space, and $T \in \mathrm{Aut}(Y, [\mu])$ be a non-singular bijection.

The construction. The Radon-Nikodym cocycle exactly encodes the measure-preservation perturbation that we want to compensate. We then define the space, the σ-algebra, and the extension $\widetilde{T}$ of T:

$$\begin{cases} X = Y \times \mathbb{R} \\ \mathcal{B}(X) = \mathcal{B}(Y) \otimes \mathcal{B}(\mathbb{R}) \\ \widetilde{T}(y,t) := (T(y), t - \ln c(T,y)). \end{cases}$$

It is easy to check that the measure λ defined by $d\lambda = d\mu \times e^t dt$ is infinite, σ-finite, and $\widetilde{T}$-invariant. The first result is the following.

Proposition B.2.1 ([Mah64, Thm. 2])**.** *T is conservative on (X, μ) if and only if $\widetilde{T}$ is conservative on (X, λ).*

Remark B.2.2. We chose here the additive notation by using the logarithm (which is more common in the literature), but it is possible to define a *multiplicative Maharam extension*, which is isomorphic to the one we just defined. For this we choose $0 < a < 2$ and consider the space

$$(X = Y \times \mathbb{R}_+^*, \mathcal{B}(X) = \mathcal{B}(Y) \otimes \mathcal{B}(\mathbb{R}_+^*), \lambda = \mu \otimes \frac{1}{t^{1+a}} dt)$$

along with the bijection $\widetilde{T}_a$ defined by

$$\widetilde{T}_a(y,t) = \left(T(y), t \times \left(c(T,y)^{\frac{1}{a}}\right)\right).$$

See *e.g.* [Roy12] for more details about this multiplicative version.

Remark B.2.3. The technique used for the Maharam extension of one bijection can easily adapt to non-singular group actions by generalizing the definition of the Radon-Nikodym cocycle:

$$c : \left\{ \begin{array}{ccl} G \times Y & \longrightarrow & \mathbb{R} \\ (g,y) & \longmapsto & c(g,y) = \dfrac{dg_*\mu}{d\mu}(y) \end{array} \right.$$

and in this case we talk about the Maharam $\mathbb{R}$-extension of $G \curvearrowright Y$. We refer to [BKV21] for more details about this construction.

An example. This example is detailed at the end of § 8.4 in [Aar97]. The space is the countable product

$$Y = \prod_{n \in \mathbb{N}^*} \{0, \dots, a_n - 1\}$$

Where $a_n = 2^n$ for any $n \in \mathbb{N}$. We let $(\gamma_1, \gamma_2, \dots) = (1, \sqrt{2}, 1, \sqrt{2}, \dots)$ and (β_n) be such that

$$\beta_n(k) := \left\{ \begin{array}{l} k\gamma_n \text{ for } 0 \leqslant k \leqslant a_n - 2 \\ 0 \text{ for } k = a_n - 1. \end{array} \right.$$

Instead of the usual Haar measure (*i.e.* the uniform measure $\prod_{n \in \mathbb{N}^*}(\frac{1}{a_n}, \dots, \frac{1}{a_n})$) on Y, we define the measure μ by

$$\mu := \prod_{n \in \mathbb{N}^*} p_n, \text{ where } p_n(k) := \frac{e^{\beta_n(k)}}{\sum_{j=1}^{a_n - 1} e^{\beta_n(j)}}.$$

We finally define T as the odometer on (Y, μ), and $\widetilde{T}$ as the $\mathbb{R}$-Maharam extension of T (which is well defined since $T_*\mu \in [\mu]$). It can be shown that $\widetilde{T}$ is ergodic on $X = Y \times \mathbb{R}$.

We have the following, allowing us to know whether or not the Maharam extension is ergodic. For the definition of the Krieger types see Section 1.2.2.

Proposition B.2.4 ([DS22, Prop. 5.2])**.** *The Maharam extension $\widetilde{T}$ of T is ergodic if and only if T is of type* III_1.

B.2.2 Maharam $\mathbb{Z}$-extension

In this section we explain the example from [EHIP14, 4.2.3], itself based on [HIK72, Sec. 3], after briefly explaining how the general $\mathbb{R}$-construction adapts.

The construction. In cases where the Radon-Nikodym cocycle can be written as $c(T,\cdot) = a^{\phi}$ for some $a \in]1,+\infty[$ and some measurable function $\phi : Y \to \mathbb{Z}$, one can define the $\mathbb{Z}$-Maharam extension of T by setting

$$\begin{cases} X = Y \times \mathbb{Z} \\ \mathcal{B}(X) = \mathcal{B}(Y) \otimes \mathcal{B}(\mathbb{Z}) \\ \widetilde{T}(y,n) := (T(y), n - \phi(y)). \end{cases}$$

One can check that the measure λ defined by $\lambda(A \times \{n\}) := a^n \mu(A)$ is $\widetilde{T}$-invariant.

An example. We closely follow [EHIP14, 4.2.3]. The base space is $Y = \{0,1\}^{\mathbb{N}^*}$ with its natural σ-algebra, but we do not consider the uniform measure. Instead, we choose $\alpha \in \mathbb{R}_+^*$, we set $\beta = \dfrac{\alpha}{1+\alpha}$, and we define

$$\mu_n := \beta\delta_0 + (1-\beta)\delta_1$$

where δ_i is the Dirac measure on $\{i\}$. In other words for any $n \in \mathbb{N}^*$ we have $\mu_n(0) = \beta$ and $\mu_n(1) = 1 - \beta$. We then endow Y with μ, which is the product measure of the μ_n, and remove from Y the μ-null set of sequences with finitely many 0 or finitely many 1. We will denote by $Y[\,j\,]$ the following cylinder set

$$Y[\,j\,] := \{(y_i)_{i\in\mathbb{N}^*} \in Y \mid y_0 = 1, y_1 = 1, \ldots y_{j-1} = 1, y_j = 0\},$$

which contains all the sequences whose first 0 is at position j. It is clear that $Y = \bigsqcup_{j\in\mathbb{N}^*} Y[\,j\,]$. We then define T_Y as the (non-singular here!) odometer on Y:

$$T_Y(Y[\,j\,]) = \{(y_i)_{i\in\mathbb{N}^*} \in Y \mid y_0 = 0, y_1 = 0, \ldots y_{j-1} = 0, y_j = 1\}$$

We have

$$\begin{cases} \mu(Y[\,j\,]) = (1-\beta)^{j-1}\beta \\ \mu(T_Y(Y[\,j\,])) = \beta^{j-1}(1-\beta) = \alpha^{j-2}\mu(Y[\,j\,]). \end{cases}$$

We have that T_Y is ergodic and that $c(T_Y,\cdot) = \alpha^{\phi}$ for $\phi : Y \to \mathbb{Z}$ defined by $\phi(y) = j_y - 2$ where j_y is the unique $j \in \mathbb{N}^*$ such that $y \in \mu(Y[\,j\,])$ (in other words $j_y = \min\{n \in \mathbb{N}^* \mid y_n = 0\}$). Therefore, we can define the 'extended'-space $X = Y \times \mathbb{Z}$, with its natural σ-algebra, and endow it with the measure λ defined by

$$\lambda(A,n) = \alpha^n \mu(A)$$

for any $(A,n) \in \mathcal{B}(Y) \times \mathbb{Z}$.

The last step is to define $\widetilde{T}$, and for any $(y,n) \in Y[\,j\,] \times \mathbb{Z}$ we set

$$\widetilde{T}(y,n) = (T_Y(y), n + j - 2).$$

A straightforward computation yields that

$$\lambda(\widetilde{T}(A,n)) = \lambda(T_Y(A), n+j-2) = \alpha^{n+j-2}\mu(T_Y(A)) = \alpha^n\mu(A) = \lambda(A,n)$$

for any $(A,n) \subseteq (Y[\,j\,],n)$, with j and n in $\mathbb{N}^*$ and $\mathbb{Z}$ respectively, so $\widetilde{T}$ is λ-preserving.

We will not prove the following proposition, which uses the Hewitt-Savage Zero-One Law. We refer to [EHIP14, 4.2.2, 4.2.3] for the proofs.

Proposition B.2.5. *The $\mathbb{Z}$-Maharam extension $\widetilde{T}$ is ergodic on (X,λ).*

Remark B.2.6. The odometer T_Y on Y that we presented is non-singular because the measure μ is not the uniform one, but one may wonder if there exists an equivalent (or just μ-absolutely continuous) measure that is T_Y-invariant. Arnold's theorem states that this cannot happen (see *e.g.* [Aar97, Thm. 1.2.9]). If we look at measures that are supported on different sets, this is another matter entirely. We present in Section B.4.2 an example of an infinite measure-preserving ergodic odometer, whose invariant measure is supported on a set which is null for the usual uniform measure.

B.3 Infinite measure rank-one bijections

Rank-one transformations, or rank-one bijections are a powerful generalization of both the odometers and the irrational rotations. Historically, the *Chacon* map, defined in 1966, opened the way to the study of these bijections, and they since then have proved to be a rich source of examples and counter-examples in ergodic theory. Sadly, they are also notorious for having many equivalent definitions, some being very technical. One can consult the impressive survey [Fer97] for more details.

A lot of authors define them in finite measure spaces only, but the definition is flexible enough to encompass the σ-finite spaces as well; the first ones to do so were most likely [AFS97] and by now even the infinite measure context has seen some substantial development (see *e.g.* [Dan04] or [RT15]). Rank-one bijections are a major source of counter-examples in the ∞-mp context as well; for instance let us quickly consider the following as a motivating example. It is known that in the pmp context, being weakly mixing, doubly ergodic, and Cartesian square ergodic are equivalent (we refer to [AS18] for a comprehensive survey on weak mixing, with a special emphasis on infinite measure bijections). In the ∞-mp context however, the authors of [BFMS01] proved the following:

Theorem B.3.1 ([BFMS01])**.** *There exists rank-one bijections $S, T \in \mathrm{Aut}(X, \lambda)$ such that S is doubly ergodic, but has non-ergodic Cartesian square and T is weak mixing, but is not doubly ergodic.*

B.3.1 Infinite Chacon map

Before giving the general construction we present the infinite measure version of the Chacon map, which has the advantage of being less heavy technically, while still providing some insight. It is of note that the *probability measure-preserving* version of Chacon's map was historically the first example of a transformation that is weakly mixing, but not (strongly) mixing (see [Cha69]). Such transformations were known to exist, since weakly mixing transformations are generic, while strong mixing ones form a set of the first category (see [Hal56]).

This section follows the exposition of [JRR18, § 1.2], and the construction is from [AFS97, § 2]. We let $X = \mathbb{R}_+$. The construction follows the iterative method of *cutting and stacking*.

First step: At first we consider the interval $[0, 1[$, cut into three subintervals of same length. We call this the *Tower 0*, and declare that it has height $h_0 = 1$. We take an extra interval of length $\frac{1}{3}$ (for instance $[1, \frac{4}{3}[$) and stack it on top of the middle part of Tower 0. We then take four (this four is $(3 \times h_0) + 1$) more intervals of length $\frac{1}{3}$ and stack them all on top of the right part of Tower 0. We call those additional parts *spacers*. At step 1, spacers have length $\frac{1}{3} = \frac{1}{3^1}$. We finally construct Tower 1 by stacking the middle part of the *modified* Tower 0 on top of the left part, and by stacking the right part on top of the obtained stack. The obtained Tower 1 has height $h_1 = 8$. Figure B.4 shows the cutting and stacking process of the first step. The transformation T maps each point of Tower 1 to the

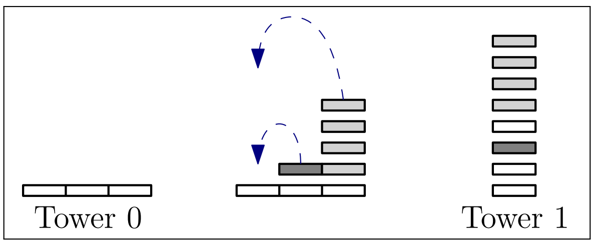

Figure B.4: The first step of the construction.

point exactly above it. Thus T is not yet defined on the top floor (floor 8).

(n+1)-th step: After n steps we have a Tower n of height h_n, made of floors which are semi-open intervals of length $\frac{1}{3^n}$. We then subdivise Tower n in three subcolumns of equal length $\text{len}(\text{Tower } n)/3 = \frac{1}{3^{n+1}}$. We add one spacer of length $\frac{1}{3^{n+1}}$ on top of the middle subcolumn, and $3h_n + 1$ spacers of length $\frac{1}{3^{n+1}}$ on top of the right subcolumn. We can pick the extra intervals successively by taking for instance the leftmost interval of desired length in the unused part of X. We can then, like in the first step, stack the second subcolumn on top of the first one, and the third one on top of it all.

Figure B.5 shows the process of this step.

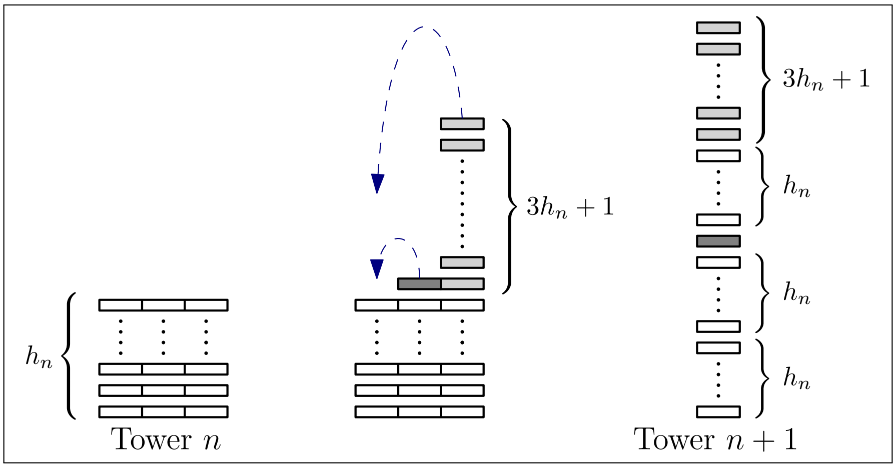

Figure B.5: The $(n+1)$-th step of the construction.

Similarly, T maps each point of Tower $n+1$ to the point exactly above it. Again, it is not defined on the top floor, which has length $\frac{1}{3^{n+1}}$.

At each step, we add two more spacers than three times the number of floors of the previous steps, in particular, the measure of the added spacers is more that the total measure of the previous floor. Clearly, the added measure of the spacers goes to infinity, so at the limit we get a $\text{Leb}_{\restriction\mathbb{R}_+}$-measure preserving bijection. We have the following.

Theorem B.3.2 ([AFS97, Thm. 2.2])**.** *The previously defined bijection T is ergodic, and in fact all of its Cartesian powers are ergodic.*

B.3.2 General geometric rank-one bijections

Here we understand *geometric* in the sense of [Fer97], who opposes this definition to the *symbolic* one. The definition we chose is the one from [Cor25b]. Another notable reference is [DV25], which presents the more general (C, F)-constructions. Their impressive work completely solves the classification of rank-one systems, which is a point we will not touch upon.

We define a rank-one transformations through its parameters.

Definition B.3.3. We define a **cutting and spacing parameter** as a tuple $(q, (\sigma_0, \ldots, \sigma_q))$ such that $q \in \mathbb{N}^*$ and $\sigma_k \in \mathbb{N}$ for any $k \leqslant q$. The parameter q will represent the 'number of parts in which we will cut each floor of the tower' and the numbers σ_n will represent the 'number of spacers added in-between each floor of the next tower'. We then define

(1) $\mathcal{P}$ the space of all possible cutting and spacing parameters;

(2) $\mathcal{P}^{\mathbb{N}}$ the space of sequences of cutting and spacing parameters;

(3) $\mathcal{P}^{<\mathbb{N}} = \bigcup_n \mathcal{P}^n$ the space of finite sequences of cutting and spacing parameters.

From a sequence $p = (q_n, (\sigma_0^n, \ldots, \sigma_{q_n}^n))_{n\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ of cutting and spacing parameters, we say that the k-th coordinate $p_k = (q_k, (\sigma_0^k, \ldots, \sigma_{q_k}^k))$ is the **k-th cutting and stacking parameter** of p. We also define

1. The **number of spacers** added at step n: $(\sigma_n)_{n\in\mathbb{N}}$, with $\sigma_n = \sum_{k=0}^{q_n} \sigma_k^n$;
2. The **height of the tower** at step n: $(h_n)_{n\in\mathbb{N}}$, with h_n defined inductively by

$$\begin{cases} h_0 := 0 \\ h_{n+1} = q_n h_n + \sigma_n. \end{cases}$$

Figure B.6 shows the role of the different parameters for the construction of a tower $n+1$ from a tower n.

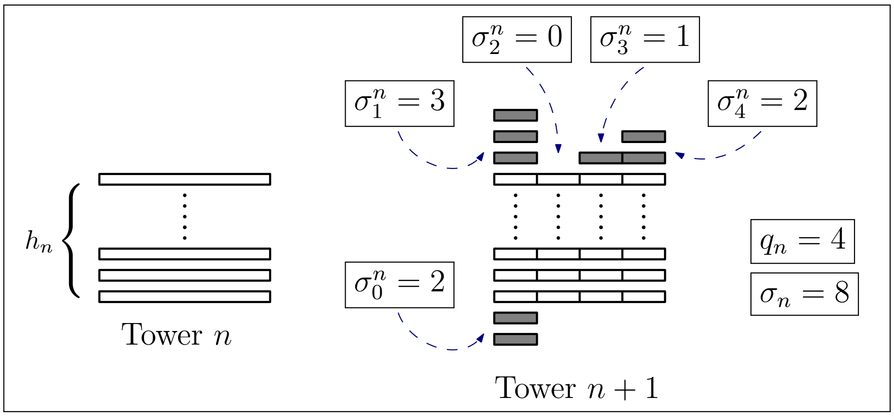

Figure B.6: Visualisation of the parameters on a example.

Remark B.3.4. In the example of Section B.3.1 we had $p = (3, (0, 0, 1, 3h_n + 1))_{n\in\mathbb{N}}$, where the height h_n of the tower at step n was defined iteratively as explained.

Lemma B.3.5 ([Cor25b, Lem. 3.3])**.** *For any $p \in \mathcal{P}^{\mathbb{N}}$, the following are equivalent:*

(i) $\displaystyle\sum_{n\in\mathbb{N}} \frac{\sigma_n}{h_{n+1}}$ *converges;*

(ii) $\displaystyle\sum_{n\in\mathbb{N}} \frac{\sigma_n}{q_0 \cdots q_n}$ *converges.*

The convergence series in the statement of the previous lemma correspond to the finiteness of the measure that will be preserved by the rank-one bijection. Hence we are of course specifically interested in sequences yielding divergent series.

Definition B.3.6. A bijection $T \in \mathrm{Aut}(X, \lambda)$ is a **rank-one bijection** if there exists $p = (q_n, (\sigma_0^n, \ldots, \sigma_{q_n}^n))_{n\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ such that there exists sequences

$$\begin{cases} (B_n)_{n\in\mathbb{N}} & \text{(the bases of the towers),} \\ \{B_{n,i} \in \mathcal{B}(X) \mid n \in \mathbb{N}, 0 \leqslant i \leqslant q_n - 1\} & \text{(the bases of the subcolumns),} \\ \{\Sigma_{n,i,j} \in \mathcal{B}(X) \mid n \in \mathbb{N}, 0 \leqslant i \leqslant q_n, 0 \leqslant j \leqslant \sigma_i^n\} & \text{(the spacers),} \end{cases}$$

of Borel subsets of X such that for any $n \in \mathbb{N}$ we have

1. $B_n, T(B_n), \dots T^{h_n-1}(B_n)$ are pairwise disjoint;
2. $B_{n,0}, \dots, B_{n,q_n-1}$ form a partition of B_n;
3. $T^{h_n}(B_{n,i}) = \begin{cases} \Sigma_{n,i+1,1} & \text{if } \sigma_i^n > 0 \\ B_{n,i+1} & \text{if } \sigma_i^n = 0 \text{ and } i < q_n - 1 \end{cases}$
4. if $\sigma_i^n > 0$ then $T(\Sigma_{n,i,j}) = \begin{cases} \Sigma_{n,i,j+1} & \text{if } j < \sigma_i^n \\ B_{n,i} & \text{if } j = \sigma_i^n, \end{cases}$
5. $B_{n+1} = \begin{cases} \Sigma_{n,0,1} & \text{if } \sigma_0^n > 0 \\ B_{n,0} & \text{if } \sigma_0^n = 0. \end{cases}$

Finally, we require that the σ-algebra generated by the $\mathcal{R}_n := (T^k(B_n))_{0 \leqslant k \leqslant h_n - 1}$ grows to the restriction of the σ-algebra $\mathcal{B}$ of the ambient space to the support of T. In particular, it separates the points.

In more casual terms, the bijection T associated to the sequence of parameters p maps each point of any 'non-top' floor of the tower to the floor above (as in the last drawing of Figure B.5), each point of the top floor of each 'non-last' subcolumn to the bottom floor of the next subcolumn (as in Figure B.7), and the measure of the last floor of the last subcolumn, where T is not defined, goes to 0.

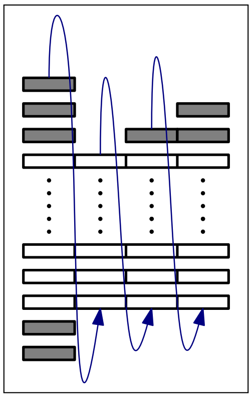

Figure B.7: Behaviour of T on the top floor of each subcolumn.

Theorem B.3.7 ([DS22, § 3.4]). *Every rank-one bijection is ergodic.*

Another remarkable result is the following. It has been known for a long time that rank-one bijections are generic in the group of measure-preserving bijections of a standard probability space endowed with the weak topology, and the result still holds in standard spaces of infinite measure.

Theorem B.3.8 ([BSSSW15, Thm. 3.1]). *Rank-one bijections are generic in* $\mathrm{Aut}(X, \lambda)$ *endowed with the weak topology.*

B.4 The three basic examples in the type II_∞ context

In this section we go back to the three basic examples that have accompanied us in the probability measure-preserving (Example 1.2.4) and measure class-preserving (Example 1.2.8) contexts, but this time in the infinite measure-preserving context, with of course a particular interest towards ergodicity. We remain very brief on irrational rotations and Bernoulli shifts, but provide a lot of details for a construction of a ∞-mp ergodic odometer, as we have not seen it fully detailed with explicit computations. The example does appear in the literature (see *e.g.* [DS22, Ex. 5.1.(iii)], [EHH98] or [AN87]), but abstract arguments are used to establish that it is of type II_∞.

B.4.1 Irrational rotations and Bernoulli shifts

For the irrational rotations, we refer to [DEJLMS23]. It has been shown by Del Junco in [Del76] that pmp irrational rotations are rank-one. Although it has been known since Schmidt's work in 1977 (see [Sch77]) that there exists infinite invariant measures for rotations on the circle, we want to spotlight [DEJLMS23] (see also [DV23]), where the authors construct *explicit* ∞-mp rank-one bijections that are isomorphic to irrational rotations. They moreover show that they are totally ergodic (T is totally ergodic if T^n is ergodic for any $n \neq 0$). We thus have the following.

Theorem B.4.1 ([DEJLMS23, Theorems 3.1, 5.1 and 6.4], see also [DV23, Thm. B])**.** *There exists infinitely many irrational angles θ for which the rotation R_θ admits an invariant atomless σ-finite measure λ, for which it is totally ergodic, and such that (R_θ, λ) is isomorphic to an explicit ∞-mp rank-one bijection (T_θ, λ).*

Let us now shift our focus towards Bernoulli shifts. The first thing to keep in mind is that the situation is quite different depending on the base space. Recall that a non-singular Bernoulli shift is an action of a countable group $G \curvearrowright (X^G, \bigotimes_{g\in G} \mu_g)$ defined by $(g \cdot x)_h = x_{g^{-1}h}$, and in the case where $G = \mathbb{Z}$ the bijection we are interested in is just the action of the generator 1. When all the μ_g are equal, the action is measure-preserving, but when it is not the case, studying the properties of the action (such as establishing ergodicity and determining the Krieger type) is a complicated problem. We will not go into details, but will state the following corollary of [BKV21, Thm. B], specifically concerning type II_∞ actions. Recall that a locally finite group is a group for which every finitely generated subgroup is finite.

Theorem B.4.2 ([BKV21, Thm. B])**.** *Let G be a countable infinite Abelian group, which is not locally finite. Any non-singular Bernoulli shift of G on a space of base $X = \{0, 1\}$ is never of type II_∞. In particular this is the case for $\mathbb{Z}$.*

It is of note also that a locally finite G admits non-singular Bernoulli actions of type II_∞, and in fact of every possible type (see [BKV21, Prop. 3.6]).

Now if the base space is not necessarily $\{0, 1\}$ but any standard Borel space, we have the following.

Theorem B.4.3 ([BV22, Thm. A],see also [DK22, Cor. 0.2])**.** *Every countable infinite amenable group G admits non-singular Bernoulli actions of any Krieger type: II_1, II_∞ and III_λ for all $\lambda \in [0, 1]$.*

In particular for $G = \mathbb{Z}$ the action of the generator provides a ∞-mp ergodic bijection.

B.4.2 An infinite measure-preserving ergodic odometer construction

We once again stress the fact that no originality is claimed for this section. The odometer that we consider is the adding machine $T : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ defined as usual by the addition with $(1, 0, 0, \ldots)$, with carry over on the right, and sending $(1, 1, 1, \ldots)$ to $(0, 0, 0, \ldots)$. Our goal is to describe an infinite measure λ on $\{0, 1\}^{\mathbb{N}}$ and explain why T is λ-measure-preserving and why it is ergodic. Let us rigorously define the space and the corresponding measure.

We describe the sequences in $\{0, 1\}^{\mathbb{N}}$ that we will be interested in by describing their behaviour in even and odd coordinates. For the even coordinates, all sequences will be allowed. We denote by X^e the following space:

$$X^e \coloneqq \{0, 1\}^{\mathbb{N}}.$$

For the odd coordinates, we will only be interested in the countable set of sequences with a finite number of non-zero coordinates. We denote by X^o the following countable space:

$$X^o \coloneqq \left\{ y = (y_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \mid \exists \text{ finitely many } n \text{ such that } y_n = 1 \right\}$$

The space X will be a countable product of copies of $X^e = \{0,1\}^{\mathbb{N}} \simeq \{0,1\}^{2\mathbb{N}}$, indexed by X^o. Any T-orbit will move through all these copies, which will provide ergodicity. We define X as follows:

$$X = \bigsqcup_{y\in X^o} X_y \coloneqq \bigsqcup_{y\in X^o} \left\{ (x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \mid (x_{2n+1})_{n\in\mathbb{N}} = y \right\} \simeq \mathbb{N} \times \{0,1\}^{2\mathbb{N}}.$$

Each space X_y contains all the sequences with all possible even coordinates, and odd coordinates fixed to be equal to y, in particular they are all zero after a certain index.

Let us now describe the measure. We endow X^o with the counting measure c, and we let μ_e be the usual product measure on X^e, *i.e.* $\mu_e = \frac{1}{2}(\delta_0 + \delta_1)^{\otimes 2\mathbb{N}}$. We finally endow X with the product measure $\lambda \coloneqq c \otimes \mu_e$. In other words $\lambda = \sum_{y\in X^o} \mu_{e\restriction X_y}$ where each $\mu_{e\restriction X_y}$ is a probability measure. In particular, (X,λ) is a standard σ-finite space. The first important remark is that this measure is not absolutely continuous with regard to the uniform product measure μ on $\{0,1\}^{\mathbb{N}}$, nor does the converse hold: indeed μ gives full weight to the sequences whose odd coordinates take infinitely many times both values by the strong law of large numbers. Note also that the sequences that are given weight by λ take infinitely many times both values in their even coordinates, and only finitely many times the value 1 in the odd coordinates.

The difficulty in describing the dynamics of T precisely is that T weaves in and out of different X_y. We give some more precision by describing this weaving phenomenon in more details.

Notation B.4.4. We define even cylinders of the form $\mathcal{C}^e_{2l} = [x_0, \dots, x_{2l}]^e$. We will write those cylinder sets more precisely when necessary, giving the information of the indices on the top line and of the coordinates on the bottom line:

$$\mathcal{C}^e_{2l} = \begin{bmatrix} 0 & 2 & \dots & 2l \\ x_0 & x_2 & \dots & x_{2l} \end{bmatrix}.$$

It clearly has measure $\mu_e(\mathcal{C}^e_{2l}) = \dfrac{1}{2^{\mathrm{len}(\mathcal{C}^e_{2l})}} = \frac{1}{2^{l+1}}$. We will use similar notations for odd sequences when necessary, using the exponent o to indicate that we look at odd coordinates. When a cylinder has no exponent, it means that we look at an even-odd cylinder, considering both even and odd coordinates.

The measure μ_e is determined by the even cylinders $\mathcal{C}^e_{2l}$. We have

$$\mathcal{C}^e_{2l} = \bigsqcup_{y\in X^o} \mathcal{C}^e_{2l} \cap X_y,$$

so $T(\mathcal{C}^e_{2l}) = \bigsqcup_{y\in X^o} T(\mathcal{C}^e_{2l} \cap X_y)$. Therefore our main goal is to understand the dynamics of T on sets of the form $\mathcal{C}^e_{2l} \cap X_y$, and to this end we introduce another notation.

Notation B.4.5. We denote by S^e_{2k} the even cylinder giving the information of the index of the first 0 appearing in even coordinates. We have

$$S^e_{2k} = \begin{bmatrix} 0 & 2 & \dots & 2k-2 & 2k \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}.$$

Similarly, S^o_{2k+1} and S_k are naturally defined as the analogous odd cylinders and even-odd cylinders.

We fix an even cylinder $\mathcal{C}^e_{2l}$ and an odd sequence $y \in X^o$. There are two cases to consider, but one is very easy to describe.

Case 1. The first zero that appears in either $\mathcal{C}^e_{2l}$ or y appears at index at most $2l+1$, *i.e.* $\mathcal{C}^e_{2l} \cap X_y \subseteq S_k$ with $k \leqslant 2l+1$.

The odometer action is very well understood on an even-odd cylinder with only ones in the first $k-1$ coordinates, and a zero at index k:

$$T\left(\begin{bmatrix} 0 & 1 & \dots & k-1 & k \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 & 1 & \dots & k-1 & k \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}\right).$$

Therefore, the T-image of $\mathcal{C}^e_{2l} \cap X_y$ is another set of the same form, in which only the coordinates of index less that $2l+1$ have changed: the even cylinder has different coordinates, but the same length, and the sequence y is changed but uniquely determined.

Case 2. The even cylinder only has coordinates equal to one, and the first zero that appears in y appears at index larger than $2l+3$, *i.e.* $\mathcal{C}^e_{2l} \cap X_y \subseteq S^o_{2k+1}$ with $2k+1 \geqslant 2l+3$.

In this case we have to look at the even coordinates of index $2l < i < 2k+1$. We provide an example to show why this case is more troublesome.

Example B.4.6. We consider $\mathcal{C}^e_2 = [1,1]^e$ and $y = (1,1,1,1,0,*,\dots)$, where y is any sequence in X^o with these first five coordinates fixed. In other words we have $l=1$ and $k=4$. Applying T to $\mathcal{C}^e_2 \cap X_y$ changes the coordinates of indices 0 to 3 from ones into zeroes, but we then have to look at the even coordinates of indices 4 to 8. We have

$$\mathcal{C}^e_2 = [1,1,0]^e \sqcup [1,1,1,0]^e \sqcup [1,1,1,1,0]^e \sqcup [1,1,1,1,1]^e.$$

It is easy to manually check that

$$\begin{aligned} T(\mathcal{C}^e_2 \cap X_y) &= T([1,1,0]^e \cap X_y) \\ &\quad \sqcup\ T([1,1,1,0]^e \cap X_y) \\ &\quad \sqcup\ T([1,1,1,1,0]^e \cap X_y) \\ &\quad \sqcup\ T([1,1,1,1,1]^e \cap X_y) \\ &= [0,0,1]^e \cap X_{y_1} \text{ where } y_1 = (0,0,1,1,0,*,\dots) \\ &\quad \sqcup [0,0,0,1]^e \cap X_{y_2} \text{ where } y_2 = (0,0,0,1,0,*,\dots) \\ &\quad \sqcup [0,0,0,0,1]^e \cap X_{y_3} \text{ where } y_3 = (0,0,0,0,0,*,\dots) \\ &\quad \sqcup [0,0,0,0,0]^e \cap X_{y_4} \text{ where } y_4 = (0,0,0,0,1,*,\dots). \end{aligned}$$

It is also easy to manually check that the measure is preserved: $\lambda(\mathcal{C}^e_2 \cap X_y) = \mu_e(\mathcal{C}^e_2) = \frac{1}{4}$, while $\lambda([0,0,1]^e \cap X_{y_1}) = \mu_e([0,0,1]^e) = \frac{1}{8}$, then the second line of the T-image has measure $\frac{1}{16}$, and the last two have measure $\frac{1}{32}$ each.

We now write the general statement. We denote by d the difference $(2k+1)-2l$ (in the previous example we had $d=3$). We have:

$$\mathcal{C}^e_{2l} \cap X_y = \left(\bigsqcup_{1 \leqslant i \leqslant d} \mathcal{C}^e_{2l} \cap S^e_{2l+2i} \right) \sqcup \begin{bmatrix} 0 & 2 & \dots & 2k-2 & 2k \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

This decomposition of $\mathcal{C}^e_{2l} \cap X_y$ keeps track of the position of the first zero, it is at index $2l+2i$ in each part of the disjoint union indexed by i, and at index $2k+1$ otherwise. Applying T is then straightforward:

$$
\begin{aligned}
&T(\mathcal{C}^e_{2l}\cap X_y)\\
&= T\left(\left(\bigsqcup_{1\leqslant i\leqslant d}\mathcal{C}^e_{2l}\cap S^e_{2l+2i}\right)\sqcup\begin{bmatrix}0&2&\dots&2k-2&2k\\1&1&\dots&1&1\end{bmatrix}\right)\\
&= \begin{bmatrix}0&\dots&2l&2l+2\\0&\dots&0&1\end{bmatrix}\cap X_{y_1}\\
&\quad\text{where } y_1=\begin{pmatrix}1&3&\dots&2l+1&2l+3&2l+5&\dots&2k-1&2k+1&2k+3&2k+5&\\0&0&\dots&0&1&1&\dots&1&0&*&*&\dots\end{pmatrix}\\
&\sqcup\begin{bmatrix}0&\dots&2l+2&2l+4\\0&\dots&0&1\end{bmatrix}\cap X_{y_2}\\
&\quad\text{where } y_2=\begin{pmatrix}1&3&\dots&2l+1&2l+3&2l+5&\dots&2k-1&2k+1&2k+3&2k+5&\\0&0&\dots&0&0&1&\dots&1&0&*&*&\dots\end{pmatrix}\\
&\dots\\
&\sqcup\begin{bmatrix}0&\dots&2k-4&2k-2\\0&\dots&0&1\end{bmatrix}\cap X_{y_{d-1}}\\
&\quad\text{where } y_{d-1}=\begin{pmatrix}1&3&\dots&2k-3&2k-1&2k+1&2k+3&2k+5&\\0&0&\dots&0&1&0&*&*&\dots\end{pmatrix}\\
&\sqcup\begin{bmatrix}0&\dots&2k-2&2k\\0&\dots&0&1\end{bmatrix}\cap X_{y_d}\\
&\quad\text{where } y_d=\begin{pmatrix}1&3&\dots&2k-1&2k+1&2k+3&2k+5&\\0&0&\dots&0&0&*&*&\dots\end{pmatrix}\\
&\sqcup\begin{bmatrix}0&\dots&2k-2&2k\\0&\dots&0&0\end{bmatrix}\cap X_{y_{d+1}}\\
&\quad\text{where } y_{d+1}=\begin{pmatrix}1&3&\dots&2k-1&2k+1&2k+3&2k+5&\\0&0&\dots&0&1&*&*&\dots\end{pmatrix}.
\end{aligned}
$$

In y_1 the first 1 appears at index $2l+3$, and there are $d-1$ ones among its $2k+1$ first coordinates, in y_2 it appears at index $2l+5$, and there are $d-2$ ones among its $2k+1$ first coordinates. In y_{d-1} it first appears at index $2k-1$, and it is the only one in the $2k+1$ first coordinates. In the d-th line, the one in the even cylinder appears so late that not a single one appears the $2k+1$ first coordinates of y_d. Finally in y_{d+1} the only one has replaced the zero at index $2k+1$.

In order to completely understand the T-action, let us also describe (for convenience) the preimage of a set of the form $\mathcal{C}^e_{2l}\cap X_y$ where $\mathcal{C}^e_{2l}=[0,0,\dots,0]^e$ and $y=(0,0,\dots,0,1,*,*,\dots)\in S^o_{2k+1}$. We still denote by d the difference $(2k+1)-2l$. Again, we advise the reader to manually test the case where for instance $\mathcal{C}^e_{2l}=[0,0]^e$ and y is of the form $(0,0,0,0,1,*,*,\dots)$. The exact same computations (we just switch the 0s and the 1s) yield that

$$
\begin{aligned}
T^{-1}(\mathcal{C}_{2l}^e \cap X_y) = &\begin{bmatrix} 0 & \dots & 2l & 2l+2 \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_1} \text{ where } y_1 = (1,1,\dots 1,0,\dots,0,1,*,*,\dots) \\
\sqcup &\begin{bmatrix} 0 & \dots & 2l+2 & 2l+4 \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_2} \text{ where } y_2 = (1,1,\dots 1,1,0,\dots,0,1,*,*,\dots) \\
\dots & \\
\sqcup &\begin{bmatrix} 0 & \dots & 2k-4 & 2k-2 \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_{d-1}} \text{ where } y_{d-1} = (1,1,\dots,1,0,1,*,*,\dots) \\
\sqcup &\begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_d} \text{ where } y_d = (1,1,\dots,1,1,*,*,\dots) \\
\sqcup &\begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 1 & \dots & 1 & 1 \end{bmatrix} \cap X_{y_{d+1}} \text{ where } y_{d+1} = (1,1,\dots,1,0,*,*,\dots).
\end{aligned}
$$

The only extra case that we have to treat is the case where there is actually not a single 1 appearing in y, *i.e.* where y is the constant sequence where all coordinates are 0. This case follows from what we have already done:

$$
\begin{aligned}
T^{-1}(\mathcal{C}_{2l}^e \cap X_y) = &\bigsqcup_{j=1}^{+\infty} \begin{bmatrix} 0 & \dots & 2l & 2l+2j \\ 1 & \dots & 1 & 0 \end{bmatrix} \cap X_{y_j} \\
&\text{where } y_j = \begin{pmatrix} 1 & \dots & 2l+2j-1 & 2l+2j+1 & 2l+2j+3 & \dots \\ 1 & \dots & 1 & 0 & 0 & \end{pmatrix}
\end{aligned}
$$

We can now manually check that the measure is preserved by T.

Proposition B.4.7. *The odometer T preserves the measure λ on X.*

Proof. As previously stated, it suffices to show that $\lambda(T(\mathcal{C}_{2l}^e \cap X_y)) = \lambda(\mathcal{C}_{2l}^e \cap X_y)$ for any set of the form $\mathcal{C}_{2l}^e \cap X_y$. We clearly have

$$
\lambda(\mathcal{C}_{2l}^e \cap X_y) = \mu_e(\mathcal{C}_{2l}^e) = \frac{1}{2^{l+1}}.
$$

If $\mathcal{C}_{2l}^e \cap X_y$ is such that we are in the Case 1, there is nothing to check as it is clear that the T-image is a set of the same form, and the measure is then only defined by the length of the even cylinder. If $\mathcal{C}_{2l}^e \cap X_y$ is such that we are in the Case 2 we have

$$
\begin{aligned}
\lambda(T(\mathcal{C}_{2l}^e \cap X_y)) = &\lambda\left(\begin{bmatrix} 0 & \dots & 2l & 2l+2 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_1}\right) \\
&+\lambda\left(\begin{bmatrix} 0 & \dots & 2l+2 & 2l+4 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_2}\right) \\
&+\dots \\
&+\lambda\left(\begin{bmatrix} 0 & \dots & 2k-4 & 2k-2 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_{d-1}}\right) \\
&+\lambda\left(\begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 0 & \dots & 0 & 1 \end{bmatrix} \cap X_{y_d}\right) \\
&+\lambda\left(\begin{bmatrix} 0 & \dots & 2k-2 & 2k \\ 0 & \dots & 0 & 0 \end{bmatrix} \cap X_{y_{d+1}}\right) \\
= &\frac{1}{2^{l+2}} + \frac{1}{2^{l+3}} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^{l+1}}.
\end{aligned}
$$

□

The next step is checking the ergodicity of our infinite dyadic odometer. As it is not a product odometer, the usual arguments do not hold, and we have to check it manually. The proof however still boils down to the fact that T acts transitively on cylinders. We provide some additional details.

Proposition B.4.8. *The odometer T is ergodic on (X, λ).*

Proof. Consider a T-invariant set A such that $\lambda(A) > 0$, and assume towards contradiction that $\lambda(X \setminus A) > 0$. By definition of λ, since there exists at least one X_{y_1} (resp. X_{y_2}) such that $\lambda(A \cap X_{y_1}) > 0$ (resp. $\lambda((X \setminus A) \cap X_{y_2}) > 0$) there exists $\mathcal{C}_{2l_1}$ and $\mathcal{C}_{2l_2}$ with $l+1 := \operatorname{len}(C_{2l_1}) = \operatorname{len}(C_{2l_2})$, as well as $p \in]0,1[$ such that

$$\begin{cases} \lambda(A \cap (\mathcal{C}_{2l_1} \cap X_{y_1})) = \mu_{e\restriction X_{y_1}}(A \cap \mathcal{C}_{2l_1}) > p\dfrac{1}{2^{l+1}} \\ \lambda((X \setminus A) \cap (\mathcal{C}_{2l_2} \cap X_{y_2})) = \mu_{e\restriction X_{y_2}}(A \cap \mathcal{C}_{2l_2}) > (1-p)\dfrac{1}{2^{l+1}}. \end{cases}$$

Without loss of generality we can assume that $2l$ is larger than the index of the last 1 appearing in both y_{k_1} and y_{k_2} (indeed by definition those are finite numbers, and we can then consider an even cylinder as long as we want for the argument). We now use the transitivity of the T-action on the cylinders: denote by $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the even-odd cylinder whose fixed even coordinates are those of $\mathcal{C}_{2l_1}$ (resp. $\mathcal{C}_{2l_2}$) and whose fixed odd coordinates are the l first coordinates of y_1 (resp. y_2). In other words, for $i = 1, 2$ we have

$$\mathcal{C}_i = \begin{bmatrix} 0 & 1 & 2 & \dots & 2l-1 & 2l \\ (\mathcal{C}_{2l_i})_0 & (y_i)_1 & (\mathcal{C}_{2l_i})_2 & \dots & (y_i)_{2l-1} & (\mathcal{C}_{2l_i})_{2l} \end{bmatrix}.$$

It is easy to send $\mathcal{C}_1$ to $\mathcal{C}_2$ via a power of T, but we insist on two main points. The first one is that our assumption that $2l$ is larger than the non-zero part of y_1 and y_2 ensures that we send X_{y_1} to X_{y_2} by applying our power of T, as long as we do not modify the coordinates after $2l$. The second point is that we can always do that by using either T or T^{-1}: indeed, if the coordinates of $\mathcal{C}_2$ are larger than the coordinates of $\mathcal{C}_1$ for the natural binary order, (*e.g.* $[0,1,1,1] > [1,1,0,1]$) then a positive power of T works, and a negative power of T works for the converse case. Denote by j this power. We thus have

$$T^j(\mathcal{C}_1 \cap X_{y_1}) = \mathcal{C}_2 \cap X_{y_2},$$

so by T-invariance of A we have

$$\lambda(T^j(A \cap (\mathcal{C}_{2l_1} \cap X_{y_1}))) = \lambda(A \cap (\mathcal{C}_{2l_2} \cap X_{y_2})),$$

and since T preserves λ, we have

$$\lambda(A\cap(\mathcal{C}_{2l_2}\cap X_{y_2})) = \lambda(T^j(A\cap(\mathcal{C}_{2l_1}\cap X_{y_1}))) = \lambda(A\cap(\mathcal{C}_{2l_1}\cap X_{y_1})) = \mu_{e\restriction X_{y_1}}(A\cap\mathcal{C}_{2l_1}) > p\frac{1}{2^{l+1}}.$$

This is a contradiction since $\lambda(\mathcal{C}_{2l_2} \cap X_{y_2}) = \mu_{e\restriction X_{y_2}}(\mathcal{C}_{2l_2}) = \frac{1}{2^{l+1}}$ but

$$\lambda((X \setminus A) \cap (\mathcal{C}_{2l_2} \cap X_{y_2})) = \mu_{e\restriction X_{y_2}}((X \setminus A) \cap \mathcal{C}_{2l_2}) > (1-p)\frac{1}{2^{l+1}}.$$

Therefore T is ergodic. $\square$

B.5 A random walk on $\mathbb{Z}$

B.5.1 Definition via the Baker's map

This section is based on [EHIP14, 4.3.2] and presents a very simple example of an ∞-mp ergodic bijection, requiring almost no preliminary knowledge, except for basic understanding of the Baker's map, of which we recall the definition.

There are several ways to define the (unfolded) Baker's map, with one of the fastest being the following:

$$B(y_1, y_2) := \left(2y_1 - \lfloor 2y_1 \rfloor, \frac{y_2 + \lfloor 2y_1 \rfloor}{2}\right) = \begin{cases} (2y_1, \frac{y_2}{2}) & \text{if } 0 \leqslant y_1 < \frac{1}{2}, \\ \left(2y_1 - 1, \frac{y_2+1}{2}\right) & \text{if } \frac{1}{2} \leqslant y_1 < 1, \end{cases}$$

where (y_1, y_2) is in the unit square $Y = [0,1[^2$. It is easy to check that $B : Y \to Y$ is an ergodic measure-preserving bijection of Y endowed with the usual (on $\mathbb{R}^2$) Lebesgue measure $\mathrm{Leb}_{\restriction Y}$. We now consider the standard σ-finite measure space

$$(X, \lambda) = (\mathbb{Z} \times Y, c \otimes \mathrm{Leb}_{\restriction Y}) \cong \left(\mathbb{R} \times [0,1[\,, \mathrm{Leb}_{\restriction \mathbb{R} \times [0,1[}\right),$$

where c is the counting measure on $\mathbb{Z}$. Our ∞-mp ergodic bijection will move the points of the 'strip' X horizontally while applying the Baker's map to each 'cell' of length 1, *i.e.* each copy of Y. We thus need to decide whether to go left or right, and this is the role of the function $\Phi : Y \to \{-1, 1\}$ defined as follows:

$$\Phi(y_1, y_2) = \begin{cases} -1 & \text{if } 0 \leqslant y_1 < \frac{1}{2}, \\ +1 & \text{if } \frac{1}{2} \leqslant y_1 < 1. \end{cases}$$

We can finally define the ∞-mp bijection as the skew-product $\widetilde{T} : X \to X$ defined by

$$\widetilde{T}(n, (y_1, y_2)) := (n + \Phi(y_1, y_2), B(y_1, y_2))$$

for any $(n, (y_1, y_2)) \in \mathbb{Z} \times Y = X$. Figure B.8 illustrates the action of $\widetilde{T}$ on one cell of X.

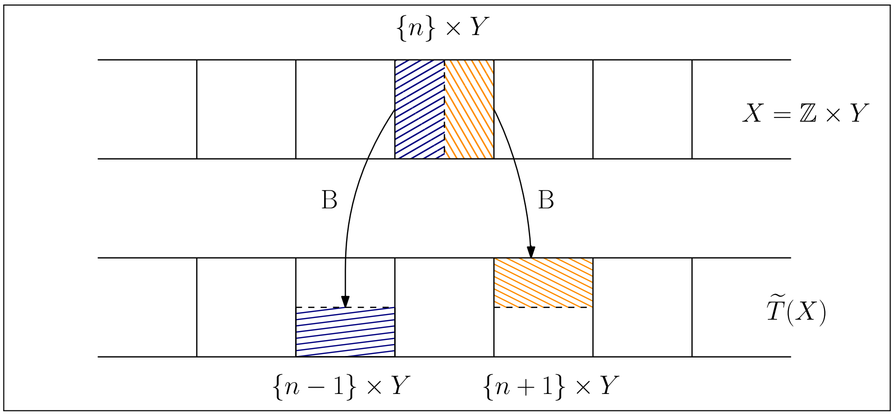

Figure B.8: Visualisation of the application of $\widetilde{T}$ on a cell of X.

Let us sketch the proof of the ergodicity of $\widetilde{T}$, so as to establish that it is indeed an ∞-mp ergodic bijection. We first notice that

$$\widetilde{T}^k(n, (y_1, y_2)) = (n + S_k(\Phi(y_1, y_2)), B^k(y_1, y_2))$$

for any $(n, (y_1, y_2)) \in X$ and any $k \in \mathbb{N}$, where S_k is the sum of k independent Bernoulli random variables. We consider a positive measure $\widetilde{T}$-invariant subset A of X. Looking at the second coordinate, it is clear by ergodicity of the Baker's map that the Y-component of A has to be the whole set Y. Now looking at the first coordinate and from what we just established, it is clear that S_k can take any value between 0 and k, so the $\mathbb{Z}$-component of A also has to be the whole set $\mathbb{Z}$, hence A is conull.

B.5.2 Definition via the two-sided shift

The reader who is familiar with it would have recognized that the Baker's transformation is just another avatar of the two-sided shift operator on $\{0,1\}^{\mathbb{Z}}$. Indeed, encoding the information of two real numbers y_1 and y_2 in $[0,1[$ by their dyadic series, *i.e.* writing

$$y_1 = \sum_{n\in\mathbb{N}} b_{-n}\frac{1}{2^{n+1}} \quad \text{and} \quad y_2 = \sum_{n\in\mathbb{N}} a_{n+1}\frac{1}{2^{n+1}}$$

and realising the sequence $(b_{-n})_{n\in\mathbb{N}}$ as the negative-index coordinates of $y \in \{0,1\}^{\mathbb{Z}}$ while $(a_{n+1})_{n\in\mathbb{N}}$ corresponds to the positive-index coordinates, we bijectively identify Y with $\{0,1\}^{\mathbb{Z}}$.

Then, we can observe in Figure B.9 the effect on the shift on a sequence, and we check that

$$2y_1 - \lfloor 2y_1 \rfloor = b_0 + \sum_{n\in\mathbb{N}} b_{-n+1}\frac{1}{2^{n+1}} - b_0 = \sum_{n\in\mathbb{N}} b_{-n+1}\frac{1}{2^{n+1}}$$

while

$$\frac{y_2 + \lfloor 2y_1 \rfloor}{2} = \sum_{n\in\mathbb{N}} a_{n+1}\frac{1}{2^{n+2}} + \frac{b_0}{2} = \frac{b_0}{2} + \sum_{n\in\mathbb{N}^*} a_n\frac{1}{2^{n+1}},$$

so it is immediate to see that the shift exactly corresponds to the unfolded Baker's transformation B applied to (y_1, y_2). With this point of view we obtain an ∞-mp ergodic bijection on the space

$$\widetilde{X} = \mathbb{Z} \times \{0,1\}^{\mathbb{Z}},$$

which is a realization of a random walk on $\mathbb{Z}$.

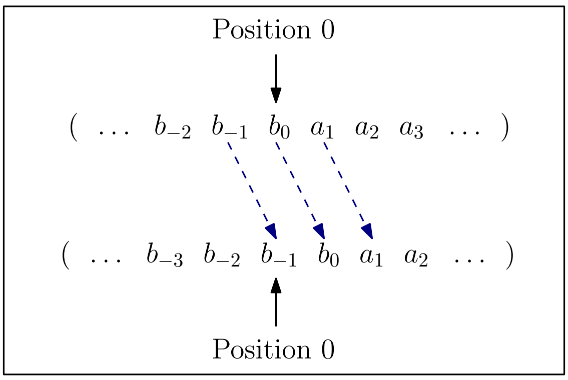

Figure B.9: Application of the shift on $y \in \{0,1\}^{\mathbb{Z}}$.

Seeing this ∞-mp ergodic bijection as a shift lets us open the door to the very interesting world of *symbolic dynamics*, which we will just understand as shifts on sequences of abstract symbols. This provides many examples of ∞-mp ergodic bijection, and we present a few, that we believe are accessible enough, at least in their basic concepts.

B.6 Markov shifts

Markov shifts are classical examples of infinite measure-preserving systems, that have in particular been studied by Kakutani and Parry in [KP63] in order to exhibit transformations with different *ergodic index*. We also refer to [Aar97, § 4.5]. This time we start with the general theory before giving an example.

B.6.1 General theory

Let P be an infinite stochastic matrix, *i.e.* a matrix $P = (P_{i,j})_{i,j\in\mathbb{Z}}$ such that all $P_{i,j} \geqslant 0$, and each line sums to one: $\sum_{j\in\mathbb{Z}} P_{i,j} = 1$. We denote by $(P^{(n)}_{i,j})_{i,j\in\mathbb{Z}}$ the coefficients of the n-th power P^n of P. We also assume that for any (i,j) there exists $n \geqslant 0$ such that $P^{(n)}_{i,j} > 0$ (it is sometimes said that P has only one **ergodic class**). The matrix P is called the **transition matrix**, giving the probabilities of going from one symbol -or state- to another.

We also assume that there exists an left eigenvector $(\lambda_n)_{n\in\mathbb{Z}}$ with eigenvalue 1 (we actually give a bit more details about the theory of infinite matrices in Section B.7.2, and refer to [Kit98, Sec. 7.1] for more on this subject), and such that

$$\begin{cases} \lambda_i \geqslant 0 \text{ for all } i \in \mathbb{Z}, \\ \sum_{i\in\mathbb{Z}} \lambda_i = +\infty. \end{cases}$$

We now define the ambient space to be $X = \mathbb{Z}^{\mathbb{Z}}$ (for more generality, one might consider $X = S^{\mathbb{Z}}$ with S being a countable set of *symbols* -or *states*-), endowed with the σ-algebra generated by the usual x-cylinder sets of the form

$$[x]_{m,n} = \{y \in X \mid \forall m \leqslant k \leqslant n \text{ we have } y_k = x_k\},$$

where x is an element of X, and $m \leqslant n$ are in $\mathbb{Z}$ (in particular this makes X into a standard Borel space). We now define the measure λ on X as follows:

$$\lambda([x]_{m,n}) = \lambda_{x_m} \prod_{k=m}^{n-1} P_{x_k,x_{k+1}},$$

(where the product is empty when $m = n$). Let us explain in details how this measure works on a concrete example. Consider $\overline{x}$ to be an element of X such that

$$\begin{cases} \overline{x}_0 = 1 \\ \overline{x}_1 = -1 \\ \overline{x}_2 = 3 \\ \overline{x}_3 = 2. \end{cases}$$

We are interested in the measure of the cylinder defined by these coefficients. In other words we want to compute $\lambda([\overline{x}]_{0,3})$. We can rewrite $[\overline{x}]_{0,3}$ as $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & -1 & 3 & 2 \end{bmatrix}$ where the first line represents the indices, and the second line the corresponding coefficients, *i.e.* the symbols. We have

$$\lambda([\overline{x}]_{0,3}) = \lambda_{\overline{x}_0} P_{\overline{x}_0,\overline{x}_1} P_{\overline{x}_1,\overline{x}_2} P_{\overline{x}_2,\overline{x}_3} = \lambda_1 P_{1,-1} P_{-1,3} P_{3,2}.$$

We interpret this as λ giving the initial weight of the 'starting point' of the cylinder, and then the transition probabilities are given by the matrix P, by following the dynamic on $\mathbb{Z}$ given by the word $\overline{x}$. Notice that the measure only takes into account the symbols, not

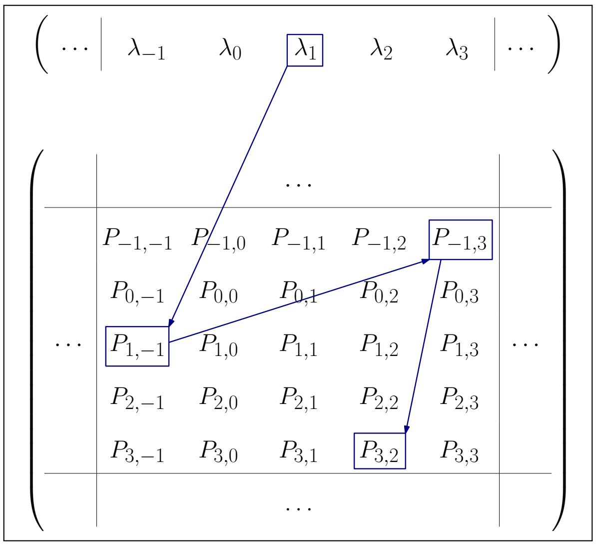

Figure B.10: Computation of the measure of $[\overline{x}]_{0,3}$.

their indices. We go from 1 to -1, then to 3 and then to 2, which reads for the measure as illustrated by Figure B.10.

We have that $X = \bigsqcup_{n\in\mathbb{Z}} X_n$, where $X_n = \{x \in X \mid x_0 = n\}$, and $\lambda(X_n) = \lambda([n]_{0,0}) = \lambda_n$. We moreover have $\lambda(X) = \sum_{n\in\mathbb{Z}} \lambda(X_n) = \sum_{n\in\mathbb{Z}} \lambda_n = +\infty$, which in particular means that (X, λ) is a standard σ-finite space.

Our measure-preserving bijection on X is the shift T, defined by $(T(x))_k = x_{k+1}$ (we advise the reader to be careful, as this shift does not go in the same direction as the one used in Section B.5). The shift T preserves λ. Indeed,

$$T([x]_{m,n}) = T\left(\begin{bmatrix} m & m+1 & \dots & n \\ x_m & x_{m+1} & \dots & x_n \end{bmatrix}\right) = \begin{bmatrix} m-1 & m & \dots & n-1 \\ x_m & x_{m+1} & \dots & x_n \end{bmatrix}$$

and so $\lambda(T([x]_{m,n})) = \lambda([x]_{m,n})$, as both the starting point and the transitions are the same, they are just not at the same index.

The two properties that will be of interest to us are the following. We give the general version, as it can be of interest, but for simplicity one might want to only consider $k = 1$ for both the conditions and the conclusion of Theorem B.6.3.

Definition B.6.1. We say that the transition matrix P is:

- **k-irreducible** if for every $i_1, \dots i_k, j_1, \dots, j_k \in \mathbb{Z}$ there exists $n \in \mathbb{N}$ such that

  $$P^{(n)}_{i_1,j_1} \times \dots \times P^{(n)}_{i_k,j_k} > 0,$$

  (for $k = 1$, we think of this as meaning that it is possible to go from i to j in n steps).

- **k-recurrent** if for some $i \in \mathbb{Z}$ (hence for all $i \in \mathbb{Z}$ if P is also k-irreducible) we have

  $$\sum_{n=1}^{+\infty} \left(P^{(n)}_{i,i}\right)^k = +\infty,$$

  (for $k = 1$, we think of this as meaning that the symbol i will be visited infinitely many times).

Remark B.6.2. The 'hence' in the previous definition holds if P is k-irreducible. Indeed, let P be k-irreducible, and i be a symbol that is k-recurrent. We want to show that

$i \neq j \in \mathbb{Z}$ is also k-recurrent. We prove it for $k = 1$. By irreducibility we let l, m be such that $P_{j,i}^{(l)} > 0$ and $P_{i,j}^{(m)} > 0$. We then have

$$\sum_{n=1}^{+\infty} P_{j,j}^{(n)} \geqslant \sum_{n=1}^{+\infty} P_{j,j}^{(l+n+m)} \geqslant \sum_{n=1}^{+\infty} P_{j,i}^{(l)} P_{i,i}^{(n)} P_{i,j}^{(m)} = +\infty.$$

The main theorem is the following.

Theorem B.6.3 ([KP63],[Aar97, Thm. 4.5.3]). *If the transition matrix P is k-irreducible and k-recurrent, then $T \times \ldots \times T$ (k times) is ergodic on $(X^k, \lambda^{\otimes k})$.*

B.6.2 The biased random walk on $\mathbb{Z}$

The example is detailed in [KP63] (but for their purposes they take the squared random walk), and originally from [Gil56]. It is a generalization of the unbiased random walk on $\mathbb{Z}$: indeed, one simply has to take $\varepsilon = 0$ in the following discussion to recover this case. We fix $-\frac{1}{2} < \varepsilon < \frac{1}{2}$ and consider the matrix P defined by

$$\begin{cases} P_{0,1} = P_{0,-1} = \frac{1}{2}; \\ P_{i,i+1} = \dfrac{1 - \frac{\varepsilon}{i}}{2} = \dfrac{i - \varepsilon}{2i}; \\ P_{i,i-1} = \dfrac{1 + \frac{\varepsilon}{i}}{2} = \dfrac{i + \varepsilon}{2i}; \\ P_{i,j} = 0 \text{ for } |i - j| \neq 1. \end{cases}$$

This transition matrix is the band matrix that appears in Figure B.12 (we represented the same block $-1 \leqslant i, j \leqslant 3$). It corresponds to the random walk represented in Figure B.11.

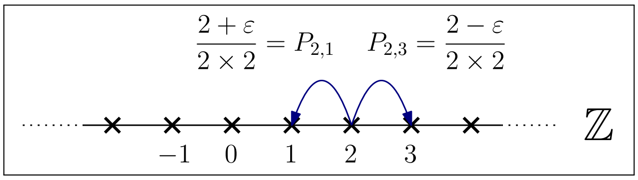

Figure B.11: The biased Random Walk of parameter ε on $\mathbb{Z}$.

$$\left(\begin{array}{c|ccccc|c}
 & & & \cdots & & & \\
\hline
 & 0 & \frac{-1-\varepsilon}{2\times(-1)} & 0 & 0 & 0 & \\
 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \\
\cdots & 0 & \frac{1+\varepsilon}{2\times 1} & 0 & \frac{1-\varepsilon}{2\times 1} & 0 & \cdots \\
 & 0 & 0 & \frac{2+\varepsilon}{2\times 2} & 0 & \frac{2-\varepsilon}{2\times 2} & \\
 & 0 & 0 & 0 & \frac{3+\varepsilon}{2\times 3} & 0 & \\
\hline
 & & & \cdots & & &
\end{array}\right)$$

Figure B.12: The transition matrix of Kakutani and Parry's example.

We now define the vector $(\lambda_n)_{n\in\mathbb{Z}}$ by

$$\begin{cases} \lambda_0 = 1 \\ \lambda_i = \lambda_{-i} = \dfrac{\Gamma(1+\varepsilon) i \Gamma(i-\varepsilon)}{\Gamma(1-\varepsilon)\Gamma(i+1+\varepsilon)} \text{ for } i \in \mathbb{N}^*. \end{cases}$$

By an immediate induction using the property of Euler's Γ function, we obtain that $\Gamma(i \pm \varepsilon) = (i-1 \pm \varepsilon)(i-2 \pm \varepsilon)\dots(2 \pm \varepsilon)(1 \pm \varepsilon)\Gamma(1 \pm \varepsilon)$, so we have

$$\lambda_i = \frac{i(i-1-\varepsilon)\dots(1-\varepsilon)}{(i+\varepsilon)(i-1+\varepsilon)\dots(1+\varepsilon)}.$$

We can now show that (λ_i) is a left eigenvector for P. The computation is straightforward, if somewhat tedious.

$$\begin{aligned}
(\lambda P)_i =& P_{i-1,i}\lambda_{i-1} + P_{i+1,i}\lambda_{i+1} \\
=& \frac{(i-1-\varepsilon)}{2(i-1)} \frac{(i-1)(i-2-\varepsilon)\dots(1-\varepsilon)}{(i-1+\varepsilon)(i-2+\varepsilon)\dots(1+\varepsilon)} \\
&+ \frac{(i+1+\varepsilon)}{2(i+1)} \frac{(i-\varepsilon)(i-1-\varepsilon)\dots(1-\varepsilon)}{(i+1+\varepsilon)(i+\varepsilon)\dots(1+\varepsilon)} \\
=& \frac{1}{2} \frac{(i-1-\varepsilon)\dots(1-\varepsilon)}{(i-1+\varepsilon)\dots(1+\varepsilon)} \left(1 + \frac{i-\varepsilon}{i+\varepsilon}\right) \\
=& \frac{1}{2} \frac{(i-1-\varepsilon)\dots(1-\varepsilon)}{(i-1+\varepsilon)\dots(1+\varepsilon)} \times \frac{2i}{(i+\varepsilon)} \\
=& \lambda_i.
\end{aligned}$$

It can moreover be checked that $\lambda_n \underset{\sim}{C} \times n^{-2\varepsilon}$ as $n \to \infty$, so the series $\sum_{n\in\mathbb{Z}} \lambda_n$ diverges whenever $\varepsilon \leqslant \frac{1}{2}$, thus making $(\mathbb{Z}^{\mathbb{Z}}, \lambda)$ a standard σ-finite space. It is clear that P is k-irreducible for every k, and it can be shown (see [Gil56] and the references therein) that P is 1-recurrent. From Theorem B.6.3 we can deduce that our example is an ∞-mp ergodic bijection as wanted.

B.7 Substitution Dynamics on infinite alphabets

We go back to presenting an example before presenting the general theory.

B.7.1 The Drunken man substitution

We stumble into the realm of substitution dynamics with the *(squared) drunken man substitution*. We will follow the exposition from [Fer06], and refer to [Que87] and [FBFMS02] for more on this vast subject, that we will barely brush. We recall the basics of a substitution dynamical system, and present Ferenczi's example. We also refer to [BJS24]. We provide quite a lot of details, as this type of construction is very different from the ones seen until here.

Definition B.7.1. We start with an infinite countable set $\mathscr{A}$, that we call an **alphabet**, and elements of $\mathscr{A}$ are called **letters** (of $\mathscr{A}$). A **word** w of length $\mathrm{len}(w) = k$ (on $\mathscr{A}$) is a finite string $w = a_1 \dots a_k$ of letters, and the **concatenation** of two words w, w' is the string $ww' := a_1 \dots a_{\mathrm{len}(w)} a'_1 \dots a'_{\mathrm{len}(w')}$. We say that a word w **occurs at place** i in an infinite string u (*i.e.* a one-sided sequence $(u_n)_{n\in\mathbb{N}^*}$) if for any $i \leqslant j \leqslant i + \mathrm{len}(w) - 1$ we have that $u_j = a_j$. For an infinite string u we moreover denote by

- $L_n(u)$ the set of words of length $n \geqslant 1$ occurring in u;
- $L(u) = \bigcup_{\mathbb{N}^*} L_n(u)$ the set of words occurring in u: the **language of** u.

A **substitution** is a map σ from $\mathscr{A}$ to the set of finite words $\mathscr{A}^* = \bigcup_{k\in\mathbb{N}^*} \mathscr{A}^k$ on $\mathscr{A}$, which we extend to a homomorphism for the concatenation by setting $\sigma(ww') = \sigma(w)\sigma(w')$. A **fixed point** for σ is an infinite string u such that $\sigma(u) = u$. We say that a word w

occurs minimally in a word v if w does not occur in any subword v' of v (*i.e.* words v' such that $v = a_1 \dots a_k v' b_1 \dots b_l$, for $k, l \in \mathbb{N}$, $a_i, b_i \in \mathscr{A}$).

For any two words v and w, we say that v is an **ancestor** (under σ) of w with multiplicity $m_w(v)$ if w occurs minimally in $\sigma(v)$ at m different places. We denote by $\mathrm{Anc}(w)$ the set of all ancestors of w.

We finally define the **(one-sided) shift** T by setting $T(x_1x_2x_3\dots) = x_2x_3x_4\dots$, for any $x_1x_2x_3\dots \in \mathscr{A}^{\mathbb{N}}$.

The existence of fixed points is important for the rest of the study, and while it can be explicitly constructed for the example we will be interested in, there is a general criterion.

Proposition B.7.2 ([Mau06])**.** *There exists a fixed point u for the substitution σ on the infinite alphabet $\mathscr{A}$ if and only if the three following conditions are satisfied:*

1. $\exists\, \iota \in \mathscr{A}, \sigma(\iota) \in \iota\mathscr{A}^*$;
2. $\mathrm{len}(\sigma(\iota)) \geqslant 2$;
3. $\mathrm{len}(\sigma(a)) \geqslant 1$ *for every* $a \in \mathscr{A}$.

We now give the construction of the drunken man substitution, and of the substitution dynamical system associated with it. For periodicity reasons, that we will not detail, the substitution we give is called the *squared* drunken man substitution. It is however the correct one to consider (see the next section), and for the sake of simplicity, we will just call it the drunken man substitution.

Definition B.7.3. Let $\mathscr{A} = 2\mathbb{Z}$ be an alphabet. The **drunken man substitution** is the map σ taking defined as follows:

$$\begin{array}{cccccc} \sigma & : & \mathscr{A} & \longrightarrow & \mathscr{A}^* \\ & & n & \longmapsto & (n-2)nn(n+2), \end{array}$$

and in order for σ to satisfy the conditions of Proposition B.7.2, we (artificially) adjoin to $\mathscr{A}$ the letter ι, and set $\sigma(\iota) = \iota 0$. We still call the alphabet $\mathscr{A}$ and the substitution σ, despite the changes. The fixed point for σ, which exists by Proposition B.7.2, will thereafter be denoted by u.

We now define X to be the subset of $\mathscr{A}^{\mathbb{N}^*}$ comprised of all the sequences $x = x_1x_2x_3\dots$ such that every word occurring in x is a word of the language $L(u)$, equivalently, such that every word occurring in x occurs in $\sigma^n(a)$ for some $a \in \mathscr{A}$ and some $n \in \mathbb{N}$. Notice now that $T(X) \subseteq X$, and as such, we say that (X, T) is the **symbolic dynamical system** associated with σ. As X is a closed subset of the (non-compact) product Polish space $\mathscr{A}^{\mathbb{N}}$ (for the discrete topology on $\mathscr{A}$), the space X is a standard Borel space. We also use the usual notion of cylinders: for a word $w = a_1 \dots a_{\mathrm{len}(w)}$, the set $[w] \subseteq X$ is the w**-cylinder** and is defined by

$$[w] = \left\{ x \in X \mid x_1 = a_1, \dots x_{\mathrm{len}(w)} = a_{\mathrm{len}(w)} \right\}.$$

Those cylinders generate both the topology and the σ-algebra on X.

Remark B.7.4. The fixed point u of σ can easily be constructed by induction, starting with the letter ι. Indeed we have

$$\begin{array}{rcl} \sigma(\iota) & = & \iota 0 \\ \sigma(\iota 0) & = & \iota 0.(-2)002 \\ \sigma(\iota 0.(-2)002) & = & \iota 0.(-2)002.(-4)(-2)(-2)0.(-2)002.(-2)002.0224 \\ & \dots & \end{array}$$

(the dots have been added for legibility). On Figure B.13 we can observe the self-similarity phenomenon occurring in u (the letter ι is not represented).

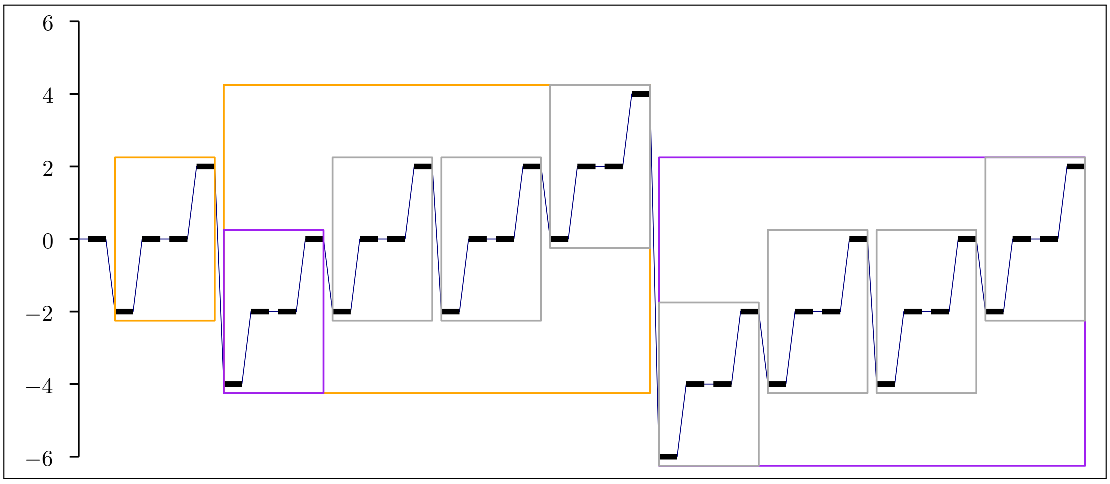

Figure B.13: The first terms of the fixed point u of σ.

We explicitly describe how the infinite measure that is preserved by T is constructed.

Definition B.7.5. We say that λ is a **natural measure** for (X,T) if it is T-invariant, and if for any cylinder $[w]$ we have

$$\lambda([w]) = \frac{1}{4} \sum_{v \in \mathrm{Anc}(w)} \lambda([v]) m_w(v). \tag{$*$}$$

Proposition B.7.6 ([Fer06, Prop. 2.11])**.** *There exists a unique infinite natural measure λ for the substitution dynamical system (X,T).*

Proof. We define λ on one-letter-word cylinders, such that the measure of the cylinder $[0]$ is equal to 1: by definition of naturality we have

$$4\lambda([2n]) = \lambda([2n-2]) + 2\lambda([2n]) + \lambda([2n+2]). \tag{$\star$}$$

Equation ($\star$) ensures that all one-letter-word cylinders have equal measure. Indeed, assume the contrary, and consider ($\star$) for $n = 0$. Without loss of generality we can assume that $\lambda([0]) > \lambda([2])$. By Equation ($\star$), the measure of any cylinder $[2n+2]$ (for $n \in \mathbb{N}^*$) has to be strictly less than that of $[2n]$. By ($\star$) again there exists $m \in \mathbb{N}^*$ such that $\lambda([2m]) < 0$, a contradiction. We then choose this constant value to be 1.

For cylinders of length 2, we define their measure recursively. Words of the form $(2n)(2n)$ or $(2n)(2n+2)$ only have ancestors of length 1, so $\lambda([(2n)(2n)]) = \frac{1}{4}$ and $\lambda([(2n)(2n+2)]) = \frac{1}{2}$. Words of the form $(2n)(2n-2p)$, for $p > 0$, only have ancestors of the form $(2n-2)(2n-2p+2)$, *i.e.* of the form $(2m)(2m-2p+4)$. Therefore, a straightforward induction shows that their measures are entirely determined by what we have already done: $\lambda([(2n)(2n-2p)]) = 2^{-p-2}$.

For cylinders of length $n \geqslant 3$, we first prove the following intermediate result ([Fer06, Lem. 2.7]), which establishes that σ is **left-determined**.

Fact: any word w in the language $L(u)$ of length at least 3 has a unique decomposition $w = w_1 \dots w_k$, where each (4-letter word) w_i is a $\sigma(a_i)$ for some $a_i \in \mathscr{A}$, except that w_1 may be only a suffix of $\sigma(a_1)$ (*i.e.* a word v such that $\sigma(a_1) = b_1 \dots b_l v$ for $1 \leqslant l \leqslant 3$ and $b_i \in \mathscr{A}$), and w_k may be only a prefix of $\sigma(a_k)$. Note that in this case the a_i ($1 \leqslant i \leqslant k-1$) are unique.

Proof of the Fact: Any word w of length at least 3 contains (at least) one of the following:

- $nn(n+2)$;

- $n(n+2)(n+2)$;
- $n(n+2)m$ for some $m \neq (n+2)$;
- $mn(n+2)$ for some $m \neq n$.

Any of those words allow us to seaparate w into the w_i in a unique way, and the last letter of a word $\sigma(a)$ completely determines a, so they are also unique. $\diamondsuit$

By the previously proved fact, all words w of length $k \geqslant 3$ only have ancestors of length at most $k-1$. It is then possible to define $\lambda([w])$ recursively thanks to ($*$).

Newt we check that it is T-invariant, *i.e.* that for any word w we have

$$\lambda([w]) = \sum_{a \in \mathscr{A}} \lambda([aw]) = \sum_{a \in \mathscr{A}} \lambda([wa]).$$

By construction of λ and Equation ($*$) we only have to check this equality for 1-letter words. Since $\lambda([(2n)(2n-2p)]) = \lambda([(2n+2p)(2n)]) = 2^{-p-2}$, we have

$$\lambda([2n]) = 1 = \sum_{p=-1}^{+\infty} \lambda([(2n)(2n-2p)]) = \sum_{p=-1}^{+\infty} \lambda([(2n+2p)(2n)]),$$

and since the measure λ is clearly infinite on X (do note also that it is atomless by construction), the proof is over. $\square$

We do not provide the proofs of the following facts. In particular the proof of the ergodicity is quite involved.

Proposition B.7.7 ([Fer06, Prop. 2.3 and 2.9])**.** *The shift T is not surjective on X, but it is bijective on a set of countable complement. In particular, T is a measure-preserving bijection on a λ-conull set.*

Proposition B.7.8 ([Fer06, Prop. 2.14])**.** *The shift T is ergodic on (X, λ).*

Remark B.7.9. It is also possible to define the whole system as a two-sided shift, *i.e.* a shift on bi-infinite words (on $\mathscr{A}^{\mathbb{Z}}$ instead of $\mathscr{A}^{\mathbb{N}}$). The two systems are measure-theoretically isomorphic, see [Fer06, Sec. 2.4]. This is also the point of view adopted in [BJS24], in which the authors develop the general theory of ∞-mp (ergodic) bijections coming from substitution dynamics.

B.7.2 General theory

In [Fer97], the author opened the door to the study of substitution dynamics on infinite alphabets, and constructed the base of the theory. We succinctly present the framework that generalizes the beginning of Section B.7.1.

Definition B.7.10. The **matrix of a substitution** σ on a countably infinite alphabet $\mathscr{A}$ is $M = (M_{i,j})_{i,j \in \mathscr{A}}$, where $M_{i,j}$ is the number of occurrences of j in the word $\sigma(i)$.

Example B.7.11. The substitution matrix of the system presented in Section B.7.1 is the one given in Figure B.14 below. As in Section B.6.2, we notice that it is a band matrix. This is in fact linked to the following property.

Definition B.7.12. A substitution σ on $\mathscr{A}$ is:

1. of **bounded length** if there exists $L \geqslant 2$ such that for any $n \in \mathscr{A}$ we have $|\sigma(n)| \leqslant L$ (we moreover say that σ is of **constant length** if $|\sigma(n)| = L$);

2. of **bounded size** if it is of bounded length and if there exists a positive integer $t \in \mathbb{Z}$ (which is independent of n) such that for any $n \in \mathbb{Z}$, all the letters of $\sigma(a)$ are in $\{n-t, n-t+1, \dots, n, \dots, n+t-1, n+t\}$, where we identify $\mathscr{A}$ and $\mathbb{Z}$. When it is the case, the smallest t verifying this is called the **size** of σ.

It is then easy to see that a substitution matrix of a σ of bounded size is a band matrix.

$$
(M_{i,j})_{i,j\in 2\mathbb{Z}} = \left(\begin{array}{c|ccccc|c}
 & & & \cdots & & & \\
\hline
 & 2 & 1 & 0 & 0 & 0 & \\
 & 1 & 2 & 1 & 0 & 0 & \\
\cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\
 & 0 & 0 & 1 & 2 & 1 & \\
 & 0 & 0 & 0 & 1 & 2 & \\
\hline
 & & & \cdots & & &
\end{array}\right)
$$

$$
(M_{\iota,j})_{j\in 2\mathbb{Z}\sqcup\{\iota\}} = \left(\begin{array}{c|ccccc|c}
\cdots & 0 & \underset{\substack{\uparrow \\ \boxed{M_{\iota,\iota}}}}{\boxed{1}} & \underset{\substack{\uparrow \\ \boxed{M_{\iota,0}}}}{\boxed{1}} & 0 & 0 & \cdots
\end{array}\right)
$$

Figure B.14: The substitution matrix of the squared drunken man substitution.

This notion of being of bounded size is used in [BJS24] to construct Bratteli-Vershik models of the shifts. As it is beyond the scope of this appendix, we do not expand on this subject.

One other notion that we will need however, is the notion of left-determinacy, already mentioned in Section B.7.1. We recall it, in greater generality, after giving a few more definitions. Although we will not really need it, we in particular use this occasion to define *subshifts* (following [BJS24]), as this is a very active area of research.

Definition B.7.13. The **language of a substitution** σ is denoted by L_σ and defined by

$$L_\sigma = \{\text{words } w \mid w \text{ occurs in } \sigma^n(a) \text{ for some } n \in \mathbb{Z}, a \in \mathscr{A}\}.$$

Notice that this time, the definition does not presuppose the existence of a fixed point for σ. We also define $X_\sigma = \left\{x \in \mathscr{A}^{\mathbb{Z}} \mid L(x) \subseteq L_\sigma\right\} \subseteq \mathscr{A}^{\mathbb{Z}}$, and then X_σ is closed in X, and T-invariant. The dynamical system (X_σ, T) is called the **subshift** on $\mathscr{A}$ associated with σ.

Definition B.7.14. A substitution σ on $\mathscr{A}$ is **left-determined** if there exists a positive $N_\sigma \in \mathbb{N}$ such that any word $w \in L_\sigma$ of length at least N_σ has a unique decomposition $w = w_1 \dots w_s$ where each $w_i = \sigma(a_i)$ for a unique $a_i \in \mathscr{A}$, except that w_1 may be only a suffix of $\sigma(a_1)$ and w_s may be only a prefix of $\sigma(a_s)$.

We have the following theorem, which fixes the topological framework for the study of subshifts of infinite alphabets.

Theorem B.7.15 ([BJS24, Thm. 3.3])**.** *If σ is left-determined, then σ is a homeomorphism from X_σ to $\sigma(X_\sigma)$.*

The next step of the theory would be to extend the Perron-Frobenius theory of matrices to countable matrices. This is done in [Kit98, Sec. 7.1], and we do not recall details. Although the terminology comes from Random Walks and Markov shifts, in the following definition, the matrix is not stochastic.

Definition B.7.16. As in Definition B.6.1, we say that an infinite matrix $M = (M_{i,j})_{i,j\in\mathscr{A}}$ is **irreducible** if for every couple $(i,j) \in \mathscr{A}^2$, there exists $n \geqslant 1$ such that $M_{i,j}^{(n)} > 0$. An irreducible matrix M has **period** d if for every $i \in \mathscr{A}$, we have that $d = \gcd\left\{n \in \mathbb{N} \mid M_{i,i}^{(n)} > 0\right\}$, and is **aperiodic** if $d = 1$. By the generalized Perron-Frobenius Theorem (see [Kit98, Thm. 7.1.3]), an irreducible aperiodic matrix M admits a left eigenvalue λ_∞ defined by

$$\lambda_\infty = \lim_{n\to\infty} \left(M_{i,i}^{(n)}\right)^{\frac{1}{n}},$$

i.e. the generalized spectral radius (it is independent of the choice of i). Note also that the eigenvectors are unique up to constant multiples. We will restrict ourself to the case of $\lambda_\infty < \infty$. We then have different possible behaviours:

1. M is **transient** if $\sum_{n\in\mathbb{Z}} M_{i,j}^{(n)} \lambda_\infty^{-n} < +\infty$ for some $i,j \in \mathscr{A}$ (equivalently all of them by irreducibility, see [Kit98, Lem. 7.1.8]), and **recurrent** otherwise. If M is recurrent, we then define for any $i,j \in \mathscr{A}$: $l_{i,j}(1) = M_{i,j}$ and $l_{i,j}(n+1) = \sum_{k\neq i} l_{i,k}(n) M_{k,j}$.

2.a) M is **null recurrent** if $\sum_{n\in\mathbb{N}} n l_{i,i}(n)\lambda_\infty^{-n} < +\infty$ for some $i \in \mathscr{A}$ (equivalently all of them, see [Kit98, Lem. 7.1.13]);

2.b) M is **positive recurrent** if $\sum_{n\in\mathbb{N}} n l_{i,i}(n)\lambda_\infty^{-n} = +\infty$.

Armed with all these definitions, we can state the main results. Ferenczi ([Fer06, Sec. 3]) classifies substitution systems through the behaviour of their substitution matrices, generalized by Bezuglyi-Jorgensen-Sanadhya ([BJS24, Sec. 7]) via generalized Bratteli diagrams techniques.

Theorem B.7.17 ([Fer06, Sec. 3])**.** *Let σ be of constant length, left-determined, with an irreducible aperiodic substitution matrix M.*

1. *If M is positive recurrent, then (X,T) admits an ergodic invariant natural probability measure μ.*

2. *If M is null recurrent, then (X,T) admits an infinite σ-finite natural measure λ.*

Theorem B.7.18 ([BJS24, Thm. 7.6])**.** *Let σ be a bounded size left-determined substitution on a countably infinite alphabet $\mathscr{A}$, and (X_σ, T) be the associated subshift. Assume that the substitution matrix M associated with σ is irreducible, aperiodic and recurrent.*

1. *There exists a T-invariant measure λ on X_σ.*

2. *The measure λ is finite if and only if the Perron-Frobenius left eigenvector $(\lambda_i)_{i\in\mathscr{A}}$ satisfies $\sum_{i\in\mathscr{A}} \lambda_i < +\infty$.*

Of course, while it is solved for probability measures, the question of ergodicity is not answered for invariant infinite measures. As pointed out by Ferenczi, the erogodicity for the squared drunken man substitution (*i.e.* Proposition B.7.8) is obtained through precise and technical considerations and estimates, which are not generalizable. Bezuglyi-Jorgensen-Sanadhya also obtain ergodicity as a corollary of their result only for the finite measure case, and in the positive recurrent case (see [BJS24, Rem. 7.8]). To our knowledge, there does not exist any general ergodicity result in infinite measure for substitution dynamics on infinite alphabets.

Remark B.7.19. We conclude this appendix by a few remarks on the '*random walk on $\mathbb{Z}$ substitution*', *i.e.* [BJS24, Ex. 8.3]. The substitution matrix (a band matrix where each line is $\ldots 0\,0\,1\,0\,1\,0\,0 \ldots$) is just the transition matrix of the usual unbiased random walk on $\mathbb{Z}$, multiplied by 2. The substitution matrix is irreducible, periodic with period 2, and null recurrent. Although the matrix is not aperiodic, it can be shown that

$$(\lambda_i)_{i\in\mathbb{Z}} = \begin{pmatrix} \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \end{pmatrix},$$

and that there exists a shift-invariant infinite σ-finite measure λ on X. In fact the whole aperiodicity hypothesis can be dispensed with (see [BJS24, Rem. 7.3]).

It is very important however to understand that this dynamical system is very different from the *usual random walk on* $\mathbb{Z}$. Indeed, both the path-space X and the measure λ are defined differently, and obey different constraints.

As it is very new, there is sadly a paucity of concrete examples (to our knowledge), and indeed the only other example of a substitution dynamical system in [BJS24] satisfying condition *2.* of Theorem B.7.18 is the already studied squared drunken man substitution (see [BJS24, Ex. 8.4]).

Bibliography

[Aar97] Jon Aaronson. *An introduction to infinite ergodic theory.* English. Vol. 50. Math. Surv. Monogr. Providence, RI: American Mathematical Society, 1997. ISBN: 0-8218-0494-4.

[AN87] Jon Aaronson and Mahendra Nadkarni. "L_∞ Eigenvalues and L_2 Spectra of Non-Singular Transformations". In: *Proceedings of the London Mathematical Society* s3-55.3 (1987), pp. 538–570. DOI: https://doi.org/10.1112/plms/s3-55.3.538. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/plms/s3-55.3.538.

[AFS97] Terrence Adams, Nathaniel Friedman, and Cesar E. Silva. "Rank-one weak mixing for nonsingular transformations". English. In: *Isr. J. Math.* 102 (1997), pp. 269–281. ISSN: 0021-2172. DOI: 10.1007/BF02773802.

[AS18] Terrence Adams and Cesar E. Silva. "Weak Mixing for Infinite Measure Invertible Transformations". In: *Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics: CIRM Jean-Morlet Chair, Fall 2016.* Ed. by Sébastien Ferenczi, Joanna Kułaga-Przymus, and Mariusz Lemańczyk. Cham: Springer International Publishing, 2018, pp. 327–349. ISBN: 978-3-319-74908-2. DOI: 10.1007/978-3-319-74908-2_17.

[Amb41] Warren Ambrose. "Representation of ergodic flows". In: *Ann. of Math. (2)* 42 (1941), pp. 723–739. ISSN: 0003-486X. DOI: 10.2307/1969259.

[AK42] Warren Ambrose and Shizuo Kakutani. "Structure and continuity of measurable flows". In: *Duke Mathematical Journal* 9.1 (1942), pp. 25–42. DOI: 10.1215/S0012-7094-42-00904-9.

[AR25] Nachi Avraham-Re'em and Emmanuel Roy. "Poissonian actions of Polish groups". In: *Advances in Mathematics* 479 (2025), p. 110437. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2025.110437.

[BK96] Howard Becker and Alexander S. Kechris. *The descriptive set theory of polish group actions.* London Mathematical Society Lecture Note Series. CUP, 1996. ISBN: 9780521576055; 0521576059.

[BV22] Tey Berendschot and Stefaan Vaes. "Nonsingular Bernoulli actions of arbitrary Krieger type". English. In: *Anal. PDE* 15.5 (2022), pp. 1313–1373. ISSN: 2157-5045. DOI: 10.2140/apde.2022.15.1313.

[BJS24] Sergey Bezuglyi, Palle E. T. Jorgensen, and Shrey Sanadhya. *Substitution-dynamics and invariant measures for infinite alphabet-path space.* 2024. arXiv: 2203.14127 [math.DS].

[BKV21] Michael Björklund, Zemer Kosloff, and Stefaan Vaes. "Ergodicity and type of nonsingular Bernoulli actions". English. In: *Invent. Math.* 224.2 (2021), pp. 573–625. ISSN: 0020-9910. DOI: 10.1007/s00222-020-01014-0.

[Bou04] Nicolas Bourbaki. *Elements of mathematics. Integration I: Chapters 1–6. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian.* English. Berlin: Springer, 2004. ISBN: 3-540-41129-1.

[BFMS01] Amie Bowles, Lukasz Fidkowski, Amy E. Marinello, and Cesar E. Silva. "Double ergodicity of nonsingular transformations and infinite measure-preserving staircase transformations". In: *Illinois Journal of Mathematics* 45.3 (2001), pp. 999–1019. DOI: 10.1215/ijm/1258138165.

[BSSSW15] Francisc Bozgan, Anthony Sanchez, Cesar E. Silva, David Stevens, and Jane Wang. "Subsequence bounded rational ergodicity of rank-one transformations". In: *Dynamical Systems* 30.1 (2015), pp. 70–84. DOI: 10.1080/14689367.2014.970518.

[Bro76] James R. Brown. *Ergodic theory and topological dynamics*. English. Vol. 70. Pure Appl. Math., Academic Press. Academic Press, New York, NY, 1976.

[CLM16] Alessandro Carderi and François Le Maître. "More Polish full groups". English. In: *Topology Appl.* 202 (2016), pp. 80–105. ISSN: 0166-8641. DOI: 10.1016/j.topol.2015.12.065.

[Cha69] R. V. Chacon. "Weakly Mixing Transformations Which are Not Strongly Mixing". In: *Proceedings of the American Mathematical Society* 22.3 (1969), pp. 559–562. ISSN: 00029939, 10886826.

[CF65] R. V. Chacon and N. Friedman. "Approximation and invariant measures". English. In: *Z. Wahrscheinlichkeitstheor. Verw. Geb.* 3 (1965), pp. 286–295. ISSN: 0044-3719. DOI: 10.1007/BF00535780.

[Che21] Ruiyuan Chen. *Notes on Quasi-Polish Spaces*. 2021. DOI: 10.48550/arXiv.1809.07440.

[CK79] J. R. Choksi and Shizuo Kakutani. "Residuality of ergodic measurable transformations and of ergodic transformations which preserve an infinite measure". English. In: *Indiana Univ. Math. J.* 28 (1979), pp. 453–469. ISSN: 0022-2518. DOI: 10.1512/iumj.1979.28.28032.

[CP83] J. R. Choksi and V. S. Prasad. "Approximation and Baire category theorems in ergodic theory". In: *Measure Theory and its Applications*. Ed. by Jean-Marc Belley, Jacques Dubois, and Pedro Morales. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983, pp. 94–113. ISBN: 978-3-540-38690-2.

[Coh13] Donald L. Cohn. *Measure theory*. English. 2nd revised ed. Birkhäuser Adv. Texts, Basler Lehrbüch. New York, NY: Birkhäuser/Springer, 2013. ISBN: 978-1-4614-6955-1; 978-1-4614-6956-8. DOI: 10.1007/978-1-4614-6956-8.

[Con00] John B. Conway. *A Course in Operator Theory*. American Mathematical Soc., 2000. ISBN: 978-0-8218-2065-0.

[Cor25a] Corentin Correia. "On quantitative orbit equivalence between amenable groups". Theses. Université Paris Cité, 2025.

[Cor25b] Corentin Correia. "Rank-one systems, flexible classes and Shannon orbit equivalence". In: *Ergodic Theory and Dynamical Systems* 45.7 (2025), pp. 2132–2182. DOI: 10.1017/etds.2024.118.

[Dan95] Alexandre I. Danilenko. "The topological structure of Polish groups and groupoids of measure space transformations". In: *Publ. Res. Inst. Math. Sci.* 31.5 (1995), pp. 913–940. ISSN: 0034-5318. DOI: 10.2977/prims/1195163723.

[Dan04] Alexandre I. Danilenko. "Infinite rank one actions and nonsingular Chacon transformations". English. In: *Ill. J. Math.* 48.3 (2004), pp. 769–786. ISSN: 0019-2082.

[Dan22] Alexandre I. Danilenko. "Haagerup property and Kazhdan pairs via ergodic infinite measure preserving actions". English. In: *Stud. Math.* 265.2 (2022), pp. 211–226. ISSN: 0039-3223. DOI: 10.4064/sm210702-27-10.

[Dan23] Alexandre I. Danilenko. "Krieger's type for ergodic non-singular Poisson actions of non-(T) locally compact groups". English. In: *Ergodic Theory Dyn. Syst.* 43.7 (2023), pp. 2317–2353. ISSN: 0143-3857. DOI: 10.1017/etds.2022.34.

[DK22] Alexandre I. Danilenko and Zemer Kosloff. "Krieger's type of nonsingular Poisson suspensions and IDPFT systems". English. In: *Proc. Am. Math. Soc.* 150.4 (2022), pp. 1541–1557. ISSN: 0002-9939. DOI: 10.1090/proc/15695.

[DP11] Alexandre I. Danilenko and Kyewon K. Park. "Rank-one flows of transformations with infinite ergodic index". In: *Proceedings of the American Mathematical Society* 139.1 (2011), pp. 201–207. ISSN: 00029939, 10886826.

[DS07] Alexandre I. Danilenko and Cesar E. Silva. "Mixing rank-one actions of locally compact Abelian groups". In: *Annales de l'Institut Henri Poincare (B) Probability and Statistics* 43.4 (2007), pp. 375–398. ISSN: 0246-0203. DOI: https://doi.org/10.1016/j.anihpb.2006.05.002.

[DS22] Alexandre I. Danilenko and Cesar E. Silva. *Ergodic Theory: Nonsingular Transformations*. 2022. arXiv: 0803.2424 [math.DS].

[DV23] Alexandre I. Danilenko and Mykyta I. Vieprik. "Explicit rank-1 constructions for irrational rotations". English. In: *Stud. Math.* 270.2 (2023), pp. 121–144. ISSN: 0039-3223. DOI: 10.4064/sm220214-13-10.

[DV25] Alexandre I. Danilenko and Mykyta I. Vieprik. *Classification of rank-one actions via the cutting-and-stacking parameters*. Preprint, arXiv:2504.04987 [math.DS] (2025). 2025.

[Del76] Andrés Del Junco. "Transformations With Discrete Spectrum are Stacking Transformations". In: *Canadian Journal of Mathematics* 28.4 (1976), pp. 836–839. DOI: 10.4153/CJM-1976-080-3.

[DEJLMS23] Hindy Drillick, Alonso Espinosa-Dominguez, Jennifer N. Jones-Baro, James Leng, Yelena Mandelshtam, and Cesar E. Silva. "Non-rigid rank-one infinite measures on the circle". In: *Dynamical Systems* 38.2 (2023), pp. 275–300. DOI: 10.1080/14689367.2023.2174412.

[Dye59] H. A. Dye. "On groups of measure preserving transformations. I". English. In: *Am. J. Math.* 81 (1959), pp. 119–159. ISSN: 0002-9327. DOI: 10.2307/2372852.

[Dye63] H. A. Dye. "On Groups of Measure Preserving Transformations. II". In: *American Journal of Mathematics* 85.4 (1963), pp. 551–576. ISSN: 00029327, 10806377.

[Eig81] S. J. Eigen. "On the simplicity of the full group of ergodic transformations". English. In: *Isr. J. Math.* 40 (1981), pp. 345–349. ISSN: 0021-2172. DOI: 10.1007/BF02761375.

[Eig82] S. J. Eigen. "The group of measure-preserving transformations of [0, 1] has no outer automorphisms". In: *Mathematische Annalen* 259.2 (June 1982), pp. 259–270. ISSN: 1432-1807. DOI: 10.1007/BF01457312.

[EHH98] Stanley Eigen, Arshag Hajian, and Kim Halverson. "Multiple recurrence and infinite measure preserving odometers". English. In: *Isr. J. Math.* 108 (1998), pp. 37–44. ISSN: 0021-2172. DOI: 10.1007/BF02783041.

[EHIP14] Stanley Eigen, Arshag Hajian, Yuji Ito, and Vidhu Prasad. *Weakly wandering sequences in ergodic theory*. English. Springer Monogr. Math. Tokyo: Springer, 2014. ISBN: 978-4-431-55107-2; 978-4-431-55108-9. DOI: 10.1007/978-4-431-55108-9.

[EG16] Amichai Eisenmann and Yair Glasner. "Generic IRS in free groups, after Bowen". English. In: *Proc. Am. Math. Soc.* 144.10 (2016), pp. 4231–4246. ISSN: 0002-9939. DOI: 10.1090/proc/13020.

[Fat78] Adil Fathi. "Le groupe des transformations de [0, 1] qui preservent la mesure de Lebesgue est un groupe simple". In: *Israel Journal of Mathematics* 29 (1978), pp. 302–308.

[Fer97] Sébastien Ferenczi. "Systems of finite rank". English. In: *Colloq. Math.* 73.1 (1997), pp. 35–65. ISSN: 0010-1354. DOI: 10.4064/cm-73-1-35-65.

[Fer06] Sébastien Ferenczi. "Substitution dynamical systems on infinite alphabets". English. In: *Ann. Inst. Fourier* 56.7 (2006), pp. 2315–2343. ISSN: 0373-0956. DOI: 10.5802/aif.2242.

[FBFMS02] N. Pytheas Fogg, Valérie Berthé, Sébastien Ferenczi, Christian Mauduit, and A. Siegel, eds. *Substitutions in dynamics, arithmetics and combinatorics*. English. Vol. 1794. Lect. Notes Math. Berlin: Springer, 2002. ISBN: 3-540-44141-7. DOI: 10.1007/b13861.

[FRW11] Matthew Foreman, Daniel J. Rudolph, and Benjamin Weiss. "The conjugacy problem in ergodic theory". English. In: *Ann. Math. (2)* 173.3 (2011), pp. 1529–1586. ISSN: 0003-486X. DOI: 10.4007/annals.2011.173.3.7.

[Fre02] D. H. Fremlin. *Measure theory 3, Measure algebras*. Measure theory. Torres Fremlin, 2002. ISBN: 9780953812936; 0953812936.

[Fre03] D. H. Fremlin. *Measure theory. Vol. 2. Broad foundations*. English. Corrected second printing of the 2001 original. Colchester: Torres Fremlin, 2003. ISBN: 0-9538129-2-8.

[Fre06] D. H. Fremlin. *Measure theory. Vol. 4. Topological measure spaces. Part I, II*. English. Corrected second printing of the 2003 original. Colchester: Torres Fremlin, 2006. ISBN: 0-9538129-4-4.

[FKSV23] Joshua Frisch, Alexander Kechris, Forte Shinko, and Zoltán Vidnyánszky. *Realizations of Countable Borel Equivalence Relations*. 2023. DOI: 10.48550/arXiv.2109.12486.

[Fur11] Alex Furman. "A survey of measured group theory". In: *Geometry, rigidity, and group actions*. Chicago Lectures in Math. Univ. Chicago Press, Chicago, IL, 2011, pp. 296–374. ISBN: 978-0-226-23788-6; 0-226-23788-5.

[Gab11] Damien Gaboriau. "Orbit equivalence and measured group theory". English. In: *Proceedings of the international congress of mathematicians (ICM 2010), Hyderabad, India, August 19–27, 2010. Vol. III: Invited lectures*. Hackensack, NJ: World Scientific; New Delhi: Hindustan Book Agency, 2011, pp. 1501–1527.

[Gao09] Su Gao. *Invariant descriptive set theory*. English. Vol. 293. Pure Appl. Math. (Boca Raton). Boca Raton, FL: CRC Press, 2009. ISBN: 978-1-58488-793-5.

[GK03] Su Gao and Alexander S. Kechris. *On the classification of Polish metric spaces up to isometry*. English. Vol. 766. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS), 2003. ISBN: 978-0-8218-3190-8; 978-1-4704-0364-5. DOI: 10.1090/memo/0766.

[Gil56] J. Gillis. "Centrally biased discrete random walk". In: *The Quarterly Journal of Mathematics* 7.1 (Jan. 1956), pp. 144–152. ISSN: 0033-5606. DOI: 10.1093/qmath/7.1.144.

[GP07] Thierry Giordano and Vladimir Pestov. "Some extremely amenable groups related to operator algebras and ergodic theory". English. In: *J. Inst. Math. Jussieu* 6.2 (2007), pp. 279–315. ISSN: 1474-7480. DOI: 10.1017/S1474748006000090.

[GTW05] E. Glasner, B. Tsirelson, and B. Weiss. "The automorphism group of the Gaussian measure cannot act pointwise". English. In: *Isr. J. Math.* 148 (2005), pp. 305–329. ISSN: 0021-2172. DOI: 10.1007/BF02775441.

[GW05] E. Glasner and B. Weiss. "Spatial and non-spatial actions of Polish groups". English. In: *Ergodic Theory Dyn. Syst.* 25.5 (2005), pp. 1521–1538. ISSN: 0143-3857. DOI: 10.1017/S0143385705000052.

[Gla03] Eli Glasner. *Ergodic theory via joinings*. English. Vol. 101. Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2003. ISBN: 0-8218-3372-3.

[HK70] A. B. Hajian and S. Kakutani. *Example of an ergodic measure preserving transformation on an infinite measure space*. English. Contrib. Ergodic Theory Probab., Lect. Notes Math. 160, 45-52 (1970). 1970.

[HI69] Arshag Hajian and Yuji Ito. "Weakly Wandering Sets and Invariant Measures for a Group of Transformations". In: *Journal of Mathematics and Mechanics* 18.12 (1969), pp. 1203–1216. ISSN: 00959057, 19435274.

[HIK72] Arshag Hajian, Yuji Ito, and Shizuo Kakutani. "Invariant measures and orbits of dissipative transformations". English. In: *Adv. Math.* 9 (1972), pp. 52–65. ISSN: 0001-8708. DOI: 10.1016/0001-8708(72)90029-1.

[Hal56] P. R. Halmos. *Lectures on ergodic theory*. English. Vol. 3. Publ. Math. Soc. Japan. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, NJ, 1956.

[HO81] Toshihiro Hamachi and Motosige Osikawa. "Ergodic groups of automorphisms and Kriegers theorem". In: Seminar on Mathematical Sciences (1981).

[Hoa25] Fabien Hoareau. *Polish full groups preserving an infinite measure*. 2025. arXiv: 2511.20786 [math.DS].

[HLM24] Fabien Hoareau and François Le Maître. *Spatial models for boolean actions in the infinite measure-preserving setup*. 2024. arXiv: 2406.02401 [math.DS].

[Hop71] Eberhard Hopf. "Ergodic theory and the geodesic flow on surfaces of constant negative curvature". English. In: *Bull. Am. Math. Soc.* 77 (1971), pp. 863–877. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1971-12799-4.

[IKSSW05] Sarah Iams, Brian Katz, Cesar E. Silva, Brian Street, and Kirsten Wickelgren. "On weakly mixing and doubly ergodic nonsingular actions". English. In: *Colloq. Math.* 103.2 (2005), pp. 247–264. ISSN: 0010-1354. DOI: 10.4064/cm103-2-10.

[Ion65] A. Ionescu Tulcea. "On the Category of Certain Classes of Transformations in Ergodic Theory". In: *Transactions of the American Mathematical Society* 114.1 (1965), pp. 261–279. ISSN: 00029947.

[JRR18] Élise Janvresse, Emmanuel Roy, and Thierry de la Rue. "Invariant measures for Cartesian powers of Chacon infinite transformation". English. In: *Isr. J. Math.* 224 (2018), pp. 1–37. ISSN: 0021-2172. DOI: 10.1007/s11856-018-1634-z.

[JŞ15] Aimee S. A. Johnson and Ayşe A. Şahin. "Directional recurrence for infinite measure preserving $\mathbb{Z}^d$ actions". In: *Ergodic Theory and Dynamical Systems* 35.7 (2015), pp. 2138–2150. DOI: 10.1017/etds.2014.17.

[KLM15] Adriane Kaïchouh and François Le Maître. “Connected Polish groups with ample generics”. In: *Bulletin of the London Mathematical Society* 47.6 (2015), pp. 996–1009.

[KP63] S. Kakutani and W. Parry. “Infinite measure preserving transformations with “mixing””. In: *Bulletin of the American Mathematical Society* 69.6 (1963), pp. 752–756.

[Kak43] Shizuo Kakutani. “Induced measure preserving transformations”. English. In: *Proc. Imp. Acad. Tokyo* 19 (1943), pp. 635–641. ISSN: 0369-9846. DOI: 10.3792/pia/1195573248.

[KM10] Steven Kalikow and Randall McCutcheon. *An outline of ergodic theory*. English. Vol. 122. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2010. ISBN: 978-0-521-19440-2.

[Kal85] Robert R. Kallman. “Uniqueness results for groups of measure preserving transformations”. In: *Proc. Amer. Math. Soc.* 95.1 (1985), pp. 87–90. ISSN: 0002-9939,1088-6826. DOI: 10.2307/2045579.

[Kea70] M. Keane. “Contractibility of the automorphism group of a nonatomic measure space”. English. In: *Proc. Am. Math. Soc.* 26 (1970), pp. 420–422. ISSN: 0002-9939. DOI: 10.2307/2037351.

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*. Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[Kec10] Alexander S. Kechris. *Global aspects of ergodic group actions*. American Mathematical Society, 2010.

[KM04] Alexander S. Kechris and Benjamin D. Miller. *Topics in Orbit Equivalence*. 1st ed. Lecture Notes in Mathematics. Springer, 2004. ISBN: 9783540226031; 3540226036.

[Kit98] Bruce P. Kitchens. *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts*. English. Universitext. Berlin: Springer, 1998. ISBN: 3-540-62738-3.

[KT10] John Kittrell and Todor Tsankov. “Topological properties of full groups”. English. In: *Ergodic Theory Dyn. Syst.* 30.2 (2010), pp. 525–545. ISSN: 0143-3857. DOI: 10.1017/S0143385709000078.

[Kre67] Ulrich Krengel. “Entropy of conservative transformations”. In: *Z. Wahrscheinlichkeitstheor. Verw. Geb.* 7 (1967), pp. 161–181. ISSN: 0044-3719. DOI: 10.1007/BF00532635.

[Kre85] Ulrich Krengel. *Ergodic theorems. With a supplement by Antoine Brunel*. English. Vol. 6. De Gruyter Stud. Math. De Gruyter, Berlin, 1985.

[Kri69] Wolfgang Krieger. “On non-singular transformations of a measure space. I, II”. English. In: *Z. Wahrscheinlichkeitstheor. Verw. Geb.* 11 (1969), pp. 83–97, 98–119. ISSN: 0044-3719. DOI: 10.1007/BF00531811.

[Kri76] Wolfgang Krieger. “On ergodic flows and the isomorphism of factors”. English. In: *Math. Ann.* 223 (1976), pp. 19–70. ISSN: 0025-5831. DOI: 10.1007/BF01360278.

[KS11] Aleksandra Kwiatkowska and Sławomir Solecki. “Spatial models of Boolean actions and groups of isometries”. English. In: *Ergodic Theory Dyn. Syst.* 31.2 (2011), pp. 405–421. ISSN: 0143-3857. DOI: 10.1017/S0143385709001138.

[LP17] Günter Last and Mathew Penrose. *Lectures on the Poisson Process*. Institute of Mathematical Statistics Textbooks. Cambridge: Cambridge University Press, 2017. ISBN: 978-1-107-08801-6. DOI: 10.1017/9781316104477.

[LM14a] François Le Maître. "Sur les groupes pleins préservant une mesure de probabilité". Theses. Ecole normale supérieure de lyon - ENS LYON, May 2014.

[LM14b] François Le Maître. "The number of topological generators for full groups of ergodic equivalence relations". In: *Invent. Math.* 198.2 (2014), pp. 261–268. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-014-0503-6.

[LM18] François Le Maître. "On a measurable analogue of small topological full groups". In: *Advances in Mathematics* 332 (2018), pp. 235–286. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2018.05.008.

[LM22] François Le Maître. "Polish topologies on groups of non-singular transformations". English. In: *J. Log. Anal.* 14 (2022). Id/No 4, p. 15. ISSN: 1759-9008. DOI: 10.4115/jla.2022.14.4.

[Mac62] George W. Mackey. "Point Realizations of Transformation Groups". In: *Illinois Journal of Mathematics* 6.2 (1962), pp. 327–335. ISSN: 0019-2082, 1945-6581. DOI: 10.1215/ijm/1255632330.

[Mah64] D. Maharam. "Incompressible transformations". English. In: *Fundam. Math.* 56 (1964), pp. 35–50. ISSN: 0016-2736. DOI: 10.4064/fm-56-1-35-50.

[Mau06] Christian Mauduit. "Arithmetics properties of substitutions and infinite automata". French. In: *Ann. Inst. Fourier* 56.7 (2006), pp. 2525–2549. ISSN: 0373-0956. DOI: 10.5802/aif.2248.

[Mic51] Ernest Michael. "Topologies on Spaces of Subsets". In: *Transactions of the American Mathematical Society* 71.1 (1951), pp. 152–182. ISSN: 0002-9947, 1088-6850. DOI: 10.1090/S0002-9947-1951-0042109-4.

[Mil04] Benjamin D. Miller. "Full groups, classification, and equivalence relations". Theses. 2004.

[Moo76] Calvin C. Moore. "Group Extensions and Cohomology for Locally Compact Groups. III". In: *Transactions of the American Mathematical Society* 221.1 (1976), pp. 1–33. ISSN: 00029947.

[Mun00] James R. Munkres. *Topology*. English. 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2000. ISBN: 0-13-181629-2.

[Mur90] Gerard J. Murphy. *C^*-algebras and operator theory*. English. Boston, MA etc.: Academic Press, Inc., 1990. ISBN: 0-12-511360-9.

[Nes19] Nikolay Nessonov. "Any Nonsingular Action of the Full Symmetric Group Is Isomorphic to an Action with Invariant Measure". In: arXiv:1811.05802 (2019). DOI: 10.48550/arXiv.1811.05802. arXiv: 1811.05802 [math].

[Orn70] Donald Ornstein. "Bernoulli shifts with the same entropy are isomorphic". In: *Advances in Math.* 4 (1970), pp. 337–352. ISSN: 0001-8708. DOI: 10.1016/0001-8708(70)90029-0.

[OW80] Donald S. Ornstein and Benjamin Weiss. "Ergodic theory of amenable group actions. I: The Rohlin lemma". English. In: *Bull. Am. Math. Soc., New Ser.* 2 (1980), pp. 161–164. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-1980-14702-3.

[Pap68] F. Papangelou. "On weak convergence and convergence in measure of transformations in topological measure spaces". English. In: *J. Lond. Math. Soc.* 43 (1968), pp. 521–526. ISSN: 0024-6107. DOI: 10.1112/jlms/s1-43.1.521.

[Par05] K. R. Parthasarathy. *Probability measures on metric spaces.* English. Reprint of the 1967 original. Providence, RI: AMS Chelsea Publishing, 2005. ISBN: 0-8218-3889-X.

[Pes06] Vladimir Pestov. *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon.* English. Vol. 40. Univ. Lect. Ser. Providence, RI: American Mathematical Society (AMS), 2006. ISBN: 0-8218-4137-8.

[Que87] Martine Queffélec. *Substitution dynamical systems - spectral analysis.* English. Vol. 1294. Lect. Notes Math. Springer, Cham, 1987. ISBN: 3-540-18692-1. DOI: 10.1007/bfb0081890.

[Ros05] Christian Rosendal. "On the non-existence of certain group topologies". English. In: *Fundam. Math.* 187.3 (2005), pp. 213–228. ISSN: 0016-2736. DOI: 10.4064/fm187-3-2.

[Ros09] Christian Rosendal. "Automatic Continuity of Group Homomorphisms". In: *The Bulletin of Symbolic Logic* 15.2 (2009), pp. 184–214. ISSN: 1079-8986. JSTOR: 25478252.

[RS07] Christian Rosendal and Sławomir Solecki. "Automatic continuity of homomorphisms and fixed points on metric compacta". English. In: *Isr. J. Math.* 162 (2007), pp. 349–371. ISSN: 0021-2172. DOI: 10.1007/s11856-007-0102-y.

[Roy12] Emmanuel Roy. "Maharam extension and stationary stable processes". English. In: *Ann. Probab.* 40.3 (2012), pp. 1357–1374. ISSN: 0091-1798. DOI: 10.1214/11-AOP671.

[Rud87] Walter Rudin. *Real and complex analysis.* English. 3rd ed. New York, NY: McGraw-Hill, 1987. ISBN: 0-07-054234-1.

[Rud76] Daniel Rudolph. "A two-valued step coding for ergodic flows". In: *Math. Z.* 150.3 (1976), pp. 201–220. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01221147.

[Ryz85] V. V. Ryzhikov. "Representation of transformations preserving the Lebesgue measure in the form of a product of periodic transformations". English. In: *Math. Notes* 38 (1985), pp. 978–981. ISSN: 0001-4346. DOI: 10.1007/BF01157015.

[RT15] V. V. Ryzhikov and J.-P. Thouvenot. "On the centralizer of an infinite mixing rank-one transformation". English. In: *Funct. Anal. Appl.* 49.3 (2015), pp. 230–233. ISSN: 0016-2663. DOI: 10.1007/s10688-015-0111-2.

[Sab19] Marcin Sabok. "Automatic continuity for isometry groups". English. In: *J. Inst. Math. Jussieu* 18.3 (2019), pp. 561–590. ISSN: 1474-7480. DOI: 10.1017/S1474748017000135.

[Sac71] Usha Sachdeva. "On category of mixing in infinite measure spaces". English. In: *Math. Syst. Theory* 5 (1971), pp. 319–330. ISSN: 0025-5661. DOI: 10.1007/BF01694076.

[Sch77] Klaus Schmidt. *Infinite invariant measures on the circle.* English. Symp. math. 21, Misure su Gruppi e su Spazi vettor., Convegno 1975, 37-43 (1977). 1977.

[Sri98] S. M. Srivastava. *A course on Borel sets.* English. Vol. 180. Grad. Texts Math. New York, NY: Springer, 1998. ISBN: 0-387-98412-7.

[Ste51] Norman Steenrod. *The topology of fibre bundles.* English. Vol. 14. Princeton Math. Ser. Princeton University Press, Princeton, NJ, 1951.

[Tör06] Asger Törnquist. “Orbit equivalence and actions of $\mathbb{F}_n$”. English. In: *J. Symb. Log.* 71.1 (2006), pp. 265–282. ISSN: 0022-4812. DOI: 10.2178/jsl/1140641174.

[Tsa24] Todor Tsankov. *Non-singular and probability measure-preserving actions of infinite permutation groups*. 2024. arXiv: 2411.04716 [math.DS].

[Tse22] Anush Tserunyan. “Pointwise ergodic theorem for locally countable quasi-pmp graphs”. English. In: *J. Mod. Dyn.* 18 (2022), pp. 609–655. ISSN: 1930-5311. DOI: 10.3934/jmd.2022019.

[Var63] V. S. Varadarajan. “Groups of automorphisms of Borel spaces”. English. In: *Trans. Am. Math. Soc.* 109 (1963), pp. 191–220. ISSN: 0002-9947. DOI: 10.2307/1993903.

[Ver12] A. M. Vershik. “Totally Nonfree Actions and the Infinite Symmetric Group”. In: *Moscow Mathematical Journal* 12.1 (2012), pp. 193–212, 216. ISSN: 1609-3321. DOI: 10.17323/1609-4514-2012-12-1-193-212.

[VO16] Marcelo Viana and Krerley Oliveira. *Foundations of ergodic theory*. English. Vol. 151. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2016. ISBN: 978-1-107-12696-1; 978-1-316-42260-1. DOI: 10.1017/CBO9781316422601.