PhD Defense
On measure-preserving actions of Polish groups on spaces of infinite measure

Fabien HOAREAU
Under the supervision of Pierre FIMA and François LE MAÎTRE

July 2, 2026

Université Paris Cité

INSTITUT DE MATHÉMATIQUES
JUSSIEU - PARIS RIVE GAUCHE

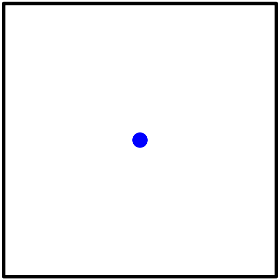
What is “ergodic theory of group actions”?

Dynamical systems

-How things evolve over time

Dynamical systems

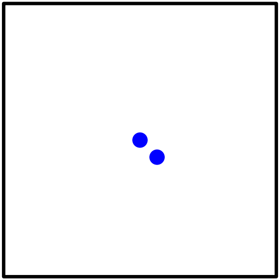
-How things evolve over time



$$t = 0$$

Dynamical systems

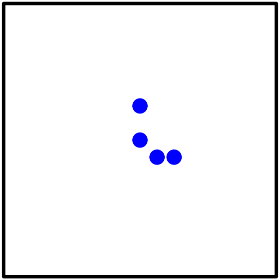
-How things evolve over time



$$t = 1$$

Dynamical systems

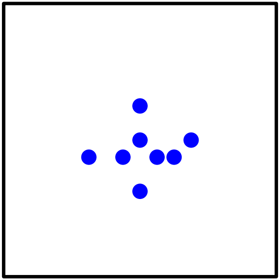
-How things evolve over time



$$t = 2$$

Dynamical systems

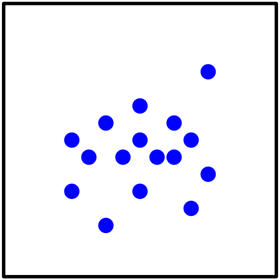
-How things evolve over time



$$t = 3$$

Dynamical systems

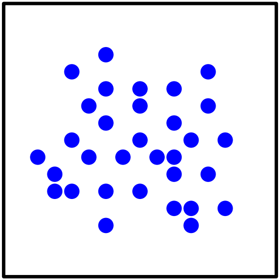
-How things evolve over time



$t = 4$

Dynamical systems

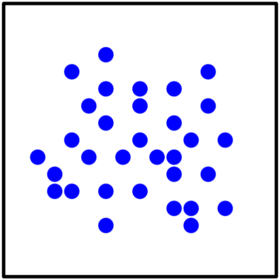
-How things evolve over time



$t = 5$

Dynamical systems

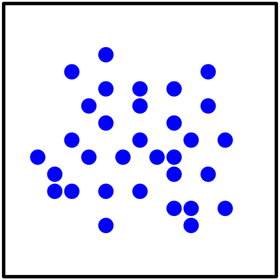
-How things evolve over time
-Very disparate systems (climate, movements, quantities...)
-Uses tools and methods from a lot of fields



$$t = 5$$

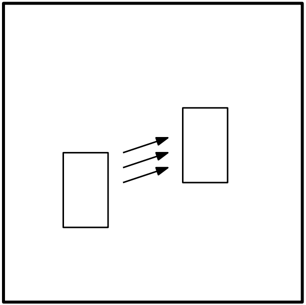
Dynamical systems

-How things evolve over time
-Very disparate systems (climate, movements, quantities...)
-Uses tools and methods from a lot of fields
-Notion of group / group action

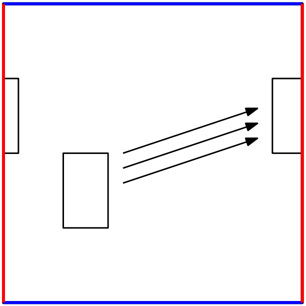


$$t = 5$$

Ergodic theory



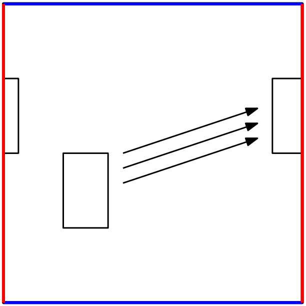
-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)



Ergodic theory

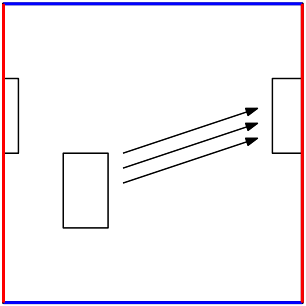
-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)

Ergodic theory



The area is preserved

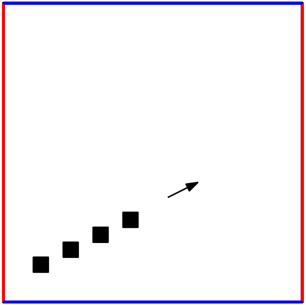
-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation



The area is preserved

Ergodic theory

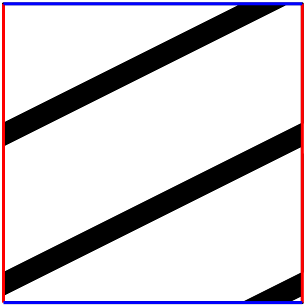
-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?



Remark: For any fixed parameters of the translation T (angle and length), applying powers of T yields a $\mathbb{Z}$-action on the space.

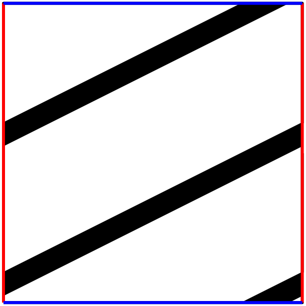
Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?



Ergodic theory

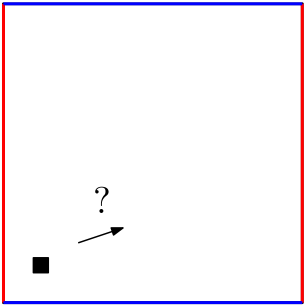
-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?



Non ergodic case

Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?



Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?

Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?

Ergodic case

Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?

Dynamical systems

-How things evolve over time
-Very disparate systems (climate, movements, quantities...)
-Uses tools and methods from a lot of fields
-Notion of group / group action

Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?

Dynamical systems

-How things evolve over time
-Very disparate systems (climate, movements, quantities...)
-Uses tools and methods from a lot of fields
-Notion of group / group action

Ergodic theory

-Relies on measure theory
-Comes from statistical mechanics (Boltzmann, 1871)
-Study of measures that are invariant under a transformation
-Long-term measure theoretic behaviour of the space?

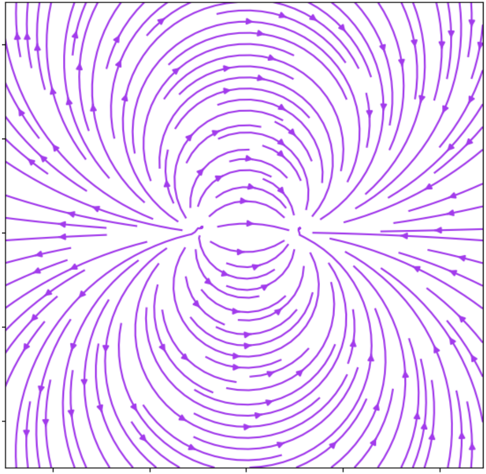
Goal: Generalizing techniques from these fields to actions of more general groups.

First generalization

Extending the "classical ergodic theory" that studies $\mathbb{Z}$-actions to actions of some uncountable groups.

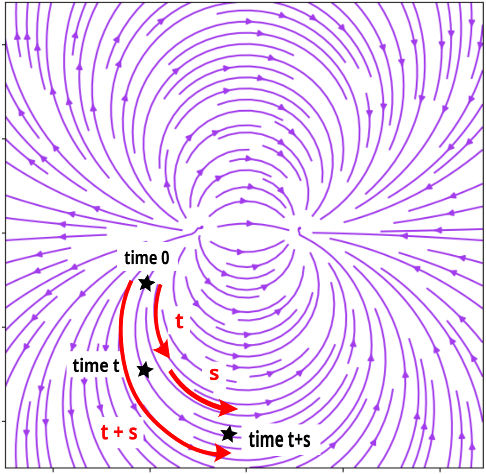
First generalization

Extending the "classical ergodic theory" that studies $\mathbb{Z}$-actions to actions of some uncountable groups.



First generalization

Extending the "classical ergodic theory" that studies $\mathbb{Z}$-actions to actions of some uncountable groups.

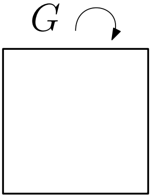


Second generalization

Extending the context of probability measure-preserving (pmp) actions to actions preserving an infinite measure (∞-mp).

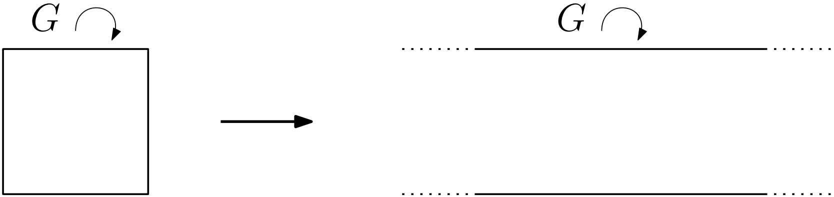
Second generalization

Extending the context of probability measure-preserving (pmp) actions to actions preserving an infinite measure (∞-mp).



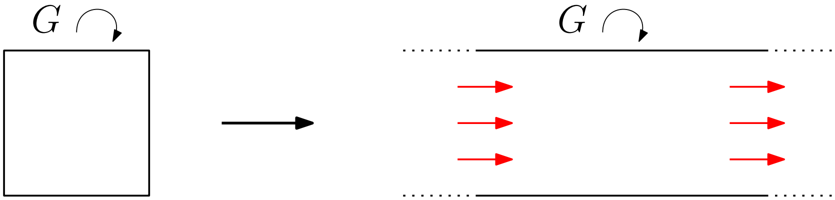
Second generalization

Extending the context of probability measure-preserving (pmp) actions to actions preserving an infinite measure (∞-mp).



Second generalization

Extending the context of probability measure-preserving (pmp) actions to actions preserving an infinite measure (∞-mp).



General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0,1], \mathrm{Leb})$.

General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0, 1], \mathrm{Leb})$.
(X, λ) will be a standard σ-finite space, *i.e.* $(X, \lambda) \simeq (\mathbb{R}, \mathrm{Leb})$.

General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0, 1], \mathrm{Leb})$.
(X, λ) will be a standard σ-finite space, *i.e.* $(X, \lambda) \simeq (\mathbb{R}, \mathrm{Leb})$.

Definition

A **Polish group** G is a topological group (G, τ_G) (*i.e.* $g \to g^{-1}$ and $(g, h) \to gh$ are continuous) such that τ_G is separable and completely metrizable.

General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0,1], \mathrm{Leb})$.
(X, λ) will be a standard σ-finite space, *i.e.* $(X, \lambda) \simeq (\mathbb{R}, \mathrm{Leb})$.

Definition

A **Polish group** G is a topological group (G, τ_G) (*i.e.* $g \to g^{-1}$ and $(g, h) \to gh$ are continuous) such that τ_G is separable and completely metrizable.

Examples

General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0,1], \mathrm{Leb})$.
(X, λ) will be a standard σ-finite space, *i.e.* $(X, \lambda) \simeq (\mathbb{R}, \mathrm{Leb})$.

Definition

A **Polish group** G is a topological group (G, τ_G) (*i.e.* $g \to g^{-1}$ and $(g, h) \to gh$ are continuous) such that τ_G is separable and completely metrizable.

Examples

- Any countable group Γ.

General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0,1], \mathrm{Leb})$.
(X, λ) will be a standard σ-finite space, *i.e.* $(X, \lambda) \simeq (\mathbb{R}, \mathrm{Leb})$.

Definition

A **Polish group** G is a topological group (G, τ_G) (*i.e.* $g \to g^{-1}$ and $(g, h) \to gh$ are continuous) such that τ_G is separable and completely metrizable.

Examples

- Any countable group Γ.
- Any locally compact second countable group.

General framework

(X, μ) will be a standard probability space, *i.e.* $(X, \mu) \simeq ([0,1], \mathrm{Leb})$.
(X, λ) will be a standard σ-finite space, *i.e.* $(X, \lambda) \simeq (\mathbb{R}, \mathrm{Leb})$.

Definition

A **Polish group** G is a topological group (G, τ_G) (*i.e.* $g \to g^{-1}$ and $(g, h) \to gh$ are continuous) such that τ_G is separable and completely metrizable.

Examples

- Any countable group Γ.
- Any locally compact second countable group.
- The group $\mathfrak{S}_\infty$ of all permutations of $\mathbb{N}$.

General framework

What does it mean that "G acts on (X, ν)"? (ν will be either μ or λ)

General framework

What does it mean that "G acts on (X, ν)"? (ν will be either μ or λ)
Notation: $T_*\nu = \nu$ means that $\forall A \subseteq X : \nu(T^{-1}(A)) = \nu(A)$

General framework

What does it meant that "G acts on (X,ν)"? (ν will be either μ or λ)

Notation: $T_*\nu = \nu$ means that $\forall A \subseteq X : \nu(T^{-1}(A)) = \nu(A)$

$\mathrm{Aut}(X,\nu) = \{T : X \to X \mid T \text{ Borel bijection such that } T_*\nu = \nu\} / \sim_\nu$

General framework

What does it mean that "G acts on (X,ν)"? (ν will be either μ or λ)
Notation: $T_*\nu = \nu$ means that $\forall A \subseteq X : \nu(T^{-1}(A)) = \nu(A)$
$\mathrm{Aut}(X,\nu) = \{T : X \to X \mid T$ Borel bijection such that $T_*\nu = \nu\} / \sim_\nu$

Definition

A ∞-mp action (resp. pmp action) of G on (X,λ) (resp. (X,μ)) is a Borel map

$$\alpha : \begin{array}{ccc} G \times X & \longrightarrow & X \\ (g,x) & \longmapsto & \alpha(g,x) \end{array}$$

General framework

What does it mean that "G acts on (X,ν)"? (ν will be either μ or λ)
Notation: $T_*\nu = \nu$ means that $\forall A \subseteq X : \nu(T^{-1}(A)) = \nu(A)$
$\mathrm{Aut}(X,\nu) = \{T : X \to X \mid T \text{ Borel bijection such that } T_*\nu = \nu\} / \sim_\nu$

Definition

A ∞-mp action (resp. pmp action) of G on (X,λ) (resp. (X,μ)) is a Borel map

$$\alpha : \begin{array}{ccc} G \times X & \longrightarrow & X \\ (g,x) & \longmapsto & \alpha(g,x) \end{array}$$

satisfying

- $\forall x \in X : \alpha(e_G, x) = x$
- $\forall x \in X, \forall h, g \in G : \alpha(h, \alpha(g,x)) = \alpha(hg, x)$

General framework

What does it mean that "G acts on (X,ν)"? (ν will be either μ or λ)
Notation: $T_*\nu = \nu$ means that $\forall A \subseteq X : \nu(T^{-1}(A)) = \nu(A)$
$\mathrm{Aut}(X,\nu) = \{T : X \to X \mid T$ Borel bijection such that $T_*\nu = \nu\} / \sim_\nu$

Definition

A ∞-mp action (resp. pmp action) of G on (X,λ) (resp. (X,μ)) is a Borel map

$$\alpha : \begin{array}{ccc} G \times X & \longrightarrow & X \\ (g,x) & \longmapsto & \alpha(g,x) \end{array}$$

satisfying

- $\forall x \in X : \alpha(e_G, x) = x$
- $\forall x \in X, \forall h, g \in G : \alpha(h, \alpha(g,x)) = \alpha(hg, x)$
- $\forall g \in G : \alpha(g,\cdot) \in \mathrm{Aut}(X,\lambda)$ (resp. $\alpha(g,\cdot) \in \mathrm{Aut}(X,\mu)$).

General framework

What does it mean that "G acts on (X, ν)"? (ν will be either μ or λ)
Notation: $T_*\nu = \nu$ means that $\forall A \subseteq X : \nu(T^{-1}(A)) = \nu(A)$
$\mathrm{Aut}(X, \nu) = \{T : X \to X \mid T$ Borel bijection such that $T_*\nu = \nu\} / \sim_\nu$

Definition

A ∞-mp action (resp. pmp action) of G on (X, λ) (resp. (X, μ)) is a Borel map

$$\alpha : \begin{array}{ccc} G \times X & \longrightarrow & X \\ (g, x) & \longmapsto & \alpha(g, x) \end{array}$$

satisfying

- $\forall x \in X : \alpha(e_G, x) = x$
- $\forall x \in X, \forall h, g \in G : \alpha(h, \alpha(g, x)) = \alpha(hg, x)$
- $\forall g \in G : \alpha(g, \cdot) \in \mathrm{Aut}(X, \lambda)$ (resp. $\alpha(g, \cdot) \in \mathrm{Aut}(X, \mu)$).

Question: Can we "classify" the orbit structure of these G-actions?

Notations

- X : standard Borel space

Notations

- X : standard Borel space
- μ : Borel atomless probability measure on X

Notations

- X : standard Borel space
- μ : Borel atomless probability measure on X
- λ : Borel atomless σ-finite infinite measure on X

Notations

- X : standard Borel space
- μ : Borel atomless probability measure on X
- λ : Borel atomless σ-finite infinite measure on X
- ν : can be either μ or λ

Notations

- X : standard Borel space
- μ : Borel atomless probability measure on X
- λ : Borel atomless σ-finite infinite measure on X
- ν : can be either μ or λ
- G : Polish group

Notations

- X : standard Borel space
- μ : Borel atomless probability measure on X
- λ : Borel atomless σ-finite infinite measure on X
- ν : can be either μ or λ
- G : Polish group
- Γ : countable group

Notations

- X : standard Borel space
- μ : Borel atomless probability measure on X
- λ : Borel atomless σ-finite infinite measure on X
- ν : can be either μ or λ
- G : Polish group
- Γ : countable group
- α and β : group actions (sometimes denoted simply by $\cdot$)

OE for pmp actions of countable groups

Definition

Two actions α and β of two countable groups Γ and Λ respectively on (X, μ) are **orbit equivalent** (denoted by OE) if there exists an element $S \in \mathrm{Aut}(X, \mu)$ such that for μ-a.e. $x \in X$ we have

$$S(\alpha(\Gamma, x)) = \beta(\Lambda, S(x)).$$

OE for pmp actions of countable groups

Definition

Two actions α and β of two countable groups Γ and Λ respectively on (X, μ) are **orbit equivalent** (denoted by OE) if there exists an element $S \in \mathrm{Aut}(X, \mu)$ such that for μ-a.e. $x \in X$ we have

$$S(\alpha(\Gamma, x)) = \beta(\Lambda, S(x)).$$

Dye proved the following for pmp group actions...

Theorem (Dye 1959)

All the ergodic pmp actions of $\mathbb{Z}$ are OE.

OE for pmp actions of countable groups

Definition

Two actions α and β of two countable groups Γ and Λ respectively on (X, μ) are **orbit equivalent** (denoted by OE) if there exists an element $S \in \mathrm{Aut}(X, \mu)$ such that for μ-a.e. $x \in X$ we have

$$S(\alpha(\Gamma, x)) = \beta(\Lambda, S(x)).$$

Dye proved the following for pmp group actions...

Theorem (Dye 1959)

All the ergodic pmp actions of $\mathbb{Z}$ are OE.

...and Ornstein and Weiss gave the following generalization.

Theorem (Ornstein–Weiss 1980)

Any two ergodic pmp actions of any two countable infinite amenable groups are OE.

Full groups

Definition

An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n\restriction A_n}$.

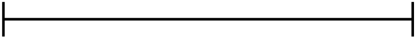
Full groups

Definition

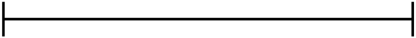
An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n\restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X,\mu)$:

$[0,1]$



$[0,1]$

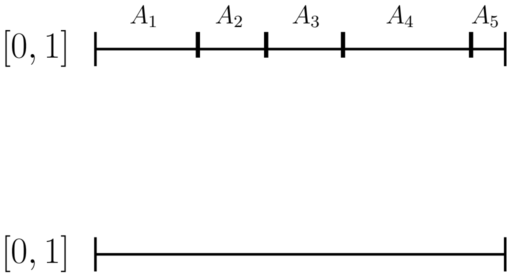


Full groups

Definition

An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n \restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X,\mu)$:

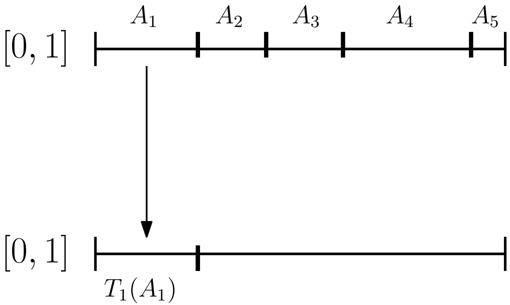


Full groups

Definition

An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n\restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X,\mu)$**:**

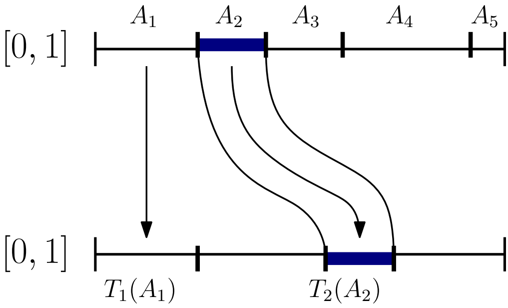


Full groups

Definition

An element T in $\mathrm{Aut}(X, \nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n \restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X, \mu)$**:**

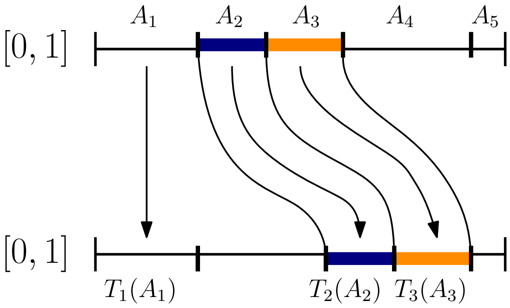


Full groups

Definition

An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n\restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X,\mu)$:

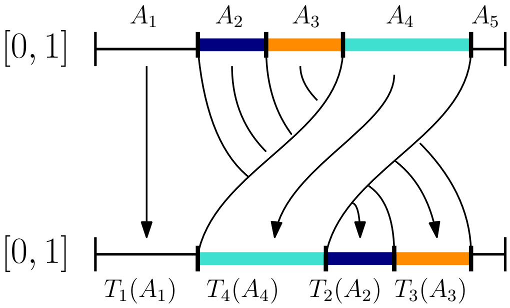


Full groups

Definition

An element T in $\mathrm{Aut}(X,\nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n\restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X,\mu)$:

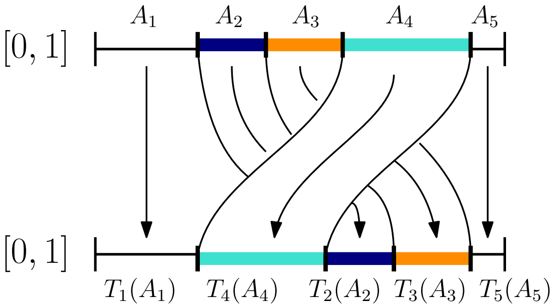


Full groups

Definition

An element T in $\mathrm{Aut}(X, \nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n \restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X, \mu)$:

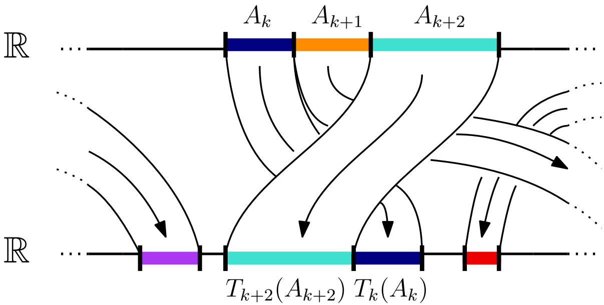


Full groups

Definition

An element T in $\mathrm{Aut}(X, \nu)$ is obtained by **cutting and pasting** a sequence (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have $T_{\restriction A_n} = T_{n \restriction A_n}$.

Cutting and pasting in $\mathrm{Aut}(X, \lambda)$:



Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation: Given a G-action on X, define the equivalence relation $\mathcal{R}_G$ by: $(x,y) \in \mathcal{R}_G \iff y \in G \cdot x$.

Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation: Given a G-action on X, define the equivalence relation $\mathcal{R}_G$ by: $(x, y) \in \mathcal{R}_G \iff y \in G \cdot x$.

Definition

The **orbit full group** associated with the action of a Polish group G on a standard measured space is the group

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X, \nu) \mid (x, T(x)) \in \mathcal{R}_G \text{ for almost all } x \in X\}.$$

Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation: Given a G-action on X, define the equivalence relation $\mathcal{R}_G$ by: $(x, y) \in \mathcal{R}_G \iff y \in G \cdot x$.

Definition

The **orbit full group** associated with the action of a Polish group G on a standard measured space is the group

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X, \nu) \mid (x, T(x)) \in \mathcal{R}_G \text{ for almost all } x \in X\}.$$

Examples

Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation: Given a G-action on X, define the equivalence relation $\mathcal{R}_G$ by: $(x,y) \in \mathcal{R}_G \iff y \in G \cdot x$.

Definition

The **orbit full group** associated with the action of a Polish group G on a standard measured space is the group

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X,\nu) \mid (x, T(x)) \in \mathcal{R}_G \text{ for almost all } x \in X\}.$$

Examples

- For the transitive action of $G = \mathbb{S}^1$ on itself, $[\mathcal{R}_G] = \mathrm{Aut}(X,\mu)$.

Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation: Given a G-action on X, define the equivalence relation $\mathcal{R}_G$ by: $(x,y) \in \mathcal{R}_G \iff y \in G \cdot x$.

Definition

The **orbit full group** associated with the action of a Polish group G on a standard measured space is the group

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X,\nu) \mid (x, T(x)) \in \mathcal{R}_G \text{ for almost all } x \in X\}.$$

Examples

- For the transitive action of $G = \mathbb{S}^1$ on itself, $[\mathcal{R}_G] = \mathrm{Aut}(X,\mu)$.
- For the transitive action of $G = \mathbb{R}$ on itself, $[\mathcal{R}_G] = \mathrm{Aut}(X,\lambda)$.

Full groups

Definition

A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \nu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Notation: Given a G-action on X, define the equivalence relation $\mathcal{R}_G$ by: $(x, y) \in \mathcal{R}_G \iff y \in G \cdot x$.

Definition

The **orbit full group** associated with the action of a Polish group G on a standard measured space is the group

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X, \nu) \mid (x, T(x)) \in \mathcal{R}_G \text{ for almost all } x \in X\}.$$

Examples

- For the transitive action of $G = \mathbb{S}^1$ on itself, $[\mathcal{R}_G] = \mathrm{Aut}(X, \mu)$.
- For the transitive action of $G = \mathbb{R}$ on itself, $[\mathcal{R}_G] = \mathrm{Aut}(X, \lambda)$.
- For an ergodic $\mathbb{Z}$-action on (X, ν), $[\mathcal{R}_\mathbb{Z}] \simeq [\mathcal{R}_{\mathrm{hyp}}] \leqslant \mathrm{Aut}(X, \nu)$.

Definition

$\mathbb{G} \leqslant \mathrm{Aut}(X, \nu)$ is **ergodic** if for every $A \subseteq X$ such that $\nu(T(A)\Delta A) = 0$ for every T in $\mathbb{G}$, we have that A is either null or conull.

Definition

$\mathbb{G} \leqslant \mathrm{Aut}(X, \nu)$ is **ergodic** if for every $A \subseteq X$ such that $\nu(T(A)\Delta A) = 0$ for every T in $\mathbb{G}$, we have that A is either null or conull.

Theorem (Dye's Reconstruction Theorem, 1963)

Let $\mathbb{G}$ and $\mathbb{H}$ be two ergodic full groups on a standard probability space (X, μ). Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some measure-preserving bijection.

Definition

$\mathbb{G} \leqslant \mathrm{Aut}(X,\nu)$ is **ergodic** if for every $A \subseteq X$ such that $\nu(T(A)\Delta A) = 0$ for every T in $\mathbb{G}$, we have that A is either null or conull.

Theorem (Dye's Reconstruction Theorem, 1963)

Let $\mathbb{G}$ and $\mathbb{H}$ be two ergodic full groups on a standard probability space (X,μ). Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some measure-preserving bijection.
In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in $\mathrm{Aut}(X,\mu)$ such that for all $T \in \mathbb{G}$, we have $\psi(T) = STS^{-1}$.

Back to OE

Definition

We say that two ∞-mp (resp. pmp) actions of two Polish groups G and H are **orbit equivalent** if there exists a conull subset $X_0 \subseteq X$ and a Borel bijection $S : X_0 \to X_0$ which verify the following:

Back to OE

Definition

We say that two ∞-mp (resp. pmp) actions of two Polish groups G and H are **orbit equivalent** if there exists a conull subset $X_0 \subseteq X$ and a Borel bijection $S : X_0 \to X_0$ which verify the following:

1. S preserves λ up to multiplication by an element of $\mathbb{R}_+^*$ (resp. μ);

Back to OE

Definition

We say that two ∞-mp (resp. pmp) actions of two Polish groups G and H are **orbit equivalent** if there exists a conull subset $X_0 \subseteq X$ and a Borel bijection $S : X_0 \to X_0$ which verify the following:

1. S preserves λ up to multiplication by an element of $\mathbb{R}_+^*$ (resp. μ);
2. for all $(x, y) \in X_0 \times X_0$, we have $(x, y) \in \mathcal{R}_G \Leftrightarrow (S(x), S(y)) \in \mathcal{R}_H$.

Back to OE

Definition

We say that two ∞-mp (resp. pmp) actions of two Polish groups G and H are **orbit equivalent** if there exists a conull subset $X_0 \subseteq X$ and a Borel bijection $S : X_0 \to X_0$ which verify the following:

1. S preserves λ up to multiplication by an element of $\mathbb{R}_+^*$ (resp. μ);
2. for all $(x, y) \in X_0 \times X_0$, we have $(x, y) \in \mathcal{R}_G \Leftrightarrow (S(x), S(y)) \in \mathcal{R}_H$.

We also say that S is an orbit equivalence between $\mathcal{R}_G$ and $\mathcal{R}_H$.

A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:

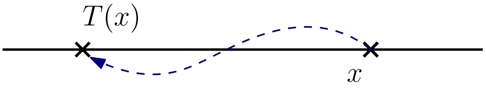
A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:

$X_1 \simeq [0, 1]$

A non-trivial example

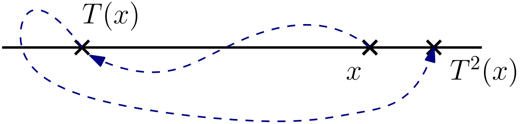
A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



$$X_1 \simeq [0,1]$$

A non-trivial example

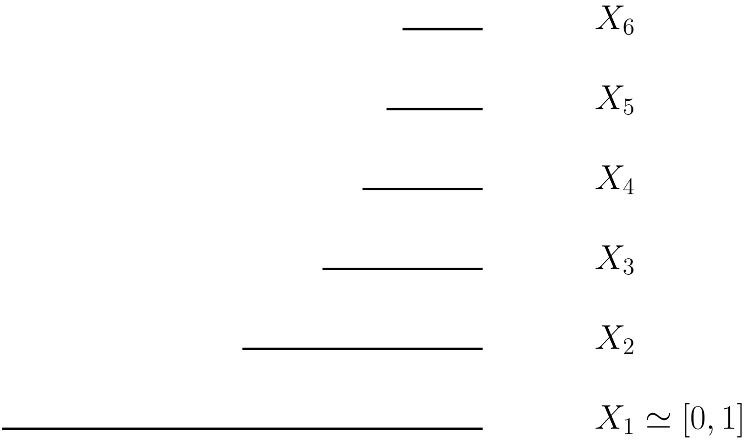
A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



$X_1 \simeq [0,1]$

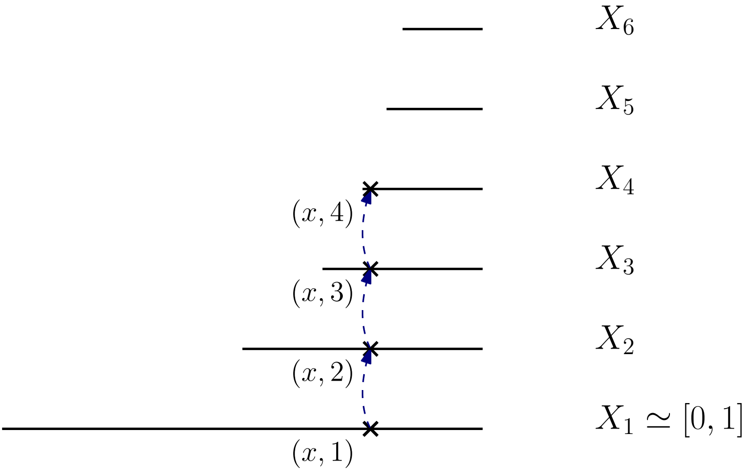
A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



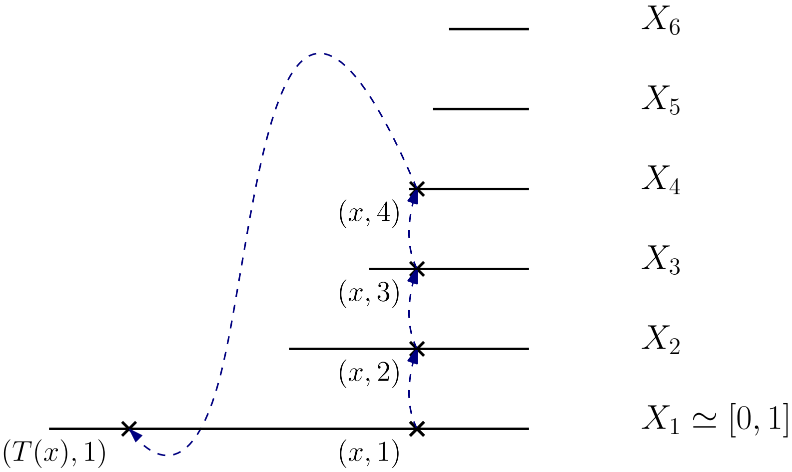
A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



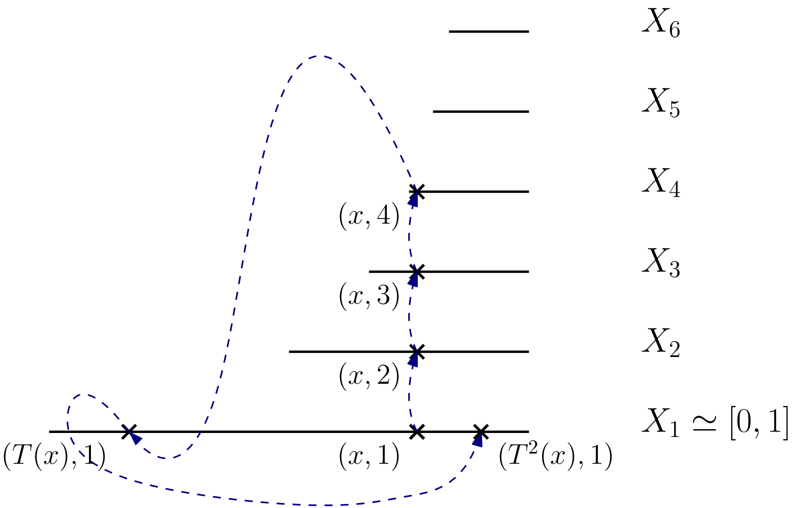
A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



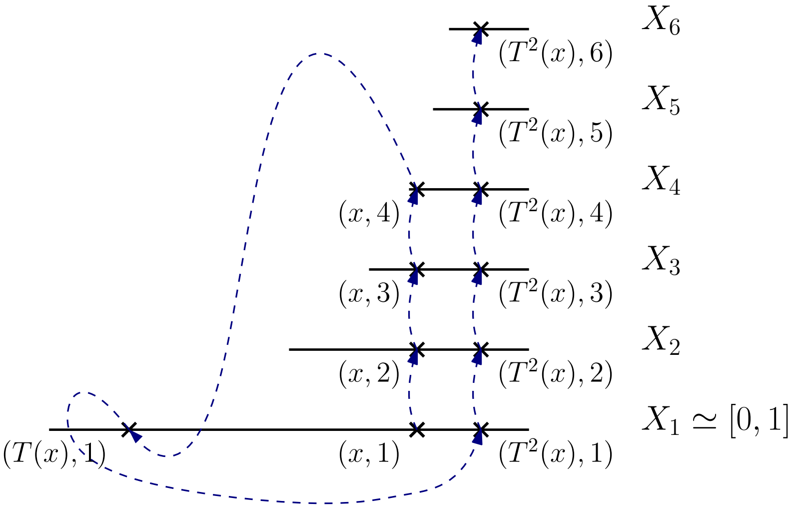
A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



A non-trivial example

A method for constructing ∞-mp ergodic $\mathbb{Z}$-actions:



A non-trivial example

Fact: In the previous construction, if T is ergodic on X_1, then the new bijection is ergodic on $Y = \bigsqcup_{n=1}^{+\infty} X_n \times \{n\}$ endowed with the product measure of the Lebesgue measure and the counting measure.

A non-trivial example

Fact: In the previous construction, if T is ergodic on X_1, then the new bijection is ergodic on $Y = \bigsqcup_{n=1}^{+\infty} X_n \times \{n\}$ endowed with the product measure of the Lebesgue measure and the counting measure.

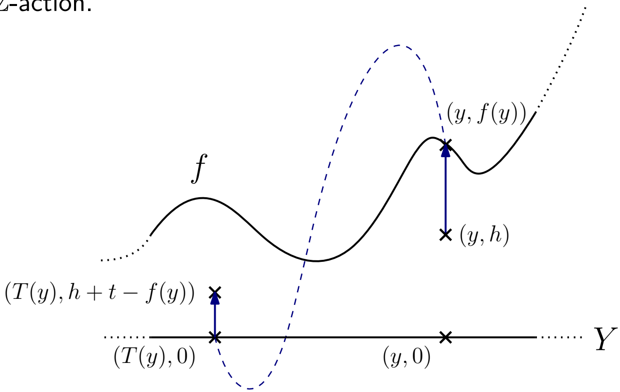
$\longrightarrow$ We can construct ∞-mp ergodic $\mathbb{Z}$-actions.

A non-trivial example

Fact: From the classical construction of a **suspension flow under a function**, we can induce an ∞-mp ergodic $\mathbb{R}$-action from the ∞-mp ergodic $\mathbb{Z}$-action.

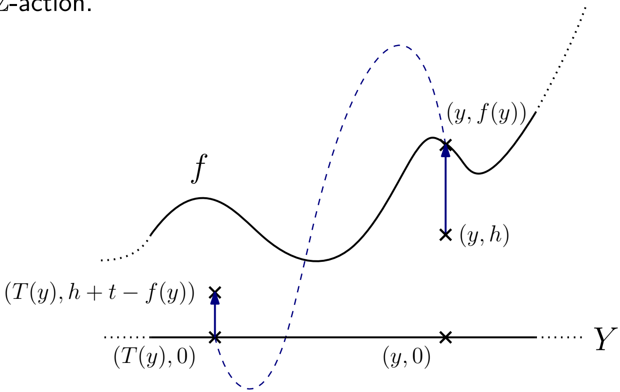
A non-trivial example

Fact: From the classical construction of a **suspension flow under a function**, we can induce an ∞-mp ergodic $\mathbb{R}$-action from the ∞-mp ergodic $\mathbb{Z}$-action.



A non-trivial example

Fact: From the classical construction of a **suspension flow under a function**, we can induce an ∞-mp ergodic $\mathbb{R}$-action from the ∞-mp ergodic $\mathbb{Z}$-action.



$\longrightarrow$ We can construct ∞-mp ergodic $\mathbb{R}$-actions, that are not transitive.

Theorem (H. 2026+)

Let G and H be two locally compact Polish groups acting in a ∞-mp ergodic manner on (X, λ). Let $\psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ be a group isomorphism.

Theorem (H. 2026+)

Let G and H be two locally compact Polish groups acting in a ∞-mp ergodic manner on (X, λ). Let $\psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ be a group isomorphism. Then there exists an orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$, such that ψ is the conjugation by S, i.e. for all $T \in [\mathcal{R}_G]$ we have

$$\psi(T) = STS^{-1} \in [\mathcal{R}_H].$$

Theorem (H. 2026+)

Let G and H be two locally compact Polish groups acting in a ∞-mp ergodic manner on (X, λ). Let $\psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ be a group isomorphism. Then there exists an orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$, such that ψ is the conjugation by S, i.e. for all $T \in [\mathcal{R}_G]$ we have

$$\psi(T) = STS^{-1} \in [\mathcal{R}_H].$$

Remark

In order to adapt the result, we rely heavily on a generalization of Dye's Reconstruction Theorem, namely Fremlin's Reconstruction Theorem.

Theorem

- *Full groups of pmp ergodic actions of countable groups on (X, μ) are complete invariants of OE.* (Dye)

Theorem

- *Full groups of pmp ergodic actions of countable groups on* (X, μ) *are complete invariants of OE.* (Dye)
- *Full groups of pmp ergodic actions of locally compact Polish groups on* (X, μ) *are complete invariants of OE.* (Carderi–Le Maître)

Theorem

- *Full groups of pmp ergodic actions of countable groups on* (X, μ) *are complete invariants of OE.* (Dye)
- *Full groups of pmp ergodic actions of locally compact Polish groups on* (X, μ) *are complete invariants of OE.* (Carderi–Le Maître)
- *Full group of* ∞*-mp ergodic actions of locally compact Polish groups on* (X, λ) *are complete invariants of OE.* (H.)

Topologies on full groups

The **weak topology** τ_w makes $\mathrm{Aut}(X,\nu)$ a Polish group.
($T_n \to T$ weakly if $\nu(T_n(A)\Delta T(A)) \to 0 \ \ (\forall A, \nu(A) < \infty)$)

Topologies on full groups

The **weak topology** τ_w makes $\mathrm{Aut}(X,\nu)$ a Polish group.
($T_n \to T$ weakly if $\nu(T_n(A)\Delta T(A)) \to 0 \ \ (\forall A, \nu(A) < \infty)$)

The **uniform topology** τ_u makes $[\mathcal{R}_\Gamma] \leqslant \mathrm{Aut}(X,\nu)$ a Polish group.
($T_n \to T$ uniformly if $\nu(\{x \in A \mid T_n(x) \neq T(x)\}) \to 0 \ \ (\forall A, \nu(A) < \infty)$)

What about more general orbit full groups $[\mathcal{R}_G]$?

Topologies on full groups

The **weak topology** τ_w makes $\mathrm{Aut}(X,\nu)$ a Polish group.
($T_n \to T$ weakly if $\nu(T_n(A)\Delta T(A)) \to 0 \;\; (\forall A, \nu(A) < \infty)$)

The **uniform topology** τ_u makes $[\mathcal{R}_\Gamma] \leqslant \mathrm{Aut}(X,\nu)$ a Polish group.
($T_n \to T$ uniformly if $\nu(\{x \in A \mid T_n(x) \neq T(x)\}) \to 0 \;\; (\forall A, \nu(A) < \infty)$)

What about more general orbit full groups $[\mathcal{R}_G]$?

Theorem (Carderi–Le Maître 2016)

Let G be a Polish group acting in a pmp way on (X,μ). There exists a topology called the ***topology of orbital convergence in measure*** *that makes $[\mathcal{R}_G] \leqslant \mathrm{Aut}(X,\mu)$ a Polish group.*

Topologies on full groups

The **weak topology** τ_w makes $\mathrm{Aut}(X, \nu)$ a Polish group.
($T_n \to T$ weakly if $\nu(T_n(A) \Delta T(A)) \to 0 \ \ (\forall A, \nu(A) < \infty)$)

The **uniform topology** τ_u makes $[\mathcal{R}_\Gamma] \leqslant \mathrm{Aut}(X, \nu)$ a Polish group.
($T_n \to T$ uniformly if $\nu(\{x \in A \mid T_n(x) \neq T(x)\}) \to 0 \ \ (\forall A, \nu(A) < \infty)$)

What about more general orbit full groups $[\mathcal{R}_G]$?

Theorem (Carderi–Le Maître 2016)

Let G be a Polish group acting in a pmp way on (X, μ). There exists a topology called the ***topology of orbital convergence in measure*** *that makes $[\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \mu)$ a Polish group.*

Is there a topology that makes $[\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ a Polish group?

Sketch of proof

Denote $\mathrm{L}^0(X,\mu,Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ f & \longmapsto & (x \mapsto f(x) \cdot x) \end{array}$$

Sketch of proof

Denote $\mathrm{L}^0(X,\mu,Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X,\mu,G) & \longrightarrow & \mathrm{L}^0(X,\mu,X) \\ f & \longmapsto & (x \mapsto f(x)\cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X,\mu,G)$ is a Polish group.

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ f & \longmapsto & (x \mapsto f(x) \cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X, \mu, G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ f & \longmapsto & (x \mapsto f(x) \cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X, \mu, G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ f & \longmapsto & (x \mapsto f(x) \cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X, \mu, G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Define $\widetilde{[\mathcal{R}_G]} = \Theta^{-1}(\mathrm{Aut}(X, \mu))$.

Sketch of proof

Denote $\mathrm{L}^0(X,\mu,Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X,\mu,G) & \longrightarrow & \mathrm{L}^0(X,\mu,X) \\ f & \longmapsto & (x \mapsto f(x)\cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X,\mu,G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Check that $(\mathrm{Aut}(X,\mu),\tau_w)$ topologically embeds into $(\mathrm{L}^0(X,\mu,X),\tau_m)$.

Define $\widetilde{[\mathcal{R}_G]} = \Theta^{-1}(\mathrm{Aut}(X,\mu))$.

Check that $[\mathcal{R}_G] \simeq \widetilde{[\mathcal{R}_G]}/\mathrm{Ker}(\Theta)$ is a Polish group. The resulting topology is the **topology of orbital convergence in measure.**

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X,\mu,G) & \longrightarrow & \mathrm{L}^0(X,\mu,X) \\ f & \longmapsto & (x \mapsto f(x)\cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X, \mu, G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Check that $(\mathrm{Aut}(X,\mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X,\mu,X), \tau_m)$.

Define $\widetilde{[\mathcal{R}_G]} = \Theta^{-1}(\mathrm{Aut}(X,\mu))$.

Check that $[\mathcal{R}_G] \simeq \widetilde{[\mathcal{R}_G]}/\mathrm{Ker}(\Theta)$ is a Polish group. The resulting topology is the **topology of orbital convergence in measure.**

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ f & \longmapsto & (x \mapsto f(x) \cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X, \mu, G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Define $\widetilde{[\mathcal{R}_G]} = \Theta^{-1}(\mathrm{Aut}(X, \mu))$.

Check that $[\mathcal{R}_G] \simeq \widetilde{[\mathcal{R}_G]}/\mathrm{Ker}(\Theta)$ is a Polish group. The resulting topology is the **topology of orbital convergence in measure.**

Sketch of proof

Denote $\mathrm{L}^0(X, \mu, Z) = \{f : X \to Z \mid f \text{ is Lebesgue-measurable}\} / \sim_\mu$. It is endowed with a topology τ_m (the topology of convergence in measure).

$$\Theta : \begin{array}{ccc} \mathrm{L}^0(X, \mu, G) & \longrightarrow & \mathrm{L}^0(X, \mu, X) \\ f & \longmapsto & (x \mapsto f(x) \cdot x) \end{array}$$

Fact (Moore): The group $\mathrm{L}^0(X, \mu, G)$ is a Polish group.

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X) and check that Θ is continuous.

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Define $\widetilde{[\mathcal{R}_G]} = \Theta^{-1}(\mathrm{Aut}(X, \mu))$.

Check that $[\mathcal{R}_G] \simeq \widetilde{[\mathcal{R}_G]}/\mathrm{Ker}(\Theta)$ is a Polish group. The resulting topology is the **topology of orbital convergence in measure.**

Problems for generalizing to $\mathrm{Aut}(X, \lambda)$

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X).

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Problems for generalizing to $\mathrm{Aut}(X, \lambda)$

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X).

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel injection $\Phi : X \hookrightarrow K$ which is G-equivariant: for all $x \in X, g \in G$, $\Phi(\alpha(g, x)) = \beta(g, \Phi(x))$.

$$\begin{tikzcd} X \arrow[r, "\alpha(g,\cdot)"] \arrow[d, hook, "\Phi"'] & X \arrow[d, hook, "\Phi"] \\ K \arrow[r, "\beta(g,\cdot)"] & K \end{tikzcd}$$

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Problems for generalizing to $\mathrm{Aut}(X, \lambda)$

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X).

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel injection $\Phi : X \hookrightarrow K$ which is G-equivariant: for all $x \in X, g \in G$, $\Phi(\alpha(g,x)) = \beta(g, \Phi(x))$.

$$\begin{tikzcd} X \arrow[r, "\alpha(g,\cdot)"] \arrow[d, hook, "\Phi"'] & X \arrow[d, hook, "\Phi"] \\ K \arrow[r, "\beta(g,\cdot)"] & K \end{tikzcd}$$

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Proposition

Let X be endowed with a compatible Polish topology τ_X.

- $(\mathrm{Aut}(X, \mu), \tau_w) \hookrightarrow (\mathrm{L}^0(X, \mu, X), \tau_m)$ *is a topological embedding.*

Problems for generalizing to $\mathrm{Aut}(X, \lambda)$

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X).

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel injection $\Phi : X \hookrightarrow K$ which is G-equivariant: for all $x \in X, g \in G$, $\Phi(\alpha(g,x)) = \beta(g, \Phi(x))$.

$$\begin{array}{ccc} X & \xrightarrow{\alpha(g,\cdot)} & X \\ \Phi \big\downarrow & & \big\downarrow \Phi \\ K & \xrightarrow{\beta(g,\cdot)} & K \end{array}$$

Check that $(\mathrm{Aut}(X,\mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X,\mu,X), \tau_m)$.

Proposition

Let X be endowed with a compatible Polish topology τ_X.

- $(\mathrm{Aut}(X,\mu), \tau_w) \hookrightarrow (\mathrm{L}^0(X,\mu,X), \tau_m)$ *is a topological embedding.*
- $(\mathrm{Aut}(X,\lambda), \tau_w) \hookrightarrow (\mathrm{L}^0(X,\lambda,X), \tau_m)$ *is a topological embedding*

Problems for generalizing to $\mathrm{Aut}(X, \lambda)$

Make $G \curvearrowright X$ continuous (for a compatible Polish topology on X).

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel injection $\Phi : X \hookrightarrow K$ which is G-equivariant: for all $x \in X, g \in G$, $\Phi(\alpha(g,x)) = \beta(g, \Phi(x))$.

$$\begin{tikzcd} X \arrow[r, "\alpha(g,\cdot)"] \arrow[d, hook, "\Phi"'] & X \arrow[d, hook, "\Phi"] \\ K \arrow[r, "\beta(g,\cdot)"] & K \end{tikzcd}$$

Check that $(\mathrm{Aut}(X, \mu), \tau_w)$ topologically embeds into $(\mathrm{L}^0(X, \mu, X), \tau_m)$.

Proposition

Let X be endowed with a compatible Polish topology τ_X.

- $(\mathrm{Aut}(X, \mu), \tau_w) \hookrightarrow (\mathrm{L}^0(X, \mu, X), \tau_m)$ *is a topological embedding.*
- $(\mathrm{Aut}(X, \lambda), \tau_w) \hookrightarrow (\mathrm{L}^0(X, \lambda, X), \tau_m)$ *is a topological embedding under the condition that λ is locally finite with regard to τ_X.*

Local finiteness

Definition

Let λ be a Borel measure on a Polish space X. We say that λ is **locally finite** if every $x \in X$ has an open neighborhood U such that $\lambda(U) < \infty$.

Local finiteness

Definition

Let λ be a Borel measure on a Polish space X. We say that λ is **locally finite** if every $x \in X$ has an open neighborhood U such that $\lambda(U) < \infty$.

Proposition

Let X be a locally compact Polish space, let λ be a Borel measure on X. Then λ is Radon if and only if it is locally finite.

Local finiteness

Definition

Let λ be a Borel measure on a Polish space X. We say that λ is **locally finite** if every $x \in X$ has an open neighborhood U such that $\lambda(U) < \infty$.

Proposition

Let X be a locally compact Polish space, let λ be a Borel measure on X. Then λ is Radon if and only if it is locally finite.

Question: Can we make ∞-mp G-actions on (X, λ) continuous, with a compatible topology τ_X such that λ is locally finite on (X, τ_X)?

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel G-equivariant injection $\Phi : X \hookrightarrow K$.

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel G-equivariant injection $\Phi : X \hookrightarrow K$.

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of a locally compact Polish group G on (X, λ). There is a continuous measure-preserving G-action β on a locally compact Polish space Y and a Borel G-equivariant injection $\Phi : X \hookrightarrow Y$ such that the pushforward measure $\Phi_* \lambda$ is a Radon measure on Y.

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel G-equivariant injection $\Phi : X \hookrightarrow K$.

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of a locally compact Polish group G on (X, λ). There is a continuous measure-preserving G-action β on a locally compact Polish space Y and a Borel G-equivariant injection $\Phi : X \hookrightarrow Y$ such that the pushforward measure $\Phi_* \lambda$ is a Radon measure on Y.

*We say that α admits a **continuous Radon model**.*

Theorem (Becker–Kechris 1993)

Let α be a Borel G-action on X. There exists a compact Polish space K with a continuous G-action β on K and a Borel G-equivariant injection $\Phi : X \hookrightarrow K$.

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of a locally compact Polish group G on (X, λ). There is a continuous measure-preserving G-action β on a locally compact Polish space Y and a Borel G-equivariant injection $\Phi : X \hookrightarrow Y$ such that the pushforward measure $\Phi_* \lambda$ is a Radon measure on Y.

*We say that α admits a **continuous Radon model**.*

Observation

There is a natural $\mathrm{Aut}(X, \lambda)$-action on $\mathrm{L}^0(X, \lambda, \mathbb{C})$ defined by

$$T \cdot f(x) = f(T^{-1}(x))$$

Observation

There is a natural $\mathrm{Aut}(X, \lambda)$-action on $\mathrm{L}^0(X, \lambda, \mathbb{C})$ defined by

$$T \cdot f(x) = f(T^{-1}(x))$$

$\mathrm{Aut}(X, \lambda) \curvearrowright \mathrm{L}^1(X, \lambda, \mathbb{C})$ continuously.

Observation

There is a natural $\mathrm{Aut}(X, \lambda)$-action on $\mathrm{L}^0(X, \lambda, \mathbb{C})$ defined by

$$T \cdot f(x) = f(T^{-1}(x))$$

$\mathrm{Aut}(X, \lambda) \curvearrowright \mathrm{L}^1(X, \lambda, \mathbb{C})$ continuously.
$\mathrm{Aut}(X, \lambda) \curvearrowright \mathrm{L}^2(X, \lambda, \mathbb{C})$ continuously.

Observation

There is a natural $\mathrm{Aut}(X,\lambda)$-action on $\mathrm{L}^0(X,\lambda,\mathbb{C})$ defined by

$$T \cdot f(x) = f(T^{-1}(x))$$

$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^1(X,\lambda,\mathbb{C})$ continuously.
$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^2(X,\lambda,\mathbb{C})$ continuously.
$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^\infty(X,\lambda,\mathbb{C})$ not continuously.

Observation

There is a natural $\mathrm{Aut}(X,\lambda)$-action on $\mathrm{L}^0(X,\lambda,\mathbb{C})$ defined by

$$T \cdot f(x) = f(T^{-1}(x))$$

$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^1(X,\lambda,\mathbb{C})$ continuously.
$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^2(X,\lambda,\mathbb{C})$ continuously.
$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^\infty(X,\lambda,\mathbb{C})$ not continuously.

Definition (Glasner–Tsirelson–Weiss 1995)

Let $G \curvearrowright (X,\lambda)$ be a ∞-mp action. We say that $f \in \mathrm{L}^\infty(X,\lambda,\mathbb{C})$ is G-**continuous** if $\|f - g \cdot f\|_\infty \to 0$ as $g \to e_G$.

Observation

There is a natural $\mathrm{Aut}(X,\lambda)$-action on $\mathrm{L}^0(X,\lambda,\mathbb{C})$ defined by

$$T \cdot f(x) = f(T^{-1}(x))$$

$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^1(X,\lambda,\mathbb{C})$ continuously.
$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^2(X,\lambda,\mathbb{C})$ continuously.
$\mathrm{Aut}(X,\lambda) \curvearrowright \mathrm{L}^\infty(X,\lambda,\mathbb{C})$ not continuously.

Definition (Glasner–Tsirelson–Weiss 1995)

Let $G \curvearrowright (X,\lambda)$ be a ∞-mp action. We say that $f \in \mathrm{L}^\infty(X,\lambda,\mathbb{C})$ (or in $\mathcal{L}^\infty(X)$) is G-**continuous** if $\|f - g\cdot f\|_\infty \to 0$ as $g \to e_G$.

Key characterization

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of G on (X, λ). TFAE

Key characterization

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of G on (X, λ). TFAE

- *α admits a **continuous Radon model**;*

Key characterization

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of G on (X, λ). TFAE

- *α admits a **continuous Radon model**;*
- *The algebra of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ contains a countable subset which separates points.*

Key characterization

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of G on (X, λ). TFAE

- *α admits a **continuous Radon model**;*
- *The algebra of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ contains a countable subset which separates points.*

- Construct a space Y as the spectrum of a well-chosen subalgebra of G-continuous functions in $\mathcal{L}^\infty(X)$.

Key characterization

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of G on (X, λ). TFAE

- *α admits a* ***continuous Radon model****;*
- *The algebra of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ contains a countable subset which separates points.*

- Construct a space Y as the spectrum of a well-chosen subalgebra of G-continuous functions in $\mathcal{L}^\infty(X)$.
- Define a Radon measure η on Y with the Riesz representation theorem.

Key characterization

Theorem (H.–Le Maître 2026)

Let α be a ∞-mp action of G on (X, λ). TFAE

- *α admits a* ***continuous Radon model****;*
- *The algebra of G-continuous functions in $\mathcal{L}^\infty(X) \cap \mathcal{L}^1(X, \lambda)$ contains a countable subset which separates points.*

- Construct a space Y as the spectrum of a well-chosen subalgebra of G-continuous functions in $\mathcal{L}^\infty(X)$.
- Define a Radon measure η on Y with the Riesz representation theorem.
- Embed X in Y in a Borel way via the evaluation map Φ, and check that $\Phi_*\lambda = \eta$ and that the G-action on (Y, η) is as wanted.

Second step

$\longrightarrow$ Find "enough" G-continuous functions.

Second step

$\longrightarrow$ Find "enough" G-continuous functions.

Our tool: Convolution.

Second step

$\longrightarrow$ Find "enough" G-continuous functions.

Our tool: Convolution.

Definition

Let G be a locally compact Polish group, and denote by m its left Haar measure. Consider a ∞-mp G-action on (X, λ). Let $f \in \mathcal{L}^\infty(X)$ and $\delta \in \mathrm{L}^1(G, m)$. The **convolution product** $\delta * f : X \to \mathbb{C}$ is defined by:

$$\delta * f(x) = \int_G \delta(g) f(g^{-1} \cdot x) dm(g).$$

Second step

$\longrightarrow$ Find "enough" G-continuous functions.

Our tool: Convolution.

Definition

Let G be a locally compact Polish group, and denote by m its left Haar measure. Consider a ∞-mp G-action on (X, λ). Let $f \in \mathcal{L}^\infty(X)$ and $\delta \in \mathrm{L}^1(G, m)$. The **convolution product** $\delta * f : X \to \mathbb{C}$ is defined by:

$$\delta * f(x) = \int_G \delta(g) f(g^{-1} \cdot x) dm(g).$$

$\longrightarrow$ Use convolution with functions δ being "approximations of the identity in G" to obtain G-continuity!

Recall the following.

Theorem (H.–Le Maître 2026)

Any ∞-mp action of a locally compact Polish group G on (X, λ) has a continuous Radon model.

Recall the following.

Theorem (H.–Le Maître 2026)

Any ∞-mp action of a locally compact Polish group G on (X, λ) has a continuous Radon model.

What about other groups?

Recall the following.

Theorem (H.–Le Maître 2026)

Any ∞-mp action of a locally compact Polish group G on (X, λ) has a continuous Radon model.

What about other groups?

Proposition

There exists a ∞-mp action of $\mathfrak{S}_\infty$ on (X, λ) which cannot admit a continuous Radon model.

Recall the following.

Theorem (Carderi–Le Maître 2016)

Let G be a Polish group acting in a pmp way on (X, μ). The orbit full group $[\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \mu)$ is a Polish group when endowed with the topology of orbital convergence in measure.

Theorem (H. 2026+)

Let G be a Polish group acting in a continuous and ∞-mp manner on a Polish space (X, τ_X) equipped with an atomless Borel σ-finite measure λ which is locally finite. The orbit full group $[\mathcal{R}_G]$, equipped with the topology of orbital convergence in measure, is a Polish group.

Theorem (H. 2026+)

Let G be a Polish group acting in a continuous and ∞-mp manner on a Polish space (X, τ_X) equipped with an atomless Borel σ-finite measure λ which is locally finite. The orbit full group $[\mathcal{R}_G]$, equipped with the topology of orbital convergence in measure, is a Polish group.

Proposition

There exists a ∞-mp action of $\mathfrak{S}_\infty$ on (X, λ), for which the associated orbit full group $[\mathcal{R}_{\mathfrak{S}_\infty}]$ is not a Polish group when endowed with the topology of orbital convergence in measure.

Theorem (H. 2026+)

Let G be a Polish group acting in a continuous and ∞-mp manner on a Polish space (X, τ_X) equipped with an atomless Borel σ-finite measure λ which is locally finite. The orbit full group $[\mathcal{R}_G]$, equipped with the topology of orbital convergence in measure, is a Polish group.

Proposition

There exists a ∞-mp action of $\mathfrak{S}_\infty$ on (X, λ), for which the associated orbit full group $[\mathcal{R}_{\mathfrak{S}_\infty}]$ is not a Polish group when endowed with the topology of orbital convergence in measure.

Theorem (H. 2026+)

Let G be a locally compact Polish group acting in a ∞-mp way on (X, λ). The orbit full group $[\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ is a Polish group when endowed with the topology of orbital convergence in measure.

Theorem (Carderi–Le Maître 2016, H. 2026+)

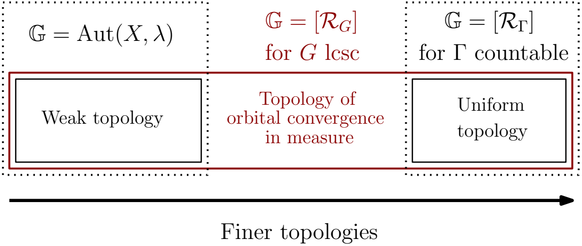
Let $\mathbb{G}$ be an ergodic full group in $\mathrm{Aut}(X, \lambda)$ (resp. $\mathrm{Aut}(X, \mu)$).

Theorem (Carderi–Le Maître 2016, H. 2026+)

Let $\mathbb{G}$ be an ergodic full group in $\mathrm{Aut}(X, \lambda)$ (resp. $\mathrm{Aut}(X, \mu)$). There is at most one Polish group topology on $\mathbb{G}$, and this topology refines the weak topology but is coarser than the uniform topology.

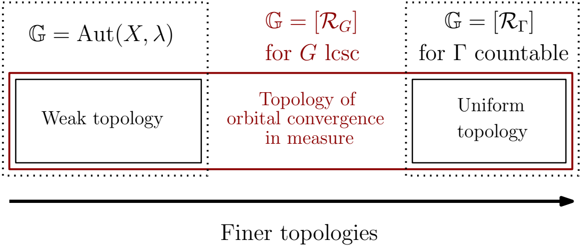
Theorem (Carderi–Le Maître 2016, H. 2026+)

Let $\mathbb{G}$ be an ergodic full group in $\mathrm{Aut}(X, \lambda)$ (resp. $\mathrm{Aut}(X, \mu)$). There is at most one Polish group topology on $\mathbb{G}$, and this topology refines the weak topology but is coarser than the uniform topology.



Theorem (Carderi–Le Maître 2016, H. 2026+)

Let $\mathbb{G}$ be an ergodic full group in $\mathrm{Aut}(X, \lambda)$ (resp. $\mathrm{Aut}(X, \mu)$). There is at most one Polish group topology on $\mathbb{G}$, and this topology refines the weak topology but is coarser than the uniform topology.



Remark

Showing that Polish topologies are coarser than the uniform topology uses an automatic continuity result of Fremlin, which extends a result of Kittrell and Tsankov for $[\mathcal{R}_\Gamma] \leqslant \mathrm{Aut}(X, \mu)$.

A non-trivial normal subgroup

Define $\mathrm{Aut}_f(X,\lambda) = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(\operatorname{supp} T) < \infty\} \lhd \mathrm{Aut}(X,\lambda)$.
Define $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda) \lhd \mathbb{G}$.

A non-trivial normal subgroup

Define $\mathrm{Aut}_f(X,\lambda) = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(\mathrm{supp}\ T) < \infty\} \triangleleft \mathrm{Aut}(X,\lambda)$.
Define $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda) \triangleleft \mathbb{G}$.

Theorem

- *For* $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ *ergodic:* $\mathbb{G}$ *is simple.* (Fathi)

A non-trivial normal subgroup

Define $\mathrm{Aut}_f(X,\lambda) = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(\mathrm{supp}\, T) < \infty\} \triangleleft \mathrm{Aut}(X,\lambda)$.
Define $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda) \triangleleft \mathbb{G}$.

Theorem

- *For* $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ *ergodic:* $\mathbb{G}$ *is simple.* (Fathi)
- *For* $\mathbb{G} = [\mathcal{R}_{\mathrm{hyp}}] \leqslant \mathrm{Aut}(X,\lambda)$*:* $\mathbb{G}$ *admits a unique non-trivial normal subgroup, which is* $\mathbb{G}_f$*.* (Eigen)

A non-trivial normal subgroup

Define $\mathrm{Aut}_f(X,\lambda) = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(\mathrm{supp}\, T) < \infty\} \triangleleft \mathrm{Aut}(X,\lambda)$.
Define $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda) \triangleleft \mathbb{G}$.

Theorem

- *For* $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ *ergodic:* $\mathbb{G}$ *is simple.* (Fathi)
- *For* $\mathbb{G} = [\mathcal{R}_{\mathrm{hyp}}] \leqslant \mathrm{Aut}(X,\lambda)$*:* $\mathbb{G}$ *admits a unique non-trivial normal subgroup, which is* $\mathbb{G}_f$*.* (Eigen)
- *For* $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ *ergodic:* $\mathbb{G}$ *admits a unique non-trivial normal subgroup, which is* $\mathbb{G}_f$*.* (H.)

A non-trivial normal subgroup

Define $\mathrm{Aut}_f(X,\lambda) = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(\mathrm{supp}\ T) < \infty\} \lhd \mathrm{Aut}(X,\lambda)$.
Define $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda) \lhd \mathbb{G}$.

Theorem

- *For $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ ergodic: $\mathbb{G}$ is simple.* (Fathi)
- *For $\mathbb{G} = [\mathcal{R}_{\mathrm{hyp}}] \leqslant \mathrm{Aut}(X,\lambda)$: $\mathbb{G}$ admits a unique non-trivial normal subgroup, which is $\mathbb{G}_f$.* (Eigen)
- *For $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ ergodic: $\mathbb{G}$ admits a unique non-trivial normal subgroup, which is $\mathbb{G}_f$.* (H.)

Question: Is the group $\mathbb{G}_f$ Polishable? In other words, can $\mathbb{G}_f$ be endowed with a Polish group topology refining that of $\mathbb{G}$?

A non-trivial normal subgroup

Define $\mathrm{Aut}_f(X,\lambda) = \{T \in \mathrm{Aut}(X,\lambda) \mid \lambda(\mathrm{supp}\, T) < \infty\} \triangleleft \mathrm{Aut}(X,\lambda)$.
Define $\mathbb{G}_f = \mathbb{G} \cap \mathrm{Aut}_f(X,\lambda) \triangleleft \mathbb{G}$.

Theorem

- *For* $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ *ergodic:* $\mathbb{G}$ *is simple.* (Fathi)
- *For* $\mathbb{G} = [\mathcal{R}_{\mathrm{hyp}}] \leqslant \mathrm{Aut}(X,\lambda)$*:* $\mathbb{G}$ *admits a unique non-trivial normal subgroup, which is* $\mathbb{G}_f$*.* (Eigen)
- *For* $\mathbb{G} \leqslant \mathrm{Aut}(X,\lambda)$ *ergodic:* $\mathbb{G}$ *admits a unique non-trivial normal subgroup, which is* $\mathbb{G}_f$*.* (H.)

Question: Is the group $\mathbb{G}_f$ Polishable? In other words, can $\mathbb{G}_f$ be endowed with a Polish group topology refining that of $\mathbb{G}$?

Theorem (Le Maître 2022)

The group $\mathrm{Aut}_f(X,\lambda) \leqslant \mathrm{Aut}(X,\lambda)$ *is not Polishable.*

Theorem (H. 2026+)

Let $\mathbb{G}$ be an ergodic full group in $\mathrm{Aut}(X, \lambda)$. The group $\mathbb{G}_f$ can be endowed with a Polish group topology iff $\mathbb{G}$ is the full group of a countable equivalence relation.

Theorem (H. 2026+)

Let $\mathbb{G}$ be an ergodic full group in $\mathrm{Aut}(X, \lambda)$. The group $\mathbb{G}_f$ can be endowed with a Polish group topology iff $\mathbb{G}$ is the full group of a countable equivalence relation.

Remark

The uniform finite topology on $\mathbb{G}_f$ is the topology induced by the following metric:

$$d_{u,f} : S, T \mapsto \lambda\left(\{x \in X \mid S(x) \neq T(x)\}\right).$$

It refines the uniform topology τ_u.

Denote by ERG and APER the ergodic and aperiodic (with only infinite orbits) elements of $\mathrm{Aut}(X, \lambda)$, and fix an ergodic full group $\mathbb{G}$.

Denote by ERG and APER the ergodic and aperiodic (with only infinite orbits) elements of $\mathrm{Aut}(X,\lambda)$, and fix an ergodic full group $\mathbb{G}$.
Sachdeva (1971) and Choksi–Kakutani (1979) proved:

Weak topology τ_w

- ERG is a τ_w-dense G_δ set in $\mathrm{Aut}(X,\lambda)$.

Uniform topology τ_u

Denote by ERG and APER the ergodic and aperiodic (with only infinite orbits) elements of $\mathrm{Aut}(X, \lambda)$, and fix an ergodic full group $\mathbb{G}$.
Sachdeva (1971) and Choksi–Kakutani (1979) proved:

Weak topology τ_w

- ERG is a τ_w-dense G_δ set in $\mathrm{Aut}(X, \lambda)$.

Uniform topology τ_u

- ERG is τ_u-nowhere dense in $\mathrm{Aut}(X, \lambda)$.

Denote by ERG and APER the ergodic and aperiodic (with only infinite orbits) elements of $\mathrm{Aut}(X, \lambda)$, and fix an ergodic full group $\mathbb{G}$.
Sachdeva (1971) and Choksi–Kakutani (1979) proved:

Weak topology τ_w

- ERG is a τ_w-dense G_δ set in $\mathrm{Aut}(X, \lambda)$.

Uniform topology τ_u

- ERG is τ_u-nowhere dense in $\mathrm{Aut}(X, \lambda)$.
- ERG is τ_u-dense in APER.

Denote by ERG and APER the ergodic and aperiodic (with only infinite orbits) elements of $\mathrm{Aut}(X, \lambda)$, and fix an ergodic full group $\mathbb{G}$.
Sachdeva (1971) and Choksi–Kakutani (1979) proved:

Weak topology τ_w

- ERG is a τ_w-dense G_δ set in $\mathrm{Aut}(X, \lambda)$.

Uniform topology τ_u

- ERG is τ_u-nowhere dense in $\mathrm{Aut}(X, \lambda)$.
- ERG is τ_u-dense in APER.
- For a fixed $T \in \mathrm{APER} \cap \mathbb{G}$, the $\mathbb{G}$-conjugacy class of T is τ_u-dense in $\mathrm{APER} \cap \mathbb{G}$.

Denote by ERG and APER the ergodic and aperiodic (with only infinite orbits) elements of $\mathrm{Aut}(X, \lambda)$, and fix an ergodic full group $\mathbb{G}$.
Sachdeva (1971) and Choksi–Kakutani (1979) proved:

Weak topology τ_w

- ERG is a τ_w-dense G_δ set in $\mathrm{Aut}(X, \lambda)$.
- For a fixed $T \in \mathrm{APER} \cap \mathbb{G}$, the $\mathbb{G}$-conjugacy class of T is τ_w-dense in $\mathrm{Aut}(X, \lambda)$.

Uniform topology τ_u

- ERG is τ_u-nowhere dense in $\mathrm{Aut}(X, \lambda)$.
- ERG is τ_u-dense in APER.
- For a fixed $T \in \mathrm{APER} \cap \mathbb{G}$, the $\mathbb{G}$-conjugacy class of T is τ_u-dense in $\mathrm{APER} \cap \mathbb{G}$.

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ be endowed with the topology of orbital convergence in measure τ. Suppose that τ is a Polish group topology on $\mathbb{G}$. TFAE

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ *be endowed with the topology of orbital convergence in measure* τ. *Suppose that* τ *is a Polish group topology on* $\mathbb{G}$. *TFAE*

(1) $\mathrm{APER} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$;

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ *be endowed with the topology of orbital convergence in measure* τ. *Suppose that* τ *is a Polish group topology on* $\mathbb{G}$. *TFAE*

(1) $\mathrm{APER} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*
(2) $\mathrm{ERG} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ *be endowed with the topology of orbital convergence in measure* τ*. Suppose that* τ *is a Polish group topology on* $\mathbb{G}$*. TFAE*

(1) $\mathrm{APER} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*

(2) $\mathrm{ERG} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*

(3) *the* $\mathbb{G}_f$*-conjugacy class of any aperiodic* $T \in \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ be endowed with the topology of orbital convergence in measure τ. Suppose that τ is a Polish group topology on $\mathbb{G}$. TFAE

(1) $\mathrm{APER} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*
(2) $\mathrm{ERG} \cap \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*
(3) *the* $\mathbb{G}_f$*-conjugacy class of any aperiodic* $T \in \mathbb{G}$ *is* τ*-dense in* $\mathbb{G}$*;*
(4) *for any essentially free measure-preserving action of a countable group* $\Gamma \curvearrowright (X, \lambda)$*, there is a dense* G_δ *set of elements of* $\mathbb{G}$ *inducing a free action of* $\Gamma * \mathbb{Z}$*.*

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ be endowed with the topology of orbital convergence in measure τ. Suppose that τ is a Polish group topology on $\mathbb{G}$. TFAE

(1) $\mathrm{APER} \cap \mathbb{G}$ *is τ-dense in $\mathbb{G}$;*
(2) $\mathrm{ERG} \cap \mathbb{G}$ *is τ-dense in $\mathbb{G}$;*
(3) *the $\mathbb{G}_f$-conjugacy class of any aperiodic $T \in \mathbb{G}$ is τ-dense in $\mathbb{G}$;*
(4) *for any essentially free measure-preserving action of a countable group $\Gamma \curvearrowright (X, \lambda)$, there is a dense G_δ set of elements of $\mathbb{G}$ inducing a free action of $\Gamma * \mathbb{Z}$.*

These equivalent conditions moreover imply the following one:

(5) *for any neighbourhood V of e_G, $\bigcup_{g \in V} \{x \in X \mid g \cdot x \neq x\}$ is conull.*

Theorem (H. 2026+, following Törnquist and Carderi–Le Maître)

Let $\mathbb{G} = [\mathcal{R}_G] \leqslant \mathrm{Aut}(X, \lambda)$ be endowed with the topology of orbital convergence in measure τ. Suppose that τ is a Polish group topology on $\mathbb{G}$. TFAE

(1) $\mathrm{APER} \cap \mathbb{G}$ *is τ-dense in $\mathbb{G}$;*
(2) $\mathrm{ERG} \cap \mathbb{G}$ *is τ-dense in $\mathbb{G}$;*
(3) *the $\mathbb{G}_f$-conjugacy class of any aperiodic $T \in \mathbb{G}$ is τ-dense in $\mathbb{G}$;*
(4) *for any essentially free measure-preserving action of a countable group $\Gamma \curvearrowright (X, \lambda)$, there is a dense G_δ set of elements of $\mathbb{G}$ inducing a free action of $\Gamma * \mathbb{Z}$.*

These equivalent conditions moreover imply the following one:

(5) *for any neighbourhood V of e_G, $\bigcup_{g \in V} \{x \in X \mid g \cdot x \neq x\}$ is conull.*

Finally, (5) implies the above conditions if the space X is endowed with a compatible Polish topology, with regard to which the measure is locally finite and the G-action is continuous.

Corollary

Let G be a locally compact Polish group acting in a ∞-mp and ergodic manner on (X,λ). Then exactly one of the following holds:

(i) *$\mathcal{R}_G$ is a countable measure-preserving equivalence relation;*

(ii) *The set $\mathrm{APER} \cap [\mathcal{R}_G]$ is τ_m^o-dense in $[\mathcal{R}_G]$.*

Corollary

Let G be a locally compact Polish group acting in a ∞-mp and ergodic manner on (X, λ). Then exactly one of the following holds:

(i) *$\mathcal{R}_G$ is a countable measure-preserving equivalence relation;*

(ii) *The set $\mathrm{APER} \cap [\mathcal{R}_G]$ is τ_m^o-dense in $[\mathcal{R}_G]$.*

Example

For $G = \mathbb{R}$ acting on itself by translation, we recover Sachdeva's results on genericity for the weak topology in $\mathrm{Aut}(X, \lambda) = [\mathcal{R}_G]$.

Thank you for your attention!